[9] G. Tenenbaum, *Estimations asymptotiques de fonctions arithmétiques liées aux diviseurs*, Thèse, Bordeaux, 30 avril 1976.

[10] — *Sur deux fonctions de diviseurs*, J. London Math. Soc. (2) 14 (1967), p. 521–526.

[11] — *Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné*, preprint.

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Special values of the dilogarithm function

by

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1. Introduction. The *dilogarithm function* defined, for suitable z, by

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2 = -\int_0^z \frac{\log(1-t)}{t}\,dt,$$

is one of the lesser transcendental functions. Nonetheless, as Lewin's treatise [3] demonstrates, it has a very respectable pedigree and a wealth of curious properties.

The present work will be concerned with some unexpected relations between the values of the dilogarithm function at certain algebraic integers. Lewin [4] has shown how interesting relations can be obtained by specializing Abel's functional equation for the dilogarithm function, but this technique does not seem to yield all the known results. Richmond and Szekeres [5] have introduced a different idea. They apply the circle method to obtain asymptotic formulae for the power series coefficients of the functions occurring on the two sides of a partition identity of Andrews and Gordon. A comparison of the two formulae then yields non-trivial numerical relations for the dilogarithm function. I shall exploit the same principle here. The investigation has produced some new partition identities of the same type as the celebrated Rogers–Ramanujan identities and some new relations between values of the dilogarithm function. In particular, I shall prove a formula conjectured by Lewin [4] which had apparently resisted more direct attacks. Despite this success, these ideas do not seem to touch the central problem here which is to explain the mechanism leading to such a profusion of identities.

2. The dilogarithm relations. It is most convenient to work with the function

$$L(z) = \mathrm{Li}_2(z) + \tfrac{1}{2}\log z\cdot\log(1-z)$$

instead of with the dilogarithm function itself. To avoid problems with complex logarithms, the argument z will be restricted to the interval

$0 < z < 1$. The function $L(z)$ satisfies the functional equations

$$L(z)+L(1-z) = \pi^2/6$$

and

$$L(z) = L\big(z/(1+z)\big)+\tfrac{1}{2}L(z^2),$$

both of which can be readily verified by differentiation.

The functional equations for $L(z)$ yield evaluations of the dilogarithm function at some special algebraic points, namely Euler's results

(L1) $$L(1) = \pi^2/6$$

and

(L2) $$L(\tfrac{1}{2}) = \pi^2/12,$$

and Landen's results

(L3) $$L\big(\tfrac{1}{2}(\sqrt{5}-1)\big) = \pi^2/10 \quad \text{and} \quad L\big(\tfrac{1}{2}(3-\sqrt{5})\big) = \pi^2/15.$$

Apparently, there are no other algebraic points at which there is such an elementary evaluation of the dilogarithm function. However, there are many identities relating the values of the dilogarithm function at various powers of algebraic numbers. A simple example is

(L4) $$6L(1/3)-L(1/9) = \pi^2/3,$$

which is easily obtained from the functional equations for $L(z)$. Again, if $a = \sqrt{2}-1$, we have the relations

(L5) $$4L(a)-L(a^2) = \pi^2/4$$

and

(L6) $$4L(a)+4L(a^2)-L(a^4) = 5\pi^2/12,$$

both obtained by Lewin [4]. A similar relation not given by Lewin is

(L7) $$12L(\beta)+3L(\beta^2)-2L(\beta^3) = 5\pi^2/6, \quad \beta = \tfrac{1}{2}(\sqrt{3}-1).$$

Watson [8] found three relations involving the roots of the cubic x^3+2x^2-x-1, as follows. If the roots of this cubic are denoted by γ, $-\delta$ and $-1/\varepsilon$, so that γ, δ and ε all lie between 0 and 1, then

(L8) $$L(\gamma)-L(\gamma^2) = \pi^2/42, \quad \gamma = 1/(2\cos 2\pi/7),$$

(L9) $$2L(\delta)+L(\delta^2) = 5\pi^2/21, \quad \delta = 1/(2\cos\pi/7),$$

and

(L10) $$2L(\varepsilon)+L(\varepsilon^2) = 4\pi^2/21, \quad \varepsilon = 2\cos 3\pi/7.$$

Watson obtained these formulae by repeated use of the basic functional equations given above. Slightly more complicated relations attach to the roots of the cubic x^3+3x^2-1. If the roots of this cubic are denoted by ζ, $-\eta$ and $-1/\theta$, so that ζ, η and θ again lie between 0 and 1, then

(L11) $$3L(\zeta)+3L(\zeta^2)-L(\zeta^3) = 7\pi^2/18, \quad \zeta = 1/(2\cos\pi/9),$$

(L12) $$6L(\eta)-9L(\eta^2)-2L(\eta^3)+L(\eta^6) = -\pi^2/9, \quad \eta = 1/(2\cos 2\pi/9)$$

and

(L13) $$6L(\theta)-9L(\theta^2)-2L(\theta^3)+L(\theta^6) = \pi^2/9, \quad \theta = 2\cos 4\pi/9.$$

Here, (L11) was discovered by the asymptotic analysis described below. Lewin observed the parallel with Watson's three identities and conjectured the relations (L12) and (L13) in [4]. (The parallel is perhaps clearer in the alternative notation

$$3\mathrm{Li}_2(\zeta)+3\mathrm{Li}_2(\zeta^2)-\mathrm{Li}_2(\zeta^3) = 7\pi^2/18-3\log^2\zeta,$$
$$3\mathrm{Li}_2(-\eta)+3\mathrm{Li}_2(\eta^2)-\mathrm{Li}_2(-\eta^3) = \pi^2/18-3\log^2\eta,$$
$$3\mathrm{Li}_2(-\theta)+3\mathrm{Li}_2(\theta^2)-\mathrm{Li}_2(-\theta^3) = -\pi^2/18.)$$

The method to be described below yields the relations (L11) and (L12), but I have not been able to obtain (L13). As far as I know, none of these three relations has a more direct proof.

3. The partition identities. The relations in the previous section will be derived by applying the circle method to certain identities which express basic hypergeometric series as infinite products. Since the analysis is rather rough, many of these identities typically yield the same dilogarithm relation and the list below is only intended to be a representative selection. Many of the identities appear in Slater's list [6] of 130 identities of this type. These will be referred to as (S1) to (S130) in what follows. In stating the identities, I use the standard abbreviations

$$(a)_n = (a;\, q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}),$$

for n a positive integer,

$$(a)_\infty = (a;\, q)_\infty = \lim_{n\to\infty}(a;\, q)_n$$

and, in general, for any real number n,

$$(a)_n = (a;\, q)_n = (a;\, q)_\infty/(aq^n;\, q)_\infty.$$

The first few identities are well known, namely Euler's identities

(P1) $$\sum_{n=0}^{\infty} q^n/(q)_n = 1/(q)_\infty$$

and

(P2 = S2) $$\sum_{n=0}^{\infty} q^{n(n+1)/2}/(q)_n = 1/(q;\, q^2)_\infty,$$

and the first Rogers–Ramanujan identity

$$\text{(P3} = \text{S18)} \qquad \sum_{n=0}^{\infty} q^{n^2}/(q)_n = 1/(q;\ q^5)_\infty (q^4;\ q^5)_\infty.$$

The following identities are variations on the same theme, becoming steadily more complex.

$$\text{(P4} = \text{S47)} \qquad \sum_{n=0}^{\infty} (-1;\ q^2)_n q^{n^2}/(q)_{2n} = (-q;\ q^2)_\infty/(q;\ q^2)_\infty,$$

$$\text{(P5} = \text{S8)} \qquad \sum_{n=0}^{\infty} (-q)_n q^{\frac{1}{2}n(n+1)}/(q)_n = (-q)_\infty/(q^2;\ q^4)_\infty,$$

$$\text{(P6} = \text{S37)} \qquad \sum_{n=0}^{\infty} (-q)_n (-q;\ q^2)_n q^{\frac{1}{2}n(n+1)}/(q)_{2n+1} = (-q)_\infty (q^3;\ q^8)_\infty (q^5;\ q^8)_\infty (q^8;\ q^8)_\infty/(q)_\infty,$$

$$\text{(P7} = \text{S76)} \qquad \sum_{n=0}^{\infty} (-q)_{n+1} (q^3;\ q^3)_n q^{\frac{1}{2}n(n+3)}/(q)_n (q)_{2n+2} = (-q)_\infty (q^3;\ q^{18})_\infty (q^{15};\ q^{18})_\infty (q^{18};\ q^{18})_\infty/(q)_\infty,$$

$$\text{(P8} = \text{S33)} \qquad \sum_{n=0}^{\infty} q^{2n^2}/(q^2;\ q^2)_n (-q)_{2n} = (q^3;\ q^7)_\infty (q^4;\ q^7)_\infty (q^7;\ q^7)_\infty/(q^2;\ q^2)_\infty,$$

$$\text{(P9} = \text{S61)} \qquad \sum_{n=0}^{\infty} q^{n^2}/(q)_n (q;\ q^2)_n = (q^6;\ q^{14})_\infty (q^8;\ q^{14})_\infty (q^{14};\ q^{14})_\infty/(q)_\infty,$$

$$\text{(P10} = \text{S82)} \qquad \sum_{n=0}^{\infty} (-q)_n q^{\frac{1}{2}n(n+3)}/(q)_{2n+1} = (-q)_\infty (q^3;\ q^7)_\infty (q^4;\ q^7)_\infty (q^7;\ q^7)_\infty (q;\ q^{14})_\infty (q^{13};\ q^{14})_\infty/(q)_\infty,$$

$$\text{(P11} = \text{S92)} \qquad \sum_{n=0}^{\infty} (q^3;\ q^3)_n q^{n(n+1)}/(q)_n (q)_{2n+1} = (q^9;\ q^{27})_\infty (q^{18};\ q^{27})_\infty (q^{27};\ q^{27})_\infty/(q)_\infty.$$

The remaining identities do not appear to be in Slater's list.

$$\text{(P12)} \qquad 1 + \sum_{n=1}^{\infty} (-q^3;\ q^3)_{n-1} q^{n(n+1)}/(-q)_{n-1} (q)_{2n} = (q;\ q^9)_\infty (q^8;\ q^9)_\infty (q^9;\ q^9)_\infty (q^7;\ q^{18})_\infty (q^{11};\ q^{18})_\infty/(q)_\infty,$$

$$\text{(P12 bis)} \qquad 1 + \sum_{n=1}^{\infty} (-q^3;\ q^3)_{n-1} q^{n^2}/(-q)_{n-1} (q)_{2n} = (q^2;\ q^9)_\infty (q^7;\ q^9)_\infty (q^9;\ q^9)_\infty (q^5;\ q^{18})_\infty (q^{13};\ q^{18})_\infty/(q)_\infty,$$

$$\text{(P13)} \qquad 1 - \sum_{n=1}^{\infty} (-q^3; q^3)_{n-2} q^{n-1} (1 - q^{n-1} - q^n - q^{n+1} + q^{3n})/(-q)_{n-2} (q)_{2n} = (q;\ q^6)_\infty (q^5;\ q^6)_\infty/(q)_\infty.$$

The final identity is the nearest that I have been able to come to finding an identity for Lewin's conjectured dilogarithm relation (L13). It seems worth including here to show the limitations of the present methods.

In the quest for the lost identity to match the relation (L13), I worked through all 130 identities in Slater's list [6]. The theorem in Section 5 below cannot be applied in every case, but when it does apply, it gives a useful independent check on the identity. The asymptotic analysis located two identities which should be corrected as follows:

$$\text{(S6)} \qquad \sum_{n=0}^{\infty} (-1)_n q^{n^2}/(q)_n (q;\ q^2)_n = (-q;\ q^3)_\infty (-q^2;\ q^3)_\infty (q^3;\ q^3)_\infty/(q)_\infty,$$

$$\text{(S10)} \qquad \sum_{n=0}^{\infty} (-1)_{2n} q^{n^2}/(q^2;\ q^2)_n (q^2;\ q^4)_n = (-q;\ q^2)_\infty (-q^2;\ q^2)_\infty (-q;\ q^4)_\infty (-q^3;\ q^4)_\infty.$$

4. Proofs of the partition identities. I will sketch the derivations of the identities (P12) and (P13), following, more or less, the method developed by Slater ([6], p. 151). This rests on the following very general series transformation first exploited by Bailey. If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \quad \text{and} \quad \gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n};$$

then, subject to proper conditions for convergence,

$$\text{(1)} \qquad \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

Proof of (P12). Following Slater, let

$$u_n = 1/(q)_n, \quad v_n = 1/(x)_n, \quad \delta_n = (y)_n (z)_n x^n/y^n z^n.$$

The series for γ_n can be summed by the basic analogue of Gauss's theorem ([1], Corollary 2.4), giving

$$\gamma_n = \frac{(x/y)_\infty (x/z)_\infty}{(x)_\infty (x/yz)_\infty} \frac{(y)_n (z)_n x^n}{(x/y)_n (x/z)_n y^n z^n}.$$

Again, following Slater, α_n and β_n are determined by specializing Bailey's summation

$$(2)\quad \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})(b)_n(c)_n(d)_n(e)_n a^{2n}q^n}{(1-a)(aq/b)_n(aq/c)_n(aq/d)_n(aq/e)_n b^n c^n d^n e^n}$$
$$= \frac{(q)_\infty(q/a)_\infty(aq)_\infty(aq/bc)_\infty(aq/bd)_\infty(aq/be)_\infty(aq/cd)_\infty(aq/ce)_\infty(aq/de)_\infty}{(q/b)_\infty(q/c)_\infty(q/d)_\infty(q/e)_\infty(aq/b)_\infty(aq/c)_\infty(aq/d)_\infty(aq/e)_\infty(a^2q/bcde)_\infty}$$

for a certain well-poised bilateral hypergeometric series. If $a = q^{1/3}$, $b = q^{-n/3}$, $c = q^{(1-n)/3}$, $d = q^{(2-n)/3}$ and $e = -q^{2/3}$, and then q is replaced by q^3, this yields

$$(3)\quad \sum_{r=-[n/3]}^{[n/3]} \frac{(1-q^{6r+1})q^{\frac12(9r^2-3r)}}{(q)_{n+3r+1}(q)_{n-3r}} = \frac{(-q^3;\ q^3)_{n-1}}{(q)_{2n}(-q)_{n-1}}.$$

Since

$$(1-q^{6r+1})q^{\frac12(9r^2-3r)} = q^{\frac12(9r^2+3r)-n}\{(1-q^{n+3r+1})-(1-q^{n-3r})\},$$

the left-hand side of (3) can be re-written as

$$\frac{1-q}{(q)_n(q)_{n+1}} + q^{-n}\sum_{r=1}^{[n/3]}\Big\{\frac{q^{\frac12(9r^2+3r)}+q^{\frac12(9r^2-3r)}}{(q)_{n+3r}(q)_{n-3r}} - \frac{q^{\frac12(9r^2+3r)}}{(q)_{n+3r+1}(q)_{n-3r-1}} - \frac{q^{\frac12(9r^2-3r)}}{(q)_{n-3r+1}(q)_{n+3r-1}}\Big\}.$$

Thus the prescriptions above are satisfied by taking

$$x = q,\ \alpha_0 = 1,\ \alpha_{3r-1} = -q^{\frac12(9r^2-3r)},\ \alpha_{3r} = q^{\frac12(9r^2+3r)}+q^{\frac12(9r^2-3r)},\ \alpha_{3r+1} = -q^{\frac12(9r^2+3r)}$$

and

$$\beta_n = (-q^3;\ q^3)_{n-1}q^n/(q)_{2n}(-q)_{n-1}.$$

Using the above values for α_n, β_n, γ_n and δ_n in (1), and letting y and z tend to infinity, leads to

$$1+\sum_{n=1}^{\infty}(-q^3;\ q^3)_{n-1}q^{n^2+n}/(q)_{2n}(-q)_{n-1}$$
$$= \frac{1}{(q)_\infty}\Big\{1-q+\sum_{r=1}^{\infty}\big(q^{\frac12(27r^2+3r)}+q^{\frac12(27r^2-3r)}-q^{\frac12(27r^2+15r+2)}-q^{\frac12(27r^2-15r+2)}\big)\Big\}$$
$$= \{(-q^{12};\ q^{27})_\infty(-q^{15};\ q^{27})_\infty(q^{27};\ q^{27})_\infty - q(-q^6;\ q^{27})_\infty(-q^{21};\ q^{27})_\infty \times (q^{27};\ q^{27})_\infty\}/(q)_\infty$$

by Jacobi's triple product identity ([1], Theorem 2.8). Finally, the right side can be expressed as a single product by means of Watson's quintuple product identity ([2], (2.1)), giving (P12). Similarly, writing

$$(1-q^{6r+1})q^{\frac12(9r^2-3r)} = q^{\frac12(9r^2-3r)}\{(1-q^{n+3r+1})-q^{6r+1}(1-q^{n-3r})\}$$

on the left side of (3) leads to the transformation

$$\alpha_{3r-1} = -q^{\frac12(9r^2-9r+2)},\quad \alpha_{3r} = q^{\frac12(9r^2-3r)}+q^{\frac12(9r^2+3r)},\quad \alpha_{3r+1} = -q^{\frac12(9r^2+9r+2)},$$
$$\beta_n = (-q^3;\ q^3)_{n-1}/(q)_{2n}(-q)_{n-1},$$

with $\alpha_0 = \beta_0 = 1$ and $x = q$, and then letting y and z tend to infinity gives (P12 bis).

Proof of (P13). For this case, let

$$u_n = 1/(q^{-1};\ q^{-1})_n = (-1)^n q^{\frac12n(n+1)}/(q)_n,$$
$$v_n = 1/(x^{-1};\ q^{-1})_n = (-1)^n x^n q^{\frac12n(n-1)}/(x)_n$$

and

$$\delta_n = (y^{-1};\ q^{-1})_n(z^{-1};\ q^{-1})_n y^n z^n(1-q/w)/x^n w^n$$
$$= (y)_n(z)_n(1-q/w)/x^n w^n q^{n^2-n}.$$

The series for γ_n can be summed in the limit when w tends to q, as follows. With the above definitions,

$$\gamma_n = \sum_{r=n}^{\infty}\frac{(y)_r(z)_r}{(q)_{r-n}(x)_{r+n}}\Big(\frac{q}{w}\Big)^r\Big(1-\frac{q}{w}\Big)x^n q^{n^2-n}$$
$$= \sum_{s=0}^{\infty}\frac{(q^ny)_s(q^nz)_s}{(q)_s(q^{2n}x)_s}\Big(\frac{q}{w}\Big)^s\Big(1-\frac{q}{w}\Big)\frac{(y)_n(z)_n x^n q^{n^2}}{(x)_{2n}w^n}.$$

By the third iterate of Heine's transformation ([1], (3.3.13)), the above is

$$\sum_{s=0}^{\infty}\frac{(q^nx/y)_s(q^nx/z)_s}{(q)_s(q^{2n}x)_s}\Big(\frac{qyz}{wx}\Big)^s\frac{(qyz/wx)_\infty(y)_n(z)_n x^n q^{n^2}}{(q^2/w)_\infty(x)_{2n}w^n}.$$

Now, it is possible to set $w = q$, whence, by the basic analogue of Gauss's theorem,

$$\gamma_n = x^n q^{n^2-n}(y)_\infty(z)_\infty/(q)_\infty(x)_\infty.$$

The sequences α_n and β_n can be determined by replacing q by q^{-1} in the derivation of (P12). Thus we take $x = q$,

$$\alpha_{3r-1} = -q^{-\frac12(9r^2-3r)}),\quad \alpha_{3r} = q^{-(9r^2+3r)/2}+q^{-(9r^2-3r)/2},\quad \alpha_{3r+1} = -q^{-\frac12(9r^2+3r)}$$

and

$$\beta_n = (-q^{-3};\ q^{-3})_{n-1} q^{-n}/(q^{-1};\ q^{-1})_{2n}(-q^{-1};\ q^{-1})_{n-1}$$
$$= (-q^3;\ q^3)_{n-1} q^{n^2+n}/(q)_{2n}(-q)_{n-1},$$

with the convention that $a_0 = \beta_0 = 1$. Now

$$\sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \frac{(-q^3;\ q^3)_{n-1}(y)_n(z)_n}{(q)_{2n}(-q)_{n-1}} \left(\frac{q}{w}\right)^n \left(1-\frac{q}{w}\right)$$
$$= 1 - \sum_{n=1}^{\infty} \left\{ \frac{(-q^3;\ q^3)_{n-2}(y)_{n-1}(z)_{n-1}}{(q)_{2n-2}(-q)_{n-2}} - \frac{(-q^3;\ q^3)_{n-1}(y)_n(z)_n}{(q)_{2n}(-q)_{n-1}} \right\} \left(\frac{q}{w}\right)^n.$$

If $w = q$ and y and z tend to zero, this gives

$$\sum_{n=0}^{\infty} \beta_n \delta_n = 1 - \sum_{n=1}^{\infty} \frac{(-q^3;\ q^3)_{n-2}}{(q)_{2n}(-q)_{n-2}} q^{n-1}(1-q^{n-1}-q^n-q^{n+1}+q^{3n}).$$

On the other hand, again letting y and z tend to zero,

$$\sum_{n=0}^{\infty} a_n \gamma_n = \sum_{n=0}^{\infty} a_n q^{n^2}/(q)_\infty^2$$

and this can be summed by the method used in the derivation of (P12) to give

$$\sum_{n=0}^{\infty} a_n \gamma_n = (q;\ q^6)_\infty (q^5;\ q^6)_\infty/(q)_\infty.$$

Thus, (P13) follows from (1).

5. Asymptotic analysis of the series. The estimation of the power series coefficients of the series appearing in the partition identities of Section 3 will be accomplished by means of the following theorem.

THEOREM. *Consider the power series*

$$\sum_{n=0}^{\infty} q^{n(an+b)/2} \Big/ \prod_{j=1}^{r} (q^{c_j};\ q^{d_j})_n^{\varepsilon_j} = \sum_{k=0}^{\infty} a_k q^k,$$

where a, b, c_j, d_j and ε_j are integers satisfying $a \geqslant 0$, $b > 0$ if $a = 0$, $a \equiv b \pmod 2$ and $d_j > 0$. Suppose that the power series expansion of each of the products $\prod_{j=1}^{r}(q^{c_j};\ q^{d_j})_n^{-\varepsilon_j}$ has positive coefficients, and that the equation $\prod_{j=1}^{r}(1-x^{d_j})^{\varepsilon_j} = x^a$ has a unique root, μ say, between 0 and 1. Then

$$(\log a_k)^2/4k \to \sum_{j=1}^{r} (\varepsilon_j/d_j) L(1-\mu^{d_j}), \quad \textit{as } k \to \infty.$$

Proof. The argument follows the method developed by Szekeres [7]. Consider first the power series

$$q^{n(an+b)/2} \Big/ \prod_{j=1}^{r} (q^{c_j};\ q^{d_j})_n^{\varepsilon_j} = \sum_{k=0}^{\infty} a_{kn} q^k.$$

The a_{kn} can be obtained by Cauchy's formula, applied to a suitable circle described by $q = e^{-\beta+i\theta}$ $(-\pi \leqslant \theta \leqslant \pi)$ with radius $e^{-\beta}$ less than 1. Thus

$$a_{kn} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big\{- \sum_{j=1}^{r} \sum_{\nu=0}^{n-1} \varepsilon_j \log(1-e^{-(\beta-i\theta)(d_j\nu+c_j)}) +$$
$$+ \big(k-\tfrac{1}{2}n(an+b)\big)(\beta-i\theta)\Big\} d\theta.$$

The radius, $e^{-\beta}$, of the circle of integration is determined by the saddle-point condition

$$\sum_{j=1}^{r} \sum_{\nu=0}^{n-1} \varepsilon_j (d_j\nu+c_j)/(e^{(d_j\nu+c_j)\beta}-1) = k - \tfrac{1}{2}n(an+b). \tag{4}$$

With this choice of β, the method of [7], Section 2, gives

$$\log a_{kn} = \big(k-\tfrac{1}{2}n(an+b)\big)\beta - \sum_{j=1}^{r} \sum_{\nu=0}^{n-1} \varepsilon_j \log(1-e^{-(d_j\nu+c_j)\beta}) + \tag{5}$$
$$+ \tfrac{1}{2}\log(\beta^3/2\pi A) + o(1),$$

where

$$A = \sum_{j=1}^{r} \frac{\varepsilon_j}{d_j} \int_0^{d_j u} \frac{t^2 e^t}{(e^t-1)^2} dt \quad \text{and} \quad u = \beta n.$$

By the Euler–Maclaurin sum formula, applied to (4),

$$\frac{1}{\beta^2} \sum_{j=1}^{r} \frac{\varepsilon_j}{d_j} \int_0^{d_j u} \frac{t}{e^t-1} dt = k - \frac{au^2}{2\beta^2} + O(k^{1/2}),$$

that is

$$k = \frac{1}{\beta^2} \Big\{ \sum_{j=1}^{r} \frac{\varepsilon_j}{d_j} \int_0^{d_j u} \frac{t}{e^t-1} dt + \tfrac{1}{2}au^2 \Big\} + O(k^{1/2}), \tag{6}$$

so that β is of order $k^{-1/2}$. Again, applying the Euler–Maclaurin sum formula to (5) in conjunction with (4),

$$\log a_{kn} = \frac{1}{\beta} \sum_{j=1}^{r} \Big\{ \frac{2\varepsilon_j}{d_j} \int_0^{d_j u} \frac{t}{e^t-1} dt - \varepsilon_j u \log(1-e^{-d_j u}) \Big\} + o(k^{1/2}). \tag{7}$$

The next step is to determine n so that this expression for $\log a_{kn}$ is maximal. Let the difference operator Δ denote differences occasioned by moving from n to $n+1$. From (4)

$$\Delta\left\{\sum_{j=1}^{r}\sum_{\nu=0}^{n-1}\frac{\varepsilon_j(d_j\nu+c_j)}{e^{(d_j\nu+c_j)\beta}-1}\right\}$$
$$= -\sum_{j=1}^{r}\sum_{\nu=0}^{n-1}\frac{\varepsilon_j(d_j\nu+c_j)^2e^{(d_j\nu+c_j)\beta}}{(e^{(d_j\nu+c_j)\beta}-1)^2}\Delta\beta+\sum_{j=1}^{r}\frac{\varepsilon_j(d_jn+c_j)}{e^{(d_jn+c_j)\beta}-1}+o(1)$$
$$= -an-\tfrac{1}{2}b$$

and, as in [7], Section 4, this leads to

$$\Delta\beta = \beta^3\left\{\sum_{j=1}^{r}\frac{\varepsilon_j(d_jn+c_j)}{e^{(d_jn+c_j)\beta}-1}+an\right\}\Big/\left\{\sum_{j=1}^{r}\frac{\varepsilon_j}{d_j}\int_0^{d_ju}\frac{t^2e^t}{(e^t-1)^2}dt\right\}+o(k^{-1}).$$

In the same way, (5) gives

$$\Delta\log a_{kn} = -\sum_{j=1}^{r}\varepsilon_j\log(1-e^{-d_ju})-au+o(1).$$

The condition for $\log a_{kn}$ to be maximal is that this last expression should vanish, that is

(8) $$\sum_{j=1}^{r}\varepsilon_j\log(1-e^{-d_ju}) = -au.$$

Since $0<e^{-u}<1$, the hypotheses of the theorem fix $e^{-u}=\mu$. Substituting (8) in (6) and (7) gives the maximum

$$\log a_{kn} = 2k^{1/2}\left\{\sum_{j=1}^{r}\frac{\varepsilon_j}{d_j}\int_0^{d_ju}\frac{t}{e^t-1}dt-\tfrac{1}{2}\varepsilon_ju\log(1-e^{-d_ju})\right\}^{1/2}+o(k^{1/2})$$

in which

$$\int_0^{d_ju}\frac{t}{e^t-1}dt-\tfrac{1}{2}d_ju\log(1-e^{-d_ju}) = L(1-\mu^{d_j}).$$

Finally, $a_k=\sum a_{kn}$, where the sum over n has at most k terms, all of them positive, so that

$$\log a_k = 2k^{1/2}\left\{\sum_{j=1}^{r}\frac{\varepsilon_j}{d_j}L(1-\mu^{d_j})\right\}^{1/2}+o(k^{1/2}).$$

6. Proofs of the dilogarithm relations. To illustrate the application of the theorem, I shall sketch the derivation of the relation (L12). The identity (P12) can be re-written as

$$1+\frac{q^2}{(1-q)(1-q^2)}\sum_{n=0}^{\infty}\frac{(q)_n(q^6;\,q^6)_nq^{n^2+3n}}{(q^2;\,q^2)_n(q^3;\,q^2)_n(q^4;\,q^2)_n(q^3;\,q^3)_n}$$
$$= (q;\,q^9)_\infty(q^8;\,q^9)_\infty(q^9;\,q^9)_\infty(q^7;\,q^{18})_\infty(q^{11};\,q^{18})_\infty/(q)_\infty.$$

Let $\sum a_kq^k$ be the power series expansion of either side of this identity. The estimate for $\log a_k$ implied by the left side of the identity is the same as that obtaining for the series

$$\sum_{n=0}^{\infty}\frac{(q)_n(q^6;\,q^6)_nq^{n^2+3n}}{(q^2;\,q^2)_n(q^3;\,q^2)_n(q^4;\,q^2)_n(q^3;\,q^3)_n}.$$

As in the theorem, let μ be the real root between 0 and 1 of the equation

$$(1-x^2)^3(1-x^3) = x^2(1-x)(1-x^6),$$

that is, $x^3-3x^2+1=0$, so that $\mu=1/(2\cos 2\pi/9)$. By the theorem,

$$(\log a_k)^2\sim 4k\{\tfrac{3}{2}L(1-\mu^2)+\tfrac{1}{3}L(1-\mu^3)-L(1-\mu)-\tfrac{1}{6}L(1-\mu^6)\},$$

as $k\to\infty$. The estimate for $\log a_k$ from the right side of the identity follows readily from the well-known estimate for the partition function; namely, if

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{k=0}^{\infty}p(k)q^k,$$

then

$$\log p(k)\sim\pi(2k/3)^{1/2},$$

as $k\to\infty$. The right side of the identity is obtained by omitting 8 residue classes modulo 18 from the last infinite product, whence

$$(\log a_k)^2\sim\tfrac{10}{18}(\log p(k))^2\sim 10\pi^2k/27,$$

as $k\to\infty$. Now (L12) follows on equating the two evaluations of $(\log a_k)^2$ and using the functional equation $L(z)+L(1-z)=\pi^2/6$ to tidy up the resulting expression.

In a similar fashion, each of the identities (P1) to (P11) yields the corresponding dilogarithm relation in the list (L1) to (L11). A propos Lewin's conjecture (L13), the coefficient of q^k in the power series

$$\sum_{n=0}^{\infty}\frac{(-q^3;\,q^3)_{n-1}q^n}{(-q)_{n-1}(q)_{2n}}$$

is asymptotic to

$$2k^{1/2}\{\tfrac{3}{2}L(1-\theta^2)+\tfrac{1}{3}L(1-\theta^3)-L(1-\theta)-\tfrac{1}{6}L(1-\theta^6)\}^{1/2},$$

with $\theta = 2\cos 4\pi/9$. However, the theorem cannot be applied to (P13) because the terms of the series there do not have positive coefficients. This makes it possible for cancellation to occur between the terms, and the identity shows that this does in fact occur to a rather surprising extent.

References

[1] G. E. Andrews, *The theory of partitions*, Addison-Wesley, 1976.
[2] W. N. Bailey, *On the simplification of some identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (3) 1 (1951), pp. 217–221.
[3] L. Lewin, *Polylogarithms and associated functions*, North-Holland, 1981.
[4] — *The dilogarithm in algebraic fields*, J. Austral. Math. Soc. Series A, 33 (1982), pp. 302–330
[5] B. Richmond and G. Szekeres, *Some formulae related to dilogarithms, the zeta function and the Rogers–Ramanujan identities*, ibid. 31 (1981), pp. 362–373.
[6] L. J. Slater, *Further identities of the Rogers–Ramanujan type*, Proc. London Math. Soc. (2) 54 (1951), pp. 147–167.
[7] G. Szekeres, *Some asymptotic formulae in the theory of partitions II*, Quart. J. Math. (2) 4 (1953), pp. 96–111.
[8] G. N. Watson, *A note on Spence's logarithmic transcendant*, Quart. J. Math. Oxford Ser. 8 (1937), pp. 39–42.

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A generalization of Hasse's generalization of the Syracuse algorithm

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1. Introduction. In 1978, H. Möller [6] discussed an algorithm due to Hasse: Let m and d be relatively prime positive integers, $d \geqslant 2$; R_d is a complete set of residues mod d, not including a representative of the multiples of d; $N_d = \{n \in Z \mid d \nmid n\}$. Then $H\colon N_d \to N_d$ is defined by

$$H(x) = \frac{mx-r}{d^a}, \tag{1.1}$$

where $mx-r = d^a M$, $a \geqslant 1$, $d \nmid M$, $r \in R_d$. (It is assumed that $r \in R_d \Rightarrow m \nmid r$, to ensure H is well-defined.)

Möller conjectured that the sequence of iterates $(H^k(n))_{k\geqslant 0}$ is periodic for all $n \in N_d$ if and only if $m < d^{d/(d-1)}$ and that the set of pure periods is finite for each choice of m, d and R_d. (See Terras [7], [8], Everett [3], Crandall [2] for the special case $d = 2$, $R_d = \{-1\}$, $m = 3$ known as the Syracuse algorithm, and Heppner [4] for the general case.)

Closely related to H is the mapping $T\colon Z \to Z$ defined by

$$T(x) = \begin{cases} (mx-r)/d & \text{if } d \nmid x, \text{ where } mx \equiv r \pmod d,\ r \in R_d, \\ x/d & \text{if } d \mid x. \end{cases} \tag{1.2}$$

In fact $H^k(n) = T^{\sigma_k}(n)$, where (using Möller's notation)

$$H^k(n) = \frac{mH^{k-1}(n) - r_{k-1}}{d^{a_k}} \quad \text{and} \quad \sigma_k = \sum_{i=0}^{k} a_i.$$

In the present paper a more symmetric mapping which generalizes T is studied. Let $d, m_1, \ldots, m_d$ be positive integers, $d \geqslant 2$, $\gcd(m_i, d) = 1$ for $i = 1, \ldots, d$; $R_d = \{x_1, \ldots, x_d\}$ is a complete set of residues mod d; $r_i \in R_d$ is defined for $i = 1, \ldots, d$ by $m_i x_i \equiv r_i \pmod d$. Then $T\colon Z \to Z$ is given by

$$T(x) = \frac{m_i x - r_i}{d} \quad \text{if} \quad x \equiv x_i \pmod d. \tag{1.3}$$