5. Proof of Theorem 4. We first rearrange the polynomial $M(s)$ by applying the celebrated identity of R. C. Vaughan [5]. Since the arguments are well-known we shall be very brief. Thus $M(s)$ can be regarded as the product of two Dirichlet's polynomials having either the shape

$$M_1(s)N_1(s) \tag{11}$$

or

$$M_2(s)L_2(s), \tag{12}$$

where $M_1(s)$, $N_1(s)$, $M_2(s)$ and $L_2(s)$ are Dirichlet's polynomials of length M_1, N_1, M_2, L_2 respectively satisfying

$$M_1N_1 \leqslant M, \quad W < N_1 < M_1,$$
$$M_2L_2 \leqslant M, \quad M_2 < W^2$$

and with coefficients bounded by the divisor function. Moreover $L_2(s)$ is a partial sum of the Riemann zeta-function. Here W is any parameter at our disposal. We choose $W = T^{1/5}$, so for polynomial (11), by Theorem 2 we get

$$\int_0^T |\zeta M_1N_1(\tfrac{1}{2}+it)|^2 dt \ll T^\varepsilon(T+T^{1/2}M^{3/4}N_1^{1/4}+T^{1/2}MN_1^{-1/2}+M^{7/4}N_1^{-1/4})$$
$$\ll T^\varepsilon(T+T^{1/2}M^{7/8}+T^{2/5}M+T^{-1/20}M^{7/4})$$

which implies (5). For the polynomial (12), if $M_2 > W$ the arguments are very similar to those of the first one, and if $M_2 < W$ we apply Theorem 3 getting

$$\int_0^T |\zeta M_2L_2(\tfrac{1}{2}+it)|^2 dt \ll T^{1+\varepsilon}.$$

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Reducibility of lacunary polynomials, IV

by

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The aim of this paper is to make a further contribution to the problem of reducibility of polynomials

$$f(x) = a_0 + \sum_{j=1}^{k} a_j x^{n_j} \quad (0 = n_0 < n_1 < \ldots < n_k,\ a_0a_k \neq 0) \tag{1}$$

for fixed integral coefficients a_j and variable exponents n_j. The non-reciprocal irreducible factors of $f(x)$ can be found by means of Theorem 2 in [3] and as to reciprocal factors the conjecture proposed in [2] implies the existence of a constant $C(a_0, a_1, \ldots, a_k)$ such that either all reciprocal irreducible factors of f are cyclotomic or $\sum_{j=1}^{k} \gamma_j n_j = 0$ for suitable integers γ_j satisfying

$$0 < \max_{1\leqslant j\leqslant k} |\gamma_j| \leqslant C(a_0, a_1, \ldots, a_k).$$

We shall prove

Theorem. *If f is given by* (1) *with a_j integral, then either all reciprocal irreducible factors of f are cyclotomic or there exist integers $\gamma_1, \ldots, \gamma_k$ satisfying*

$$\sum_{j=1}^{k} \gamma_j n_j = 0, \tag{2}$$

$$0 < \max_{j=1,\ldots,k} |\gamma_j| \leqslant \max_{0\leqslant j\leqslant k} \frac{\log a_j^2}{\log 2} \tag{3}$$

and the number of reciprocal non-cyclotomic factors of f does not exceed the total number of prime factors of (a_0, a_k) or finally the following system of *inequalities is fulfilled*

$$(4)\quad \begin{cases} \sum_{j=0}^{k-1} |a_j||n_j-n_i| > |a_k|(n_k-n_i) & \text{if} \quad n_i < n_k/2, \\ \sum_{j=1}^{k} |a_j||n_j-n_i| > |a_0|n_i & \text{if} \quad n_i > n_k/2, \\ \sum_{j=1}^{k-1} |a_j|\left|n_j-\frac{n_k}{2}\right| > \big||a_k|-|a_0|\big| \frac{n_k}{2}. \end{cases}$$

This theorem supersedes Lemma 15 of [3] and implies the following

COROLLARY. *If* $\sum_{j=0}^{k} \xi^{2^j} = 0$ *and* $|\xi| = 1$ *then* ξ *is a root of unity.*

The corollary answers in the negative Problem 1 of Mahler [1]. The proof of the theorem is based on two lemmata.

LEMMA 1. *For every positive real number* $r \neq 1$ *and real numbers* n, m *satisfying* $|n| > |m|$ *we have*

$$h(m,n,r) = \left|\frac{r^m - r^{-m}}{r^n - r^{-n}}\right| < \left|\frac{m}{n}\right|.$$

Proof. Since $h(m,n,r)$ is an even function of m and n and

$$h(m,n,r^{-1}) = h(m,n,r)$$

it is enough to prove the lemma for $n > m \geqslant 0$, $r > 1$. Now, the function $g(r) = m(r^n - r^{-n}) - n(r^m - r^{-m})$ satisfies

$$g'(r) = mnr^{-1}[(r^n + r^{-n}) - (r^m + r^{-m})] = mnr^{-1}(r^n - r^m)(1 - r^{-n-m}) > 0$$

for all $r > 1$ hence for such r $g(r) > g(1) = 0$ and

$$h(m,n,r) = \frac{m}{n} - \frac{g(r)}{r^n - r^{-n}} < \frac{m}{n}.$$

LEMMA 2. *Let f be given by* (1) *with a_j arbitrary complex numbers. If*

$$(5)\qquad f(\xi) = f(\bar{\xi}^{-1}) = 0$$

then either $|\xi| = 1$ *or the system* (4) *is fulfilled.*

Proof. Let $\xi = re^{i\varphi}$ (r, φ real) and let ϱ be a real number. From (5) we infer that

$$\sum_{j=0}^{k} a_j r^{n_j-\varrho} e^{i\varphi n_j} = 0 = \sum_{j=0}^{k} a_j r^{\varrho-n_j} e^{i\varphi n_j}$$

hence

$$\sum_{j=0}^{k} a_j (r^{n_j-\varrho} - r^{\varrho-n_j}) e^{i\varphi n_j} = 0.$$

Taking

$$\varrho = n_i \quad \text{and} \quad \nu = \begin{cases} k & \text{if} \quad n_i < n_k/2, \\ 0 & \text{if} \quad n_i > n_k/2, \end{cases}$$

we get

$$|n_j - n_i| \leqslant |n_\nu - n_i| \quad (0 \leqslant j \leqslant k)$$

also

$$\sum_{j=0, j\neq\nu}^{k} a_j (r^{n_j-n_i} - r^{n_i-n_j}) e^{i\varphi n_j} = -a_\nu (r^{n_\nu-n_i} - r^{n_i-n_\nu}) e^{i\varphi n_\nu}$$

hence dividing by $r^{n_\nu-n_i} - r^{n_i-n_\nu}$ and using Lemma 1 we get the first two inequalities of (4). The last inequality is obtained similarly on taking $\varrho = n_k/2$.

Remark. This lemma supersedes Lemma 14 of [3].

Proof of Theorem. Suppose that f has a reciprocal irreducible factor g that is not cyclotomic. Let η be a zero of g. By Kronecker's theorem either η has a conjugate ξ with $|\xi| \neq 1$ or η is not an algebraic integer. In the former case we use Lemma 2 and get the conditions (4). In the latter case we use Lemma 13 of [3] and get the conditions (2) and (3). Also the product of the leading coefficients of all reciprocal non-cyclotomic factors of f must divide (a_0, a_k). Since all these coefficients are greater than 1 their number does not exceed the total number of prime factors of (a_0, a_k).

Proof of Corollary. Let g be a minimal polynomial of ξ. Since $g(\xi^{-1}) = g(\xi) = 0$, g is reciprocal. We apply the theorem to the polynomial $f(x) = \sum_{j=0}^{k} x^{2^j-1}$. Since this polynomial does not satisfy the conditions (4) (for $i = 0$) and $(a_0, a_k) = 1$ the number of its reciprocal non-cyclotomic factors is 0. Hence g is cyclotomic and ξ is a root of unity.

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