[4] L. J. Goldstein, *Analogues of Artin's conjecture,* Trans. Amer. Math. Soc. 149 (1970), pp. 431–442.
[5] H. Hasse, *Über die Dichte der Primzahlen p, für die eine vorgegebene ganzrationale Zahl $a \neq 0$ von gerader bzw. ungerader Ordnung* mod $p$ *ist,* Math. Ann. 166 (1966), pp. 19–23.
[6] — *Über die Dichte der Primzahlen p, für die eine vorgegebene ganzrationale Zahl $a \neq 0$ von durch eine vorgegebene Primzahl $l \neq 2$ teilbarer bzw. unteilbarer Ordnung* mod $p$ *ist,* ibid. 162 (1965), pp. 74–76.
[7] C. Hooley, *On Artin's conjecture,* J. Reine Angew. Math. 225 (1967), pp. 209–220.
[8] J. Lagarias and A. Odlyzko, *Effective versions of the Chebotarev density theorem, Algebraic number fields* (ed. Fröhlich), Academic Press, London–New York 1977, pp. 409–464.
[9] E. Landau, *Über Ideale und Primideale in Idealklassen,* Math. Zeitschr. 2 (1918), pp. 52–154.
[10] K. Prachar, *Primzahlverteilung,* Berlin 1957.
[11] A. Schinzel, *A refinement of a theorem of Gerst on power residues,* Acta Arith. 17 (1970), pp. 161–168.
[12] H. M. Stark, *Some effective cases of the Brauer–Siegel theorem,* Inventiones Math. 23 (1974), pp. 135–152.
[13] P. J. Stephens, *Prime divisors of second-order linear recurrences,* J. Number Theory 8 (1976), pp. 313–345.
[14] — *An average result for Artin's conjecture,* Mathematika 16 (1969), pp. 178–188.
[15] R. Warlimont, *On Artin's conjecture,* J. London Math. Soc. (2), 5 (1972), pp. 91–94.
[16] K. Wiertelak, *On the density of some sets of primes, I,* Acta Arith. 34 (1978), pp. 183–196.
[17] — *On the density of some sets of primes, II,* ibid. 34 (1978), pp. 197–210.
[18] — *On the density of some sets of primes, III,* Studies in Pure Mathematics, To the Memory of Paul Turán, pp. 761–773.

INSTITUTE OF MATHEMATICS
OF THE ADAM MICKIEWICZ UNIVERSITY
Poznań

*Received on 12. 3. 1982*
*and in revised form on 30. 8. 1982* (1295)

ACTA ARITHMETICA
XLIII (1984)

# Multidimensional covering systems of congruences

by

J. Fabrykowski (Warszawa)

**1. Introduction.** Covering systems of congruences in one variable have been studied for many years. The aim of the present paper is to extend the results obtained for such systems to multidimensional systems introduced recently by A. Schinzel [8]. We begin by defining the principal notions.

Definition 1. A system of congruences

$$b_{i0} + \sum_{j=1}^{k} b_{ij} x_j \equiv 0 \bmod m_i \quad (1 \leqslant i \leqslant n) \tag{1}$$

*covers a set* $S \subset \mathbf{Z}^k$ if every vector $[x_1, \ldots, x_k] \in S$ satisfies one of the congruences of the system.

Definition 2. A congruence of the system (1) is called *essential* if there exists an integral vector $[x_1, \ldots, x_k] \in \mathbf{Z}^k$ which satisfies this and only this congruence.

Definition 3. A system of the form (1) is called *regular* if all congruences are essential.

Definition 4. A system of the form (1) is called *covering* if it covers the set $\mathbf{Z}^k$, and *disjoint covering* if it is regular and every vector in $\mathbf{Z}^k$ satisfies one and only one congruence of this system.

For one dimensional systems ($k = 1$) it is usual to take $b_{i1} = 1$ which can be relaxed to $(b_{i1}, m_i) = 1$. Here are principal results concerning such systems.

Theorem A (see [7], Theorems 2–4). *For a disjoint covering system $x \equiv a_i \bmod m_i$ $(1 \leqslant i \leqslant n)$, where $1 < m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n$ we have*

$$\sum_{i=1}^{n} 1/m_i = 1, \quad m_{n-1} = m_n,$$

*for every $i = 1, 2, \ldots, n$ there exists a $t \neq i$ such that $m_i | m_t$,*
*if $p$ is the least prime factor of $m_n$ then $m_n = m_{n-1} = \ldots = m_{n-p+1}$.*

THEOREM B (Znám [9]). *If a system* $x \equiv a_i \bmod m_i$ $(1 \leqslant i \leqslant n)$ *is covering, the congruence* $x \equiv a_r \bmod m_r$ *is essential, and*

$$m_r = \prod_{\tau=1}^{s} p_\tau^{a_\tau},$$

*then*

$$n \geqslant 1 + \sum_{\tau=1}^{s} a_\tau(p_\tau - 1).$$

THEOREM C (Crittenden, Vanden Eynden [1]). *If a system* $x \equiv a_i \bmod m_i$ $(1 \leqslant i \leqslant n)$ *covers the segment* $1 \leqslant x \leqslant 2^n$, *then it is a covering one.*

In Section 2 we give analogues of Theorems A, B, C for general covering systems and in Section 3 their refinements for homogeneous covering systems, i.e. systems of the form (1) in which $b_{i0} = 0$ for all $i \leqslant n$. The latter section contains our principal results (Theorems 4 and 6). As a tool we use the Jacobsthal function and also the following result of Frobenius.

THEOREM D ([2]). *The number of solutions of the congruence*

$$a_0 + \sum_{i=1}^{n} a_i x_i \equiv 0 \bmod m$$

*equals* $(m, a_1, \ldots, a_n) m^{n-1}$ *provided* $(m, a_1, \ldots, a_n) \mid a_0$.

**2. General covering systems.** If the system (1) is regular we can rewrite it in the following form

$$(2) \qquad a_{i0} + \sum_{j=1}^{k} a_{ij} x_j \equiv 0 \bmod m_i, \qquad (a_{i1}, \ldots, a_{ik}, m_i) = 1 \qquad (1 \leqslant i \leqslant n)$$

where $m_i$ are not necessarily the same as before.

For systems of this form we have the following analogues of Theorems A, B and C.

THEOREM 1. *For a disjoint covering system of the form* (2), *where* $1 < m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n$ *we have*

(i) $\sum_{i=1}^{n} 1/m_i = 1$,

(ii) $m_{n-1} = m_n$,

(iii) *for every* $i \leqslant n$ *there exists an* $l \neq i$ *such that* $m_i \mid m_l$,

(iv) *if* $p$ *is the least prime divisor of* $m_n$ *then* $m_n = m_{n-1} = \ldots = m_{n-p+1}$.

THEOREM 2. *If a system of the form* (2) *is covering, the congruence*

$$a_{r0} + \sum_{j=1}^{k} a_{rj} x_j \equiv 0 \bmod m_r$$



*is essential and*

$$m_r = \prod_{\tau=1}^{s} p_\tau^{a_\tau},$$

*then*

$$n \geqslant 1 + \sum_{\tau=1}^{s} a_\tau(p_\tau - 1).$$

THEOREM 3. *If a system of the form* (2) *covers a* $k$-*dimensional cube* $C_k \subset \boldsymbol{Z}^k$ *with the side length* $2^n$, *then it is a covering one.*

Theorems 1 and 2 are deduced from Theorems A and B by the same method, therefore we shall prove only Theorem 2.

We begin by proving a simple lemma which shows that any covering system has a shifting property.

LEMMA 1. *Suppose that* (2) *is a covering system. Then for every index* $h$, $1 \leqslant h \leqslant n$, *and every integer* $r_h$ *there exists a covering system with the same moduli* $m_1, \ldots, m_n$, *with the same congruences essential and with*

$$a_{h0} \equiv r_h \bmod m_h.$$

Proof. Let us consider the congruence

$$r_h - a_{h0} + \sum_{j=1}^{k} a_{hj} y_j \equiv 0 \bmod m_h.$$

Since $(a_{h1}, a_{h2}, \ldots, a_{hk}, m_h) = 1$ it follows that our congruence is solvable. Let an integral vector $[\lambda_1, \lambda_2, \ldots, \lambda_k]$ be one of its solutions. The system:

$$a_{i0} + \sum_{j=1}^{k} a_{ij} x_j - \sum_{j=1}^{k} a_{ij} \lambda_j \equiv 0 \bmod m_i, \qquad 1 \leqslant i \leqslant n$$

is a covering one. Indeed, if a vector $[x_1 + \lambda_1, \ldots, x_k + \lambda_k]$ is not covered by the system above, the vector $[x_1, \ldots, x_k]$ will not be covered by the system (2), contrary to the assumption. Now in the system in question, the congruence with index $h$ is of the form:

$$r_h + \sum_{j=1}^{k} a_{hj} x_j \equiv 0 \bmod m_h.$$

Proof of Theorem 2. Let us put in Lemma 1 $h = r$, $r_h = 1$ and

$$K_s = \Big\{[x_1, \ldots, x_k] \in \boldsymbol{Z}^k,\ a_{s0} + \sum_{j=1}^{k} a_{sj} x_j \equiv 0 \bmod m_s\Big\} \quad \text{for } 1 \leqslant s \leqslant n.$$

Under the assumption there exists a vector $[x_1^0, x_2^0, \ldots, x_k^0] \in K_r \setminus \bigcup_{\substack{i=1\\ i\neq r}}^{n} K_i$ and we have

$$\sum_{j=1}^{k} a_{rj}x_j^0 + 1 \equiv 0 \bmod m_r. \tag{3}$$

Putting in the system (2) $x_j = tx_j^0$ for $1 \leqslant j \leqslant k$ we obtain

$$t\sum_{j=1}^{k} a_{ij}x_j^0 + a_{i0} \equiv 0 \bmod m_i, \quad 1 \leqslant i \leqslant n. \tag{4}$$

Let

$$A_i = \sum_{j=1}^{k} a_{ij}x_j^0, \quad \Delta_i = (A_i, m_i), \quad 1 \leqslant i \leqslant n,$$

$$\mathscr{F} = \{i\colon 1 \leqslant i \leqslant n,\ \Delta_i \mid a_{i0}\}.$$

We can rewrite (4) in the form

$$t \equiv c_i \bmod \frac{m_i}{\Delta_i} \tag{5}$$

for certain integers $c_i$. In virtue of (3) $\Delta_r = 1$, thus $r \in \mathscr{F}$ and the $r$th congruence is essential. The system (5) is obviously covering. Applying to (5) Theorem B we obtain:

$$|\mathscr{F}| \geqslant 1 + \sum_{\tau=1}^{s} a_\tau(p_\tau - 1), \quad \text{where } |\mathscr{F}| \leqslant n.$$

**Proof of Theorem 3.** Without loss of generality we can assume that $C_k = \langle 1, 2^n\rangle^k$ (1). Let us suppose that our system is not a covering one. Let

$$\mathscr{A} = \{[x_1, \ldots, x_k] \in \mathbf{Z}^k\colon x_i > 0, 1 \leqslant i \leqslant n, [x_1, \ldots, x_k] \notin \bigcup_{i=1}^{n} K_i\}$$

where $\mathscr{A} \neq \emptyset$ and $K_i$ has the same meaning as in the proof of Theorem 2. Let the vector $\boldsymbol{t} = [t_1, \ldots, t_k] \in \mathscr{A}$ be such that $\sum_{j=1}^{k} t_i$ is minimal. Since $\boldsymbol{t} \notin \bigcup_{i=1}^{n} K_i$ in virtue of the assumption, there exists an $r$, $1 \leqslant r \leqslant k$ such that $t_r > 2^n$. Thus for every $1 \leqslant x \leqslant 2^n$ we have $[t_1, \ldots, t_{r-1}, x, t_{r+1}, \ldots, t_k] \in \bigcup_{i=1}^{n} K_i$.

(1) Here and in the sequel the length of a segment is the number of integers contained in it.



We can assume that $r = 1$ and consider the system of congruences in one variable $x$:

$$a_{i1}x + \sum_{j=2}^{k} a_{ij}t_j + a_{i0} \equiv 0 \bmod m_i, \quad 1 \leqslant i \leqslant n. \tag{6}$$

Putting $w_i = a_{i0} + \sum_{j=2}^{k} a_{ij}t_j$, $\Delta_i = (a_{i1}, m_i)$, $1 \leqslant i \leqslant n$ and $\mathscr{F} = \{i\colon 1 \leqslant i \leqslant n, \Delta_i \mid w_i\}$, we can rewrite (6) in the following form:

$$x \equiv c_i \bmod (m_i/\Delta_i), \quad i \in \mathscr{F}.$$

Since $|\mathscr{F}| \leqslant n$ and the system above covers the segment $1 \leqslant x \leqslant 2^n$, in virtue of Theorem C it is a covering one. Thus $\boldsymbol{t} \in \bigcup_{i=1}^{n} K_i$ contrary to the assumption.

## 3. Homogeneous covering systems

DEFINITION 5. A system of the form (2) is called *homogeneous* if for every $i$, $1 \leqslant i \leqslant n$, $a_{i0} = 0$.

For the systems in question we shall obtain a stronger form of Theorem 2, namely

THEOREM 4. *Let* (2) *be a homogeneous covering system in which the $r$-th congruence is essential. Then we have*

$$n \geqslant 1 + \sum_{\tau=1}^{s} \{a_\tau(p_\tau - 1) + 1\},$$

*where* $m_r = \prod_{\tau=1}^{s} p_\tau^{a_\tau}$, $a_\tau \geqslant 1$.

LEMMA 2. *Let $S$ be a regular covering system of the form* (2). *For a given prime number $q$ dividing $\prod_{i=1}^{n} m_i$ and a given $\gamma > 0$, let $n_\gamma$ denote the number of indexes $1 \leqslant i \leqslant n$ such that $q^\gamma \| m_i$ and let $\alpha = \min_{n_\gamma \neq 0} \gamma$, $\beta = \max_{n_\gamma \neq 0} \gamma$. Then*

$$\frac{n_\alpha}{q^\alpha} + \ldots + \frac{n_\beta}{q^\beta} \geqslant 1.$$

**Remark.** By a similar argument one can prove that

$$\frac{n_\delta}{q^\alpha} + \ldots + \frac{n_\beta}{q^{\beta-\delta+\alpha}} \geqslant 1 \quad (\alpha \leqslant \delta \leqslant \beta).$$

This improves inequality (51) in [8] which has the same form with $\alpha$ replaced by 1.

**Proof.** Let us number the congruences in our system so that $(m_i, q) = 1$ for $1 \leqslant i \leqslant n'$ precisely. Suppose first that $n' \geqslant 1$. Since $S$ is regular, its proper subset $S' \subset S$:

$$a_{i0}+\sum_{j=1}^{k} a_{ij}x_j \equiv 0 \bmod m_i, \quad 1 \leqslant i \leqslant n',$$

cannot be a covering one. So let $[\tilde{x}_1, \ldots, \tilde{x}_k]$ be a vector that satisfies none of the congruences of $S'$. Putting in $S$: $x_j = Mt_j+\tilde{x}_j$, $1 \leqslant j \leqslant k$, where $M = \text{l.c.m.}\{m_1, \ldots, m_{n'}\}$, we make all congruences of $S'$ contradictory and thus get the covering system $S''$ of the form:

$$a''_{i0}+M\sum_{j=1}^{k} a_{ij}t_j \equiv 0 \bmod m_i, \quad n'+1 \leqslant i \leqslant n,$$

where $q \mid m_i$. Let us now replace the moduli $m_i$ occurring in $S''$ by $q^\gamma$ where $q^\gamma \| m_i$. The system will be still a covering one and since $(M, q) = 1$ it can be written in the form:

$$a'''_{i0}+\sum_{j=1}^{k} a_{ij}t_j \equiv 0 \bmod q^\gamma, \quad q^\gamma \| m_i$$

for $n'+1 \leqslant i \leqslant n$. Applying Theorem D to any covering system with moduli $m_1, m_2, \ldots, m_n$ we obtain $\sum_{i=1}^{n} 1/m_i \geqslant 1$. In our case it means that $\sum_{\gamma=a}^{\beta} n_\gamma/q^\gamma \geqslant 1$. Suppose now that $n' = 0$. Then for every $i$, $q \mid m_i$ and it suffices to replace the moduli $m_i$ in the system $S$ by $q^\gamma$ where $q^\gamma \| m_i$. In this way we obtain a covering system and proceeding as before we easily get the inequality in question.

LEMMA 3. *Let $p_1, \ldots, p_s$ be distinct primes and $A_i, B_i$ $(1 \leqslant i \leqslant s)$, $A, B$ integers such that*

$$\Big(A, B, \prod_{\tau=1}^{s} p_\tau\Big) = 1.$$

*Let us assume that there exists an integral vector $[\tilde{x}, \tilde{y}]$ satisfying the following conditions:*

(i) $A\tilde{x}+B\tilde{y} \equiv 0 \bmod \prod_{\tau=1}^{s} p_\tau$,

(ii) $A_i\tilde{x}+B_i\tilde{y} \not\equiv 0 \bmod p_i, 1 \leqslant i \leqslant s$.

*Then there exists a common solution $[a, \beta]$ of the congruences*

$$A_ix+B_iy \equiv 0 \bmod p_i, \quad 1 \leqslant i \leqslant s$$

*such that*

$$\Big(Aa+B\beta, \prod_{\tau=1}^{s} p_\tau\Big) = 1.$$



Proof. Let us suppose that every solution $[a, \beta]$ of the congruence $A_ix+B_iy \equiv 0 \bmod p_i$ satisfies $Aa+B\beta \equiv 0 \bmod p_i$. Since $(A, B, p_i) = 1$ the above congruences would have the same solutions, contrary to (i) and (ii). Thus there exists a vector $[a_i, \beta_i]$ such that

$$A_ia_i+B_i\beta_i \equiv 0 \bmod p_i,$$

$$Aa_i+B\beta_i \not\equiv 0 \bmod p_i.$$

In virtue of the Chinese Remainder Theorem one can find a vector $[a, \beta]$ such that $a \equiv a_i \bmod p_i$, $\beta \equiv \beta_i \bmod p_i$ for $1 \leqslant i \leqslant s$ satisfying the assertion.

Proof of Theorem 4. Let us denote by $K_i$ $(1 \leqslant i \leqslant n)$ the set of solutions of the $i$th congruence of our system and let

$$[x_1^0, \ldots, x_k^0] \in K_r \setminus \bigcup_{\substack{i=1\\ i\neq r}}^{n} K_i.$$

We can assume that $(x_1^0, \ldots, x_k^0) = 1$. Suppose otherwise that $(x_1^0, \ldots, x_k^0) = d > 1$ and that no vector $[x_1, \ldots, x_k]$ with $(x_1, \ldots, x_k) = 1$ has the desired property, i.e. if $[x_1, \ldots, x_k] \in K_r$, there exists an $i$, $1 \leqslant i \leqslant n$, $i \neq r$ such that $[x_1, \ldots, x_k] \in K_i$. We have

$$[x_1^0, \ldots, x_k^0] = d\left[\frac{x_1^0}{d}, \ldots, \frac{x_k^0}{d}\right], \quad \text{where } \left(\frac{x_1^0}{d}, \ldots, \frac{x_k^0}{d}\right) = 1,$$

thus there exists an $i \neq r$ such that $\left[\frac{x_1^0}{d}, \ldots, \frac{x_k^0}{d}\right] \in K_i$. Since the system is homogeneous, we have $[x_1^0, \ldots, x_k^0] \in K_i$ contrary to the choice of $[x_1^0, \ldots, x_k^0]$. In order to prove the theorem we proceed by induction with respect to $k$.

Assume that $k = 2$. Let us consider a homogeneous covering system:

$$A_ix+B_iy \equiv 0 \bmod m_i, \quad 1 \leqslant i \leqslant n, \tag{7}$$

where $1 < m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n$, $(A_i, B_i, m_i) = 1$, $1 \leqslant i \leqslant n$ and the congruence $A_rx+B_ry \equiv 0 \bmod m_r$ is essential. Omitting if need be not essential congruences we can assume that our system is regular.

Let $p$ be a prime number, $p \mid m_r$ and $Z_p = \{i \neq r, 1 \leqslant i \leqslant n, p \mid m_i\}$, $Z_p^* = \{i \neq r, 1 \leqslant i \leqslant n, p \nmid m_i\}$. In virtue of Lemma 2 $|Z_p| \geqslant p-1$.

Since (7) is a covering system, the following one has the same property:

$$(8) \quad \begin{aligned} &A_ix+B_iy \equiv 0 \bmod m_i, && i \in Z_p^*,\\ &A_ix+B_iy \equiv 0 \bmod p, && i \in Z_p,\\ &A_rx+B_ry \equiv 0 \bmod m_r. && \end{aligned}$$

Let $K_i$ denote the set of solutions of the $i$th congruence of the system (7), and let $[x^0, y^0] \in K_r \setminus \bigcup_{\substack{i=1\\ i\neq r}}^{n} K_i$. We shall prove that for every $p \mid m_r$ there exists an $i \in Z_p$ such that $A_i x^0 + B_i y^0 \not\equiv 0 \bmod p$. For this purpose, suppose that there exists a $p \mid m_r$ such that for every $i \in Z_p$, $A_i x^0 + B_i y^0 \equiv 0 \bmod p$. Let

$$\tilde{K}_i = \{[x, y] \in \mathbf{Z}^2\colon A_i x + B_i y \equiv 0 \bmod p\} \quad \text{for} \quad i \in Z_p \cup \{r\}.$$

Obviously $\tilde{K}_r \supset K_r$, so $[x^0, y^0] \in \tilde{K}_r$. The congruences

$$A_i x + B_i y \equiv 0 \bmod p \quad \text{for} \quad i \in Z_p \cup \{r\}$$

have the common solution $[x^0, y^0]$ and since $(x^0, y^0, p) = 1$,

$$\tilde{K}_i = \{\lambda[x^0, y^0] + p[\mu, \nu]\colon \lambda = 0, 1, \ldots, p-1, [\mu, \nu] \in \mathbf{Z}^2\} \quad \text{for} \quad i \in Z_p \cup \{r\}.$$

Thus we infer that the system:

$$\text{(9)} \qquad \begin{aligned} &A_i x + B_i y \equiv 0 \bmod m_i, \quad i \in Z_p^*,\\ &A_r x + B_r y \equiv 0 \bmod p \end{aligned}$$

is a covering one. Now we can omit not essential congruences in (9), obtaining a regular system and applying Lemma 2 with $q = p$ we have a contradiction.

Let now $m_r = \prod_{\tau=1}^{s} p_\tau^{a_\tau}$, $p_1 < p_2 < \ldots < p_s$, $a_\tau \geqslant 1$. Let us consider first the prime $p_1$. There exists an index $i(p_1) \in Z_{p_1}$ such that $A_{i(p_1)} x^0 + B_{i(p_1)} y^0 \not\equiv 0 \bmod p_1$. Since (7) is a covering system, so is the following one:

$$\text{(10)} \qquad \begin{aligned} &A_i x + B_i y \equiv 0 \bmod m_i, \quad 1 \leqslant i \leqslant n,\ i \neq i(p_1),\ i \neq r,\\ &A_{i(p_1)} x + B_{i(p_1)} y \equiv 0 \bmod p_1,\\ &A_r x + B_r y \equiv 0 \bmod m_r. \end{aligned}$$

The argument above shows that in the latter system the $r$th congruence is essential. Let us consider now the prime $p_2$, and put:

$$\begin{aligned} Z_{p_2} &= \{i\colon i \neq r,\ i \neq i(p_1),\ 1 \leqslant i \leqslant n,\ p_2 \mid m_i\},\\ Z_{p_2}^* &= \{i\colon i \neq r,\ i \neq i(p_1),\ 1 \leqslant i \leqslant n,\ p_2 \nmid m_i\}, \end{aligned}$$



where $Z_{p_2} \neq \emptyset$ in virtue of Lemma 2. Proceeding with (10) as we did with (7) we shall get a covering system of the form:

$$\text{(11)} \qquad \begin{aligned} &A_i x + B_i y \equiv 0 \bmod m_i, \quad i \in Z_{p_2} \cup Z_{p_2}^* \setminus \{i(p_2)\},\\ &A_{i(p_1)} x + B_{i(p_1)} y \equiv 0 \bmod p_1,\\ &A_{i(p_2)} x + B_{i(p_2)} y \equiv 0 \bmod p_2,\\ &A_r x + B_r y \equiv 0 \bmod m_r \end{aligned}$$

with the $r$th congruence essential.

Repeating this proceeding $s$ times we shall obtain the following covering system:

$$\text{(12)} \qquad \begin{aligned} &A_{i(p_\tau)} x + B_{i(p_\tau)} y \equiv 0 \bmod p_\tau, \quad 1 \leqslant \tau \leqslant s,\\ &A_r x + B_r y \equiv 0 \bmod m_r,\\ &A_i x + B_i y \equiv 0 \bmod m_i, \end{aligned}$$

where $i \in \langle 1, n\rangle \setminus \bigcup_{\tau=1}^{s} \{i(p_\tau)\} \setminus \{r\}$, with the $r$th congruence essential.

Let us now apply Lemma 3 to the system (12). In virtue of the lemma there exists a vector $[a, b]$ such that:

$$A_{i(p_\tau)} a + B_{i(p_\tau)} b \equiv 0 \bmod p_\tau, \quad 1 \leqslant \tau \leqslant s,$$
$$(A_r a + B_r b, m_r) = 1.$$

Putting in (12) $x = at + x^0$, $y = bt + y^0$, where

$$A_{i(p_\tau)} x^0 + B_{i(p_\tau)} y^0 \not\equiv 0 \bmod p_\tau, \quad 1 \leqslant \tau \leqslant s,$$

we get a covering system in one variable $t$, in which the first $s$ congruences are contradictory. In virtue of Theorem B we obtain:

$$\text{(13)} \qquad n - s \geqslant 1 + \sum_{\tau=1}^{s} a_\tau (p_\tau - 1).$$

Let now $k \geqslant 3$ and let in the covering homogeneous system (2) the $r$th congruence be essential. Let $[x_1^0, \ldots, x_k^0] \in K_r \setminus \bigcup_{\substack{i=1\\ i\neq r}}^{n} K_i$ and let us choose arbitrary integers $A_j$ $(1 \leqslant j \leqslant k)$ such that:

$$\text{(14)} \qquad \sum_{j=1}^{k} a_{rj} A_j \equiv 1 \bmod m_r.$$

Let us now make in our system the substitution $x_j = A_j x + B_j y$, where $B_j$ $(1 \leqslant j \leqslant k)$ are defined so that $[x_1^0, \ldots, x_k^0]$ corresponds to the vector $[x^0, y^0] = [1, 1]$. In this way we obtain the system:

$$x\sum_{j=1}^{k} a_{ij}A_j + y\sum_{j=1}^{k} a_{ij}B_j \equiv 0 \bmod m_i, \quad 1 \leqslant i \leqslant n. \tag{15}$$

It is easy to see that the system is a covering one with the $r$th congruenc essential and by (14) we have

$$\Big(\sum_{j=1}^{k} a_{rj}A_j, \sum_{j=1}^{k} a_{rj}B_j, m_r\Big) = 1.$$

Applying (13) to the system (15) we get the assertion of the theorem.

Remark. The estimate occurring in Theorem 4 is the best possible as the following example shows:

$$y \equiv 0 \bmod p_\tau, \qquad 1 \leqslant \tau \leqslant s,$$
$$x - tp_1^{w-1}y \equiv 0 \bmod p_1^w, \qquad 1 \leqslant w \leqslant a_1,\ 1 \leqslant t \leqslant p_1 - 1,$$
$$x - tp_1^{a_1}p_2^{w-1}y \equiv 0 \bmod p_1^{a_1}p_2^w, \qquad 1 \leqslant w \leqslant a_2,\ 1 \leqslant t \leqslant p_2 - 1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x - tp_1^{a_1}p_2^{a_2}\dots p_{s-1}^{a_{s-1}}p_s^{w-1}y \equiv 0 \bmod p_s^w\prod_{\tau=1}^{s-1}p_\tau^{a_\tau}, \qquad 1 \leqslant w \leqslant a_s,\ 1 \leqslant t \leqslant p_s - 1,$$
$$x \equiv 0 \bmod \prod_{\tau=1}^{s} p_\tau^{a_\tau}.$$

Now we shall return to the problem considered in Theorem 3, but for homogeneous covering systems. We shall prove the main theorem of this paper, Theorem 6, formulated below. We have been unable to prove that a homogeneous system of the form (2) which covers a $k$-dimensional cube $C_k \subset \mathbf{Z}^k$ with the side length $2^{n-2}+2$, and such that $\mathbf{0} = [0, 0, \dots, 0] \in C_k$ is a covering one. In this connection we shall show that the length $2^{n-2}+1$ of the side of our cube is not sufficient for the assertion and we shall prove the latter for the cube with the side length $2^{n-1}$ $(n \geqslant 2)$.

Let us consider the system of $n$ congruences:

$$y \equiv 0 \bmod 2,$$
$$x \equiv 0 \bmod 2,$$
$$x + (2^i - 1)y \equiv 0 \bmod 2^{i+1}, \qquad 1 \leqslant i \leqslant n-3,$$
$$x - y \equiv 0 \bmod p, \qquad \text{where } p \text{ is an arbitrary odd prime.}$$

There is no difficulty in showing that this system covers the square $\langle 0, 2^{n-2}\rangle^2$ and the vector $[1, 2^{n-2}+1]$ satisfies none of these congruences.



Theorem 5. *If $n \geqslant 2$ and a homogeneous system of the form (2) covers a cube $C_k \subset \mathbf{Z}^k$ with the side length $2^{n-1}$ and $\mathbf{0} = [0, 0, \dots, 0] \in C_k$, it is a covering system.*

Proof. From the assumption we infer that the system in question covers all the vectors $e_j = [0, 0, \dots, 0, \underset{j}{1}, 0, \dots, 0]$ for $1 \leqslant j \leqslant k$. Let

$$\mathscr{A} = \{[x_1, \dots, x_k] \in \mathbf{Z}^k;\ x_j \geqslant A_j + 1,\ 1 \leqslant j \leqslant k, \text{ and } [x_1, \dots, x_k] \notin \bigcup_{i=1}^{n} K_i\}$$

where $C_k = I_1 \times \dots \times I_k$, $I_j = \langle A_j + 1, A_j + 2^{n-1}\rangle$, $1 \leqslant j \leqslant k$ and $K$ has the same meaning as in the proof of Theorem 2.

Let us assume that $\mathscr{A} \neq \emptyset$ and choose a vector $\tilde{x} \in \mathscr{A}$ with minimal $\sum_{j=1}^{k}\tilde{x}_j$. By the assumption there exists an $r$ such that $x_r > A_r + 2^{n-1}$. We can assume, as we did before, that $r = 1$. Let us consider the following system of congruences in one variable $x_1$:

$$a_{i1}x_1 + \sum_{j=2}^{k} a_{ij}\tilde{x}_j \equiv 0 \bmod m_i, \quad 1 \leqslant i \leqslant n. \tag{16}$$

By the choice of the vector $\tilde{x}$ this system covers the segment $A_1 + 1 \leqslant x_1 \leqslant A_1 + 2^{n-1}$ and since the vector $e_1$ satisfies one of the congruences of the system (16), there exists an index $i_0$ such that $a_{i_01} \equiv 0 \bmod m_{i_0}$. Therefore the $i_0$-th congruence of the system (16) is contradictory. Thus we have got a noncovering system of $n-1$ congruences, which covers a segment of $2^{n-1}$ consecutive numbers, contrary to Theorem C.

Theorem 6. *If $n \geqslant 5$ and a homogeneous system of the form (2) covers a $k$-dimensional cube $C_k \subset \mathbf{Z}^k$ with the side length $2^{n-1}$, it is a covering system.*

Remark. The following example shows that $2^{n-1}$ cannot be replaced by $2^{n-1} - 1$:

$$y \equiv 0 \bmod 2,$$
$$x + 2^i y \equiv 0 \bmod 2^{i+1}, \qquad 0 \leqslant i \leqslant n-2.$$

Moreover, for $n \leqslant 4$ the assertion of the theorem does not hold. The systems

$$x \equiv 0 \bmod 2; \quad x \equiv 0 \bmod 2, \quad x \equiv 0 \bmod 3; \quad x \equiv 0 \bmod 2,$$
$$x \equiv 0 \bmod 3, \quad x \equiv 0 \bmod 5; \quad x \equiv 0 \bmod 2, \quad x \equiv 0 \bmod 3,$$
$$x \equiv 0 \bmod 5, \quad x \equiv 0 \bmod 7;$$

cover the segments $\langle 2\rangle$, $\langle 2, 4\rangle$, $\langle 2, 6\rangle$, $\langle 2, 10\rangle$ of length: 1, 3, 5, 9, respectively.

Before proceeding to the proof of Theorem 6 we shall remind the definition of the Jacobsthal function introduced in [4].

DEFINITION 6. The *Jacobsthal function* $g(m)$ is the least positive integer $M$ such that every sequence of $M$ consecutive integers contains a number prime to $m$.

LEMMA 4. *Suppose that $S_1, S_2, \ldots, S_t$ are sets of integers such that $S_i$ consists exactly of $k_i$ residue classes modulo $b_i$, $i = 1, 2, \ldots, t$, and that $(b_i, b_j) = 1$ if $i \neq j$. Let $N$ be the number of integers $x$ in a segment of $M$ consecutive integers such that $x$ is in none of the $S$'s. Then if $1 \leqslant s \leqslant t$ we have*

$$N > 1 + M\Big(1 - \sum_{i=s+1}^{t} \frac{k_i}{b_i}\Big) \prod_{i=1}^{s} \Big(1 - \frac{k_i}{b_i}\Big) - \Big(1 + \sum_{i=s+1}^{t} k_i\Big) \prod_{i=1}^{s} (1 + k_i).$$

Proof. See [1] for $M = 2^n$. The general case does not differ in anything.

LEMMA 5. *If the number $\omega(m)$ of distinct prime factors of $m$ is greater than 4, then*

(17) $$g(m) < 2^{\omega(m)-1}.$$

Proof. It has been shown by Jacobsthal [5] that

$$\max_{\omega(m)=5} g(m) = 14, \qquad \max_{\omega(m)=6} g(m) = 22,$$

and by Kanold [6] that

$$g(m) \leqslant \omega(m)^2 \quad \text{if} \quad \omega(m) \leqslant 12.$$

Since $14 < 2^4$, $22 < 2^5$ and

$$\omega(m)^2 < 2^{\omega(m)-1} \quad \text{if} \quad \omega(m) \geqslant 7,$$

inequality (17) is true for $\omega(m) \leqslant 12$. Further proof will be performed first for $m$ odd. Let

$$m = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}, \qquad 2 < p_1 < p_2 < \ldots < p_t.$$

Take in Lemma 4

$$b_i = p_i, \quad S_i = \{x \equiv 0 \bmod p_i\}, \quad k_i = 1 \ (1 \leqslant i \leqslant t),$$
$$M = 2^{t-1}, \quad s = [t/3]+2.$$

Then

$$\Big(1 + \sum_{i=s+1}^{t} k_i\Big) \prod_{i=1}^{s} (1 + k_i) \leqslant \tfrac{2}{3} t \cdot 2^{t/3+2}.$$

Moreover, it is easy to verify that for every $t$

$$1 - \sum_{i=s+1}^{t} \frac{k_i}{b_i} \geqslant 1 - \frac{t-s}{p_{s+1}} \geqslant \frac{13}{23},$$



thus it suffices to show that

$$2^{t-1} \cdot \frac{13}{23} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \Big(\frac{12}{13}\Big)^{t/3-2} \geqslant \frac{2}{3} t \cdot 2^{t/3+2},$$

i.e., $\Big(\frac{48}{13}\Big)^{t/3} \cdot \frac{1}{t} \geqslant 19.34\ldots$

This inequality holds for $t \geqslant 13$.

Assume now that $m$ is even, $\omega(m) > 12$, $m = 2^a m_1$, $m_1$ odd. Then $\omega(m_1) \geqslant 12$ and by what has just been proved

$$g(m) = 2g(m_1) \leqslant 2^{\omega(m_1)} = 2^{\omega(m)-1}.$$

Remark. An estimate for $g(m)$ much stronger than (17) for $\omega(m)$ large has been given by H. Iwaniec [3], however it cannot be used for our purposes since it involves an unspecified constant.

Proof of Theorem 6. Let us assume contrary to the assertion that for an $n \geqslant 5$ there exist homogeneous systems of the form (2) covering a $k$-dimensional cube $C_k \subset \mathbf{Z}^k$ which are not covering.

Let us consider that system $\mathcal{U}$ for which $\sum_{i=1}^{n} m_i$ is the least and let for a given prime $p$

$$Z_p = \{i\colon 1 \leqslant i \leqslant n,\ p \mid m_i\}, \quad k_p = |Z_p|, \quad \beta_p = \max_{1 \leqslant i \leqslant n} \operatorname{ord}_p m_i.$$

We shall prove five lemmata about the system $\mathcal{U}$.

Let one of the cubes covered by $\mathcal{U}$ be $C_k = I_1 \times \ldots \times I_k$, where $I_j = \langle A_j, A_j + 2^{n-1} - 1\rangle$, $A_j \in \mathbf{Z}$ $(1 \leqslant j \leqslant k)$. Since $\mathcal{U}$ is not covering, there exist vectors $[x_1, \ldots, x_k]$ not covered by $\mathcal{U}$ and satisfying $x_j \geqslant A_j$ $(1 \leqslant j \leqslant k)$. From among the vectors with this property we choose a vector $\xi = [\xi_1, \ldots, \xi_k]$ for which $\sum_{j=1}^{k} \xi_j$ is the least.

LEMMA 6. *Every modulus $m_i$ in $\mathcal{U}$ is a prime power.*

Proof. Suppose that for a certain $i$

$$m_i = \prod^{s} p_\tau^{a_\tau}, \quad a_\tau \geqslant 1, \ s > 1.$$

Since $p_\tau^{a_\tau} < m_i$ the system obtained from $\mathcal{U}$ on replacing $m_i$ by $p_\tau^{a_\tau}$ is covering. However

$$\sum_{j=1}^{k} a_{lj} \xi_j \not\equiv 0 \bmod m_l \quad (l \neq i),$$

hence

$$\sum_{j=1}^{k} a_{ij} \xi_j \equiv 0 \bmod p_\tau^{a_\tau} \quad (1 \leqslant \tau \leqslant s).$$

It follows that

$$\sum_{j=1}^{k} a_{ij}\xi_j \equiv 0 \bmod \prod_{\tau=1}^{s} p_\tau^{a_\tau},$$

contrary to the choice of $\xi$.

LEMMA 7. *There exists an index $s \leqslant k$ such that $a_{is} \not\equiv 0 \bmod m_i$ for all $i \leqslant n$, the system*

$$a_{is}x + \sum_{j\neq s} a_{ij}\xi_j \equiv 0 \bmod m_i \quad (1 \leqslant i \leqslant n)$$

*covers the segment $\langle A_s, A_s+2^{n-1}-1\rangle$ but not $\xi_s$, and no proper subsystem has the same property.*

Proof. Since $\xi \notin C_k$, there exists an index $s$ such that

$$\xi_s \geqslant A_s + 2^{n-1}.$$

Now $\mathscr{U}$ covers all the vectors $[\xi_1, \ldots, \xi_{s-1}, x, \xi_{s+1}, \ldots, \xi_k]$, where $x \in \langle A_s, A_s+2^{n-1}-1\rangle$. Otherwise there would be a noncovered vector with the $j$th component $\geqslant A_j$ $(1 \leqslant j \leqslant k)$ and the sum of components less than $\sum_{j=1}^{k}\xi_j$, contrary to the choice of $\xi$.

If for an $i$ we had $a_{is} \equiv 0 \bmod m_i$, the congruence

$$a_{is}x + \sum_{j\neq s} a_{ij}\xi_j \equiv 0 \bmod m_i$$

not satisfied by $x = \xi_s$ would be contradictory. Omitting it we would get a system of $n-1$ congruences

$$a_{ls}x + \sum_{j\neq s} a_{lj}\xi_j \equiv 0 \bmod m_l \quad (1 \leqslant l \leqslant n,\ l \neq i)$$

covering the segment $I_s$ of length $2^{n-1}$, but not covering $\xi_s$. The same would follow from the existence of a subsystem considered in the lemma. This would contradict Theorem C.

LEMMA 8. *If $k_p, \beta_p$ have the meaning as above and $\beta_p \geqslant 2$ then*

$$k_p - 1 \geqslant (\beta_p - 1)(p-1) + 1. \tag{18}$$

Proof. By the choice of the system $\mathscr{U}$, changing in it the modulus $p^{\beta_p}$ for $p^{\beta_p-1}$ we obtain a covering system $\mathscr{U}'$. Let $c = \operatorname{l.c.m.}_{1\leqslant i\leqslant n}\{m_i\}/p^{\beta_p}$ and let us make in $\mathscr{U}'$ the substitution $x_j = ct_j + \xi_j$, $1 \leqslant j \leqslant k$. So all the congruences which have the moduli indivisible by $p$ become contradictory. Let us multiply both sides of our congruences by $\bar{c}$ and put $s_j = t_j + \bar{c}\xi_j$, $1 \leqslant j \leqslant k$, where $c\bar{c} \equiv 1 \bmod p^{\beta_p}$. In this way we obtain the following covering homogeneous system:



$$\sum_{j=1}^{k} a_{ij}s_j \equiv 0 \bmod p^{a_i}, \quad 1 \leqslant i \leqslant k_p - 1,$$

$$\sum_{j=1}^{k} a_{k_pj}s_j \equiv 0 \bmod p^{\beta_p-1}.$$

The last congruence of this system is essential since only this one is satisfied by the vector $[\bar{c}\xi_1, \bar{c}\xi_2, \ldots, \bar{c}\xi_k]$. Now, applying to this system Theorem 4, we get the assertion of the lemma.

LEMMA 9. *For primes $p$, for which $Z_p \neq \emptyset$ we have $\beta_p = 1$.*

Proof. (i) Let first $p = 2$ and assume that $\beta_p \geqslant 2$. Putting in $\mathscr{U}$ $x_j = 2^{\beta_2}t_j + \xi_j$ we obtain $k_2$ contradictory congruences. $[x_1, \ldots, x_k] \in C_k$ holds if and only if for every $1 \leqslant j \leqslant k$:

$$A_j \leqslant 2^{\beta_2}t_j + \xi_j \leqslant A_j - 1 + 2^{n-1},$$

thus certainly if for suitable $\lambda_j \in \mathbf{Z}$:

$$\lambda_j \leqslant t_j \leqslant \lambda_j + 2^{n-\beta_2-1} - 1.$$

Therefore the new system of congruences in the variables $t_j$, $1 \leqslant j \leqslant k$ covers a $k$-dimensional cube with the side length $2^{n-\beta_2-1}$. In virtue of (18) $2^{n-\beta_2-1} \geqslant 2^{n-k_2}$, so by Theorem 3 this system is a covering one, contrary to the assumption following which the vector $\mathbf{0}$ is not covered.

(ii) Let $p = 3$ and $\beta_p \geqslant 2$. If we substitute $x_j = 3^{\beta_3}t_j + \xi_j$, $1 \leqslant j \leqslant k$ in $\mathscr{U}$, we shall obtain $k_3$ contradictory congruences and a new system of $n - k_3$ congruences in the variables $t_1, t_2, \ldots, t_k$, which covers a cube with the side length $[2^{n-1}/3^{\beta_3}]$. Since by (18), $k_3 \geqslant 2\beta_3$ it suffices to prove that:

$$2^{n-1}/3^{\beta_3} > 2^{n-2\beta_3}.$$

This inequality is equivalent to the following one:

$$\frac{\log 3}{\log 2} < 2 - \frac{1}{\beta_3}.$$

It holds for $\beta_3 \geqslant 3$. The case of $\beta_3 = 2$, $k_3 \geqslant 5$ is obvious since $2^{n-1}/9 > 2^{n-5}$.

Let now $\beta_3 = 2$, $k_3 = 4$. Let us first consider the case where there is exactly one modulus of $\mathscr{U}$ which is divisible by 9. Changing, if need be, the numeration of congruences and variables and putting $x_j = \xi_j$, $2 \leqslant j \leqslant k$, we obtain a system of the following form:

$$a_{i1}x_1 + \sum_{j=2}^{k} a_{ij}\xi_j \equiv 0 \bmod 3, \quad 1 \leqslant i \leqslant 3$$

$$a_{41}x_1+\sum_{j=2}^{k}a_{4j}\xi_j\equiv 0 \bmod 9,$$

$$a_{i1}x_1+\sum_{j=2}^{k}a_{ij}\xi_j\equiv 0 \bmod m_i,\quad 5\leqslant i\leqslant n,$$

where by Lemma 5 allowing for permutation of variables $a_{i1}\not\equiv 0 \bmod m_i$ $(1\leqslant i\leqslant n)$.

Our system contains three congruences with modulus 3, and is not a covering one. Hence two of them are equivalent. So, if we omit not essential congruences, we shall obtain at most $n-1$ congruences covering $2^{n-1}$ consecutive numbers, but not all, which contradicts Theorem C.

Let now, among the moduli $m_i$ for $i\in Z_3$, at least two be divisible by 9. Changing one of those for 3 we obtain a covering system. None of the remaining congruences with modulus 9 can be essential, otherwise we should have by (18) $k_3\geqslant 6$. On the other hand, by Theorem 4 there is no homogeneous covering system of at most three congruences with modulus 3 each.

(iii) It remains to consider the case $p\geqslant 5$, $\beta_p\geqslant 2$. As in (ii) it suffices to prove that:

$$2^{n-1}/p^{\beta_p}>2^{n-k_p}\quad\text{or}\quad 2^{k_p-1}>p^{\beta_p}.$$

In virtue of Lemma 8:

$$\beta_p\leqslant\frac{k_p-2}{p-1}+1 \tag{19}$$

as well as

$$p\leqslant k_p-1. \tag{20}$$

Moreover, for $m\geqslant 5$ we have

$$2^m>m^3/4. \tag{21}$$

Applying (19), (20), and (21) successively we obtain:

$$p^{\beta_p}<2^{\frac{p+3}{2}\left(\frac{k_p-2}{p-1}+1\right)}\leqslant 2^{\frac{11}{12}k_p-\frac{5}{6}}\leqslant 2^{k_p-1}.$$

LEMMA 10. *For every prime $p$ we have $2^{k_p-1}<p$ and for $p<2^{n-1}$ even $k_p\leqslant 1$.*

Proof. Let us assume that there exists a prime $p$ that satisfies $2^{k_p-1}\geqslant p$. Let us put in the system $\mathcal{U}$ $x_j=pt_j+\xi_j$, $1\leqslant j\leqslant k$. According to Lemma 9 we obtain $k_p$ contradictory congruences. The remaining $n-k_p$ congruences cover a $k$-dimensional cube and form a noncovering system.



Using Theorem 3 we infer that $[2^{n-1}/p]<2^{n-k_p}$. So, we have:

$$2^{n-1}<p\cdot 2^{n-k_p}\leqslant 2^{k_p-1}\cdot 2^{n-k_p}=2^{n-1},$$

i.e. a contradiction.

Let us now assume that there exists a prime $p$ with $p<2^{n-1}$ and $k_p\geqslant 2$. Since $p<2^{n-1}$, there exists a vector $[x_1^0, x_2^0, \ldots, x_k^0]$ such that $x_j\in I_j$ and $x_j^0\equiv 0 \bmod p$ for $1\leqslant j\leqslant k$. Putting now in our system $\mathcal{U}$ $x_j=x_j^0$ for $j\neq s$, where $s$ has the meaning of Lemma 7, we obtain a system that covers $2^{n-1}$ consecutive numbers with at least two congruences equivalent to $x_s\equiv 0 \bmod p$. Removing not essential congruences, we get a regular system of at most $n-1$ congruences, i.e. by Theorem C a covering one. It is clear that every covering system of dimension one with all moduli greater than 1 has the property that a certain prime $q$ divides at least $q$ of them. Hence $k_q\geqslant q$ contrary to the first assertion of the lemma.

Completion of the proof of Theorem 6. By Lemmas 6 and 9 the $m_i$ are primes. Without loss of generality we may assume that $m_i<2^{n-1}$ for $i\leqslant n_0$ precisely. Let $s$ have the meaning of Lemma 7. By that lemma for every $i>n_0$ there exists an integer $x\in I_s$ such that

$$a_{is}x+\sum_{j\neq s}a_{ij}\xi_j\equiv 0 \bmod m_i, \tag{22}$$

besides $a_{is}\not\equiv 0 \bmod m_i$. The congruence $a_{is}x\equiv a_{is}y \bmod m_i$ implies $x\equiv y \bmod m_i$ and $x=y$ since $m_i>2^{n-1}=|I_s|$. Thus $x$ is determined uniquely and we can denote it by $x_i$.

Let us take distinct primes $p_{n_0+1},\ldots,p_n$ different from $m_1,\ldots,m_{n_0}$ and consider the system of congruences

$$a_{is}x+\sum_{j\neq s}a_{ij}\xi_j\equiv 0 \bmod m_i\quad (i\leqslant n_0),$$

$$x\equiv x_i \bmod p_i\quad (i>n_0).$$

Since by Lemma 7 $a_{is}\not\equiv 0 \bmod m_i$ this system is solvable in virtue of the Chinese remainder theorem; let $x_0$ be a solution. We assert that every integer $t$ in the segment $\langle A_s-x_0, A_s-x_0+2^{n-1}-1\rangle$ has a common factor with

$$\prod_{i=1}^{n_0}m_i\prod_{i=n_0+1}^{n}p_i.$$

Indeed, $t+x_0\in\langle A_s, A_s+2^{n-1}-1\rangle$ hence $t+x_0$ satisfies a congruence (22) for a suitable $i\leqslant n$. If $i\leqslant n_0$, we have

$$a_{is}t\equiv a_{is}(t+x_0)+\sum_{j\neq s}a_{ij}\xi_j-a_{is}x_0-\sum_{j\neq s}a_{ij}\xi_j\equiv 0 \bmod m_i,$$

thus

$$t \equiv 0 \bmod m_i.$$

If $i > n_0$, we have $t+x_0 = x_i$, hence

$$t = x_i - x_0 \equiv 0 \bmod p_i.$$

It follows that

$$g\Big(\prod_{i=1}^{n_0} m_i \prod_{i=n_0+1}^{n} p_i\Big) > 2^{n-1},$$

contrary to Lemma 5.

Remark. It is clear from the above proof that for $n \leqslant 4$ the numb[er] $2^{n-1}$ should be replaced in Theorem 6 by $\max\limits_{\omega(m)=n} g(m)$, i.e. by 2, 4, 6 or [...] for $n = 1, 2, 3, 4$ respectively.

### References

[1] R. Crittenden and C. L. Vanden Eynden, *Any n arithmetic progressi[ons] covering the first $2^n$ integers cover all integers*, Proc. Amer. Math. Soc. 24 (197[0]) pp. 475–481.

[2] G. Frobenius, *Theorie der linearen Formen mit ganzen Coefficienten*, J. Re[ine] Angew. Math. 86 (1878), pp. 146–208.

[3] H. Iwaniec, *On the problem of Jacobsthal*, Demonstr. Math. 11 (1978), pp. 225–2[..]

[4] E. Jacobsthal, *Über Sequenzen ganzer Zahlen, von denen keine zu n teilerfre[md] ist, I, II, III*, Norske Vid. Selsk. Forh. (Trondheim) 33 (1960), pp. 117–1[..], 125–131, 132–139.

[5] — *Über Sequenzen ganzer Zahlen, von denen keine zu n teilerfremd ist, IV*, ibid. 34 (1961), pp. 1–7, 110–115.

[6] H. Kanold, *Über eine zahlentheoretische Funktion von Jacobsthal*, Math. An[n.] 170 (1967), pp. 314–326.

[7] B. Novák and S. Znám, *Disjoint covering systems*, Amer. Math. Monthly [..] (1974), pp. 42–45.

[8] A. Schinzel, *Abelian binomials, power residues and exponential congruenc[es]*, Acta Arith. 32 (1972), pp. 245–274.

[9] S. Znám, *On properties of systems of arithmetic sequences*, ibid. 26 (1975), p[p.] 279–283.

*Received on 15. 5. 1982*
*and in revised form on 22. 6. 1982* (131[..])