5. Proof of Theorem 4. We first rearrange the polynomial $M(s)$ by applying the celebrated identity of R. C. Vaughan [5]. Since the arguments are well known, we shall be very brief. Thus $M(s)$ can be regarded as the product of two Dirichlet's polynomials having either the shape

\begin{equation}
M_1(s)N_1(s)
\end{equation}

or

\begin{equation}
M_3(s)L_3(s),
\end{equation}

where $M_1(s), N_1(s), M_3(s)$ and $L_3(s)$ are Dirichlet's polynomials of length $M_1, N_1, M_3, L_3$ respectively satisfying

\[ M_1N_1 \leq M, \quad W < N_1 < M, \]
\[ M_3L_3 \leq M, \quad M_3 < W^2 \]

and with coefficients bounded by the divisor function. Moreover $L_3(s)$ is a partial sum of the Riemann zeta-function. Here $W$ is any parameter at our disposal. We choose $W = T^{12}$, so for polynomial (11), by Theorem 2 of

\[ \int_0^T |\zeta M_1 N_1(\frac{1}{2}+it)|^2 dt \leq T^4(T + T^{12}M^4 N_1^{14} + T^{12}M N_1^{-12} + M^7 N_1^{-14}) \]

\[ \leq T^4(T + T^{12}M^{-12} + T^{12}M + T^{-12}M^7) \]

which implies (5). For the polynomial (12), if $M \geq W$ the arguments are very similar to those of the first one, and if $M_3 < W$ we apply Theorem 3 getting

\[ \int_0^T |\zeta M_3 L_3(\frac{1}{2}+it)|^2 dt \leq T^{1+\epsilon}. \]

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Reducibility of lacunary polynomials, IV

by

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The aim of this paper is to make a further contribution to the problem of reducibility of polynomials

\[ f(x) = a_0 + \sum_{j=1}^k a_j x^{n_j} \quad (0 = n_0 < n_1 < \ldots < n_k, a_j \neq 0) \]

for fixed integral coefficients $a_j$ and variable exponents $n_j$. The non-reciprocal irreducible factors of $f(x)$ can be found by means of Theorem 2 in [4] and as to reciprocally factors the conjecture proposed in [2] implies the existence of a constant $C(a_0, a_1, \ldots, a_k)$ such that all reciprocal irreducible factors of $f$ are cyclotomic or if

\[ \sum_{j=1}^k \gamma_j n_j = 0 \]

for suitable integers $\gamma_j$ satisfying

\[ 0 < \max_{i<j<k} |\gamma_j| \leq C(a_0, a_1, \ldots, a_k). \]

We shall prove

**Theorem.** If $f$ is given by (1) with $a_j$ integral, then either all reciprocal irreducible factors of $f$ are cyclotomic or there exist integers $\gamma_1, \ldots, \gamma_k$ satisfying

\[ \sum_{j=1}^k \gamma_j n_j = 0, \]

\[ 0 < \max_{j=1,\ldots,k} |\gamma_j| \leq \max \frac{\log a_j^2}{\log 2} \]

and the number of reciprocal non-cyclotomic factors of $f$ does not exceed the total number of prime factors of $(a_0, a_k)$ or finally the following system of
inequalities is fulfilled
\[
\begin{align*}
&\sum_{j=0}^{k} |a_j| |n_j - n_i| > |a_k| (n_k - n_i) \quad \text{if} \quad n_i < n_k/2, \\
&\sum_{j=0}^{k} |a_j| |n_j - n_i| > |a_i| n_i \quad \text{if} \quad n_i > n_k/2, \\
&\sum_{j=0}^{k} |a_j| \left| n_j - \frac{n_k}{2} \right| > \frac{|a_k|}{2} \left| n_k - \frac{n_k}{2} \right|
\end{align*}
\]  
(4)

This theorem supersedes Lemma 16 of [3] and implies the following

**Corollary.** If \( \sum_{j=0}^{k} \xi^j = 0 \) and \( |\xi| = 1 \) then \( \xi \) is a root of unity.

The corollary answers the negative Problem 1 of Mahler [1].

The proof of the theorem is based on two lemmata.

**Lemma 1.** For every positive real number \( r \neq 1 \) and real numbers \( n, m \) satisfying \( |n| > |m| \) we have
\[
h(m, n, r) = \frac{r^m - r^n}{r^m - r^n} \left| n \right| < \left| m \right| \cdot
\]

**Proof.** Since \( h(m, n, r) \) is an even function of \( m \) and \( n \)
\[
h(m, n, r^{-1}) = h(m, n, r)
\]

it is enough to prove the lemma for \( n > m > 0, \ r > 1 \). Now, the function
\[
g(r) = n \left( r^n - r^m \right) - m \left( r^m - r^n \right)
\]

satisfies
\[
g'(r) = mn^{-1} [(r^n - r^m) - r(r^n - r^m)] = mn^{-1} (r^m - r^n) (1 - r^{-m}) > 0
\]

for all \( r > 1 \) hence for such \( r \) \( g(r) > g(1) = 0 \) and
\[
h(m, n, r) = \frac{m}{n} - \frac{g(r)}{r^m - r^n} < \frac{m}{n}.
\]

**Lemma 2.** Let \( f \) be given by (1) with \( a_0 \) arbitrary complex numbers. If
\[
\sum_{j=0}^{k} a_j r^{n_j} = f(\xi) = f(\xi^{-1}) = 0
\]

then either \( |\xi| = 1 \) or the system (4) is fulfilled.

**Proof.** Let \( \xi = r^\theta \) (\( r, \theta \) real) and let \( q \) be a real number. From (5) we infer that
\[
\sum_{j=0}^{k} a_j r^{n_j - q} \xi^{m_j} = 0 = \sum_{j=0}^{k} a_j r^{n_j - q} \xi^{-m_j}
\]

hence
\[
\sum_{j=0}^{k} a_j (r^{n_j - q} - r^{n_j - q}) \xi^{m_j} = 0.
\]

Taking \( q = n_i \) and
\[
\nu = \begin{cases} 
2 & \text{if} \ n_i < n_k/2, \\
0 & \text{if} \ n_i > n_k/2, 
\end{cases}
\]

we get
\[
|n_j - n_i| \leq |n_k - n_i| \quad (0 \leq j \leq k)
\]

hence dividing by \( r^{n_k - n_1} - r^{n_k - n} \) and using Lemma 1 we get the first two inequalities of (4). The last inequality is obtained similarly on taking \( q = n_k/2 \).

**Remark.** This lemma supersedes Lemma 14 of [3].

**Proof of Theorem.** Suppose that \( f \) has a reciprocal irreducible factor \( g \) that is not cyclotomic. Let \( \eta \) be a zero of \( g \). By Kronecker's theorem either \( \eta \) has a conjugate \( \xi \) with \( |\xi| > 1 \) or \( \eta \) is not an algebraic integer. In the former case we use Lemma 2 and get the conditions (4). In the latter case we use Lemma 13 of [3] and get the conditions (3) and (3). Also the product of the leading coefficients of all reciprocal non-cyclotomic factors of \( f \) must divide \( (a_0, a_k) \). Since all these coefficients are greater than 1 their number does not exceed the total number of prime factors of \( (a_0, a_k) \).

**Proof of Corollary.** Let \( g \) be a minimal polynomial of \( \xi \). Since \( g(\xi^{-1}) = g(\xi) = 0, \ g \) is reciprocal. We apply the theorem to the polynomial \( f(x) = \sum_{j=0}^{k} \xi^{m_j} \). Since this polynomial does not satisfy the conditions (4) for \( i = 0 \) and \( (a_0, a_k) = 1 \) the number of its reciprocal non-cyclotomic factors is 0. Hence \( g \) is cyclotomic and \( \xi \) is a root of unity.

**References**


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