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# Primitive newforms of weight 3/2

by

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**0. Introduction.** In [12], Vigneras shows that if $F(z) = \sum_{n\geqslant 0} a(n)\exp(2\pi inz)$ is a modular form of half-integral weight for some congruence subgroup of $\mathrm{SL}(2, \mathbf{Z})$ such that $a(n) = 0$ unless $n = tm^2$ for $t$ an element of a finite set of positive square-free integers and $m$ an integer, then the weight of $F$ is $1/2$ or $3/2$ and $F$ can be realized as a linear combination of certain explicit theta series. Serre and Stark [9] established that all modular forms of weight $1/2$ are so distinguished, however it is well known that not all forms of weight $3/2$ are. For an odd Dirichlet character $\psi$, Shimura [11] defines the cusp form $h_\psi = \frac{1}{2}\sum_{-\infty}^{\infty}\psi(m)m\exp(2\pi im^2z)$ which has weight $3/2$ and is obviously distinguished as above. In fact as Gelbart and Piatetski-Shapiro [5] point out, the $h_\psi$ are "essentially" the only forms of weight $3/2$ which satisfy Vigneras' Theorem. Thus we restrict our attention to Shimura's $h_\psi$.

There is intrinsic interest in the $h_\psi$. Under the Shimura lifting of modular forms of half-integral weight to modular forms of integral weight, the image of the orthogonal complement (in the space of cusp forms of weight $3/2$) of the space generated by the $h_\psi$ is cuspidal. This was conjectured by Shimura [11] and first proven by Gelbart and Piatetski-Shapiro [4], [5] using representation-theoretic methods (see also Flicker [3]) and, using "classical" methods, by Cipra [2] and Kojima [8]. It is relevant to inquire whether the $h_\psi$ are newforms since newforms of a given level are, in an explicit sense, fundamental to the construction of modular forms of higher levels. One can also ask whether $h_\psi$ is a primitive form in the sense of [1] or [6].

In this paper, we establish that $h_\psi$ is a cuspidal newform by means of a trace operator and give necessary and sufficient conditions that $h_\psi$ be a primitive newform.

**1. Notation and terminology.** For $z \in C$, put $e(z) = \exp(2\pi i z)$ with $i = \sqrt{-1}$ and define $\sqrt{z} = z^{1/2}$ with $-\pi/2 < \arg z^{1/2} \leqslant \pi/2$. Further put $z^{\varkappa/2} = (z^{1/2})^{\varkappa}$ for every $\varkappa \in Z$. Let $\mathfrak{Z}$ denote the upper half-plane $\{z \in C \mid \operatorname{Im}(z) > 0\}$. Denote by $\Gamma_0(N)$ the group defined by

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2, Z) \mid \ c \equiv 0 \ (\mathrm{mod}\ N)\right\}.$$

For $t \in Z$ we define a quadratic symbol $\left(\frac{t}{*}\right)$ exactly as in [11]. All characters are assumed primitive so the product of two characters $\chi$, $\psi$ is the primitive character associated with $n \to \chi(n)\psi(n)$. For $z \in \mathfrak{Z}$ let $\theta(z) = \sum_{-\infty}^{\infty} e(n^2 z)$ be the standard theta function and if $A \in \Gamma_0(4)$ set $j(A, z) = \theta(Az)/\theta(z)$, the theta multiplier of $A$.

We shall be concerned exclusively with cusp forms of weight 3/2 defined on congruence subgroups $\Gamma_0(N)$ where $N$ is always assumed to be *divisible by* 4. If $\chi$ is a Dirichlet character modulo $N$, then in addition to holomorphy conditions (see [11]) a modular form $F$ of weight 3/2 and character $\chi$ on $\Gamma_0(N)$ satisfies the functional equation $F(Az) = \chi(d) j(A, z)^3 F(z)$ for every $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N)$. The space of all such cusp forms is denoted $\mathscr{S}(N, 3/2, \chi)$. We say $N$ is the *exact level* of a modular form $F$ if $F$ has level $N$ but does not have level $N'$ for any $N' < N$.

Finally for a primitive character $\psi$ of conductor $r$ and $a \in Z/rZ$, we define the Gauss sum:

$$\mathfrak{g}_\psi(a) = \sum_{b \bmod r} \psi(b) e(ab/r).$$

Put $\mathfrak{g}(\psi) = \mathfrak{g}_\psi(1)$. It is well known that $\mathfrak{g}_\psi(a) = \bar{\psi}(a)\mathfrak{g}(\psi)$ and $|\mathfrak{g}(\psi)| = \sqrt{r}$.

**2. The newform $h_\psi$.** For an odd Dirichlet character $\psi$ of conductor $r$, put

$$h_\psi(z) = \sum_{m=1}^{\infty} \psi(m) m e(m^2 z).$$

By a remark following Proposition 2.2 of [11], $h_\psi \in \mathscr{S}\left(4r^2, \tfrac{3}{2}, \left(\frac{-1}{*}\right)\psi\right)$. In this section we show that $h_\psi$ is a cuspidal newform of level $4r^2$. Having first established that $h_\psi$ is an eigenform for all of the Hecke operators, the result will follow from Theorem 5.2 of [10] which characterizes the space generated by cuspidal newforms by means of a trace operator. We shall also need two other operators to achieve our goal: the slash operator denoted by $|$ and for a positive integer $N$, the symmetry operator $W(N)$. For the definition and properties enjoyed by these operators, the reader is referred to § 3 of [9].



We begin with

Proposition 2.1. *$h_\psi$ is an eigenform for all the Hecke operators, $T(p^2)$, and if $r$ is the conductor of $\psi$ and we put $h_\psi | T(p^2) = \omega_p h_\psi$, then*

$$\omega_p = \begin{cases}(1+p)\psi(p) & \text{if} \quad p \nmid 2r, \\ p\psi(p) & \text{if} \quad p \mid 2r.\end{cases}$$

Proof. Let $h_\psi | T(p^2) = \sum_{n \geqslant 1} b(n) e(nz)$. By Lemma 1 of [9], $b(n) = 0$ if $n$ is not a square and

$$b(m^2) = \begin{cases} mp\psi(mp) & \text{if} \quad p \mid 2r \\ mp\psi(mp) + \psi(p)\left(\dfrac{m^2}{p}\right) m\psi(m) + p\left(\dfrac{-1}{p^2}\right)\psi(p^2)\left\{\dfrac{m}{p}\right\}\psi\left(\left\{\dfrac{m}{p}\right\}\right) & \\ & \text{if} \quad p \nmid 2r,\end{cases}$$

where $\left\{\dfrac{m}{p}\right\} = 0$ if $p \nmid m$ or $m/p$ if $p \mid m$. This reduces to

$$b(m^2) = \begin{cases} p\psi(p) m\psi(m) & \text{if} \quad p \mid 2r, \\ \big(p\psi(p) + \psi(p)\big) m\psi(m) & \text{if} \quad p \nmid 2r\end{cases}$$

from which the proposition follows.

Corollary 2.2. *The Hecke eigenvalues of $h_\psi$ determine the character $\psi$. In particular, if $h_\psi$ and $h_\varphi$ have the same eigenvalues for all but a finite number of the Hecke operators, then $\varphi = \psi$.*

Proof. By the previous proposition, for all but a finite number of primes $p$ we have $(1+p)\varphi(p) = (1+p)\psi(p)$. The corollary follows from this and Dirichlet's theorem on primes in arithmetic progressions.

Recall the definition and basic properties of the trace operator. Let $q$ be a prime with $4q \mid N$ and write $\Gamma_0(N/q)$ as a disjoint union of right cosets modulo $\Gamma_0(N)$, say

$$\Gamma_0(N/q) = \bigcup_{j=1}^{\mu} \Gamma_0(N) A_j \quad \text{where } A_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j\end{bmatrix}$$

and where $\mu = [\Gamma_0(N/q) \colon \Gamma_0(N)]$. If $\chi$ is a Dirichlet character definable modulo $N/q$, we define the trace operator $\operatorname{Tr}(\chi) = \operatorname{Tr}(\chi, N, q)$ on $\mathscr{S}(N, 3/2, \chi)$ as follows. Let $F \in \mathscr{S}(N, 3/2, \chi)$. Then in the above notation,

$$F | \operatorname{Tr}(\chi) = \sum_{k=1}^{\mu} \chi(a_k) j(A_k, z)^{-3} F(A_k z) \in \mathscr{S}(N/q, 3/2, \chi),$$

where $j(A_k, z)$ is the theta multiplier of $A_k$. One easily verifies that the definition is independent of the choice of $A_k$'s. Moreover, $\mathrm{Tr}(\chi)$ commutes with the Hecke operators $T(p^2)$ for $p \nmid N$ and if $F \in \mathscr{S}(N/q, 3/2, \chi)$, then $F|\mathrm{Tr}(\chi) = \mu F$. For more details of the trace operator, the reader is referred to § 5 of [10].

We now fix some notation for the remainder of the paper. Let $\psi$ be an odd Dirichlet character of conductor $r$ and set $\chi = \left(\frac{-1}{*}\right)\psi$. In view of Proposition 2.1 of this paper and Theorem 5.2 of [10], to prove $h_\psi$ is a cuspidal newform in $\mathscr{S}(4r^2, 3/2, \chi)$ we need only establish that for each prime $q|r$,

$$h_\psi|\mathrm{Tr}(\chi, 4r^2, q) = 0 \quad \text{and} \quad h_\psi|W(4r^2)|\mathrm{Tr}(\bar{\chi}, 4r^2, q) = 0.$$

For a prime $q|r$ a complete set of right coset representatives of $\Gamma_0(4r^2)$ in $\Gamma_0(4r^2/q)$ is given by:

$$A_v = \begin{bmatrix} 1 & 0 \\ av & 1 \end{bmatrix}, \quad v = 0, 1, \ldots, q-1; \; a = 4r^2/q.$$

We note that the theta multiplier of $A_v$, $j(A_v, z)$, is simply $(avz+1)^{1/2}$.

LEMMA 2.3. $h_\psi|\mathrm{Tr}(\chi, 4r^2, q) = 0$.

Proof. In the above notation we have

$$h_\psi|\mathrm{Tr}(\chi, 4r^2, q) = \sum_{v=0}^{q-1} h_\psi\big(z/(avz+1)\big)(avz+1)^{-3/2}.$$

By Proposition 2.3 of [11], we have

$$h_\psi\big(z/(avz+1)\big)(avz+1)^{-3/2} = \frac{-1}{2r}\sum_{m=-\infty}^{\infty} m\xi(m, v)e(m^2 z)$$

where

$$\xi(m, v) = \sum_{k=1}^{r}\sum_{g=1}^{r}\psi(k)e\big((gm+gk-g^2vr/q)/r\big).$$

Thus

$$h_\psi|\mathrm{Tr}(\chi, 4r^2, q) = \frac{-1}{2r}\sum_{-\infty}^{\infty} m\lambda(m)e(m^2 z) \quad \text{where} \quad \lambda(m) = \sum_{v=0}^{q-1}\xi(m, v).$$

Now

$$\lambda(m) = \sum_{g=1}^{r} e(mg/r)\sum_{v=0}^{q-1} e(-g^2v/q)\sum_{k=1}^{r}\psi(k)e(gk/r)$$
$$= \sum_{g=1}^{r}\bar{\psi}(g)\mathfrak{g}(\psi)e(mg/r)\sum_{v=0}^{q-1} e(-g^2v/q).$$



Since the conductor of $\psi$ is $r$, $\bar{\psi}(g) = 0$ if $(g, r) > 1$. On the other hand, if $(g, r) = 1$ then since $q|r$, $e(-g^2/q)$ is a primitive $q$th root of unity. Thus $\sum_{v=0}^{q-1} e(-g^2v/q) = 0$, hence $\lambda(m) = 0$ for all integers $m$ and so the lemma is proved.

LEMMA 2.4. $h_\psi|W(4r^2)|\mathrm{Tr}(\bar{\chi}, 4r^2, q) = 0$.

Proof. By Proposition 2.3 of [11], $h_\psi|W(4r^2) = \varkappa h_{\bar{\psi}}$ where $\varkappa$ is a constant. The lemma is now immediate from the preceding one.

THEOREM 2.5. *$h_\psi$ is a cuspidal newform in* $\mathscr{S}\left(4r^2, 3/2, \left(\frac{-1}{*}\right)\psi\right)$.

Proof. By Proposition 2.1, $h_\psi$ is an eigenform for all of the Hecke operators so all we need show is that $h_\psi$ is in the orthogonal complement of the space generated by the cuspidal oldforms. Using Theorem 5.2 of [10], this is accomplished by Lemmas 2.3 and 2.4.

**3. Primitive forms.** A final question which can be asked about $h_\psi$ is the conditions under which it is primitive. If $F(z) = \sum a(n)e(nz)$ is a modular form, the character twist of $F$ by the Dirichlet character $\psi$, denoted $F^\psi$, is the modular form given by $F^\psi(z) = \sum a(n)\psi(n)e(nz)$ (see [10]). Recall that a cusp form is primitive if it is not the character twist of a cuspidal newform which has level lower than the original cusp form. Primitive forms of integral weight were studied in [1], [6] and [7]. On a tangential note, character twists can also be used to provide an alternate means of proof of Theorem 2.5.

Throughout this section, $\psi$ is an odd Dirichlet character of conductor $r$. One may write $\psi$ in a unique way as $\psi = \prod_{p|r}\psi_p$, the product over all primes dividing $r$ where $\psi_p$ is the $p$th component of $\psi$ having conductor $r_p$ equal to the highest power of $p$ dividing $r$. The question of when $h_\psi$ is primitive is completely answered by

THEOREM 3.1. *$h_\psi$ is primitive if and only if each $\psi_p$ is an odd Dirichlet character of conductor $p$ ($4$ if $p = 2$).*

Proof (only if). We prove the contrapositive. Suppose some $\psi_p$ is even. Then $\psi_p = \varphi_p^2$ for some $\varphi_p$ having conductor $r_p$ ($2r_p$ if $p = 2$). Let $\psi' = \prod_{\substack{q|r \\ q\neq p}}\psi_q$. Then $\psi = \psi'\varphi_p^2$ and $h_\psi = (h_{\psi'})^{\varphi_p}$ (i.e., the character twist of $h_{\psi'}$ by $\varphi_p$). By Theorem 2.5 $h_{\psi'}$ is a cuspidal newform in $\mathscr{S}\left(4(r/r_p)^2, 3/2, \left(\frac{-1}{*}\right)\psi'\right)$ and hence $h_\psi$ is not primitive. Next suppose that each $\psi_p$ is odd, but some $\psi_p$ ($p \neq 2$) does not have prime conductor (i.e., $p^2|r_p$). Then we may write $\psi_p = \varepsilon_p\varphi_p^2$, where $\varepsilon_p$ is an odd character mod $p$ and $\varphi_p$ is primitive mod $r_p$. Letting $\psi' = \varepsilon_p\prod_{\substack{q|r \\ q\neq p}}\psi_q$ we see that $h_\psi = (h_{\psi'})^{\varphi_p}$.

By Theorem 2.5, $h_{\psi'}$ is a cuspidal newform of level $4(rp/r_p)^2 < 4r^2$, so $h_\psi$ is not primitive. Finally if each $\psi_p$ is odd and $\psi_2$ had conductor divisible by 8, then $\psi_2 = \left(\frac{-1}{*}\right)\varphi_2^2$ where $\varphi_2$ is primitive mod $2r_2$. The rest of the proof is analogous to the previous case.

(if) We prove this direction by contradiction. Suppose each $\psi_p$ is odd and of conductor $p$ (4 if $p = 2$), and suppose that $h_\psi$ is not primitive, that is $h_\psi = F^\varphi$ for some cuspidal newform $F \in \mathscr{S}(N, 3/2, \lambda)$ with $N < 4r^2$ and some Dirichlet character $\varphi$. Let $s$ be the conductor of $\varphi$ and decompose $\varphi$ into $p$th components: $\varphi = \prod_{p|s} \varphi_p$. Since $h_\psi$ has character $\left(\frac{-1}{*}\right)\psi$ and $F^\varphi$ has character $\lambda\varphi^2$ we have $\left(\frac{-1}{*}\right)\psi = \lambda\varphi^2$. Now $F$ is a cuspidal newform in $\mathscr{S}(N, 3/2, \lambda)$ and by Theorem 2.5, $F^\varphi = h_\psi$ is a cuspidal newform in $\mathscr{S}(4r^2, 3/2, \lambda\varphi^2)$. Let $t$ be the conductor of $\psi\bar{\varphi}^2$ and $t_p$ the conductor of $\psi_p\bar{\varphi}_p^2$. We consider two cases. If $r|t$ then $F^{\varphi\bar{\varphi}} = h_\psi^{\bar{\varphi}} = h_{\psi\bar{\varphi}^2}$ is (by Theorem 2.5) a cuspidal newform in $\mathscr{S}(4t^2, 3/2, \lambda)$ and if we set $F(z) = \sum_{n\geqslant 1} a(n)e(nz)$ then

$$(F - F^{\varphi\bar{\varphi}})(z) = \sum_{(n,s)>1} a(n)e(nz) \in \mathscr{S}(4t^2, 3/2, \lambda).$$

By Theorem 1 of [9], $F - F^{\varphi\bar{\varphi}}$ is an element of the space generated by the cuspidal oldforms of level $4t^2$ $(\mathscr{S}^{\text{old}}(4t^2, 3/2, \lambda))$. Since $N < 4r^2 \leqslant 4t^2$, $F \in \mathscr{S}^{\text{old}}(4t^2, 3/2, \lambda)$ and so $F^{\varphi\bar{\varphi}} \in \mathscr{S}^{\text{old}}(4t^2, 3/2, \lambda)$. But $F^{\varphi\bar{\varphi}} = h_{\psi\bar{\varphi}^2}$ is a cuspidal newform in $\mathscr{S}(4t^2, 3/2, \lambda)$. This provides the desired contradiction in the case $r|t$. If $r\nmid t$ then there exists a prime $p|r$ with $t_p < r_p (t_p | r_p)$. Since $r$ is square-free (except possibly $r_2 = 4$) we must have for this $p$, $\psi_p\bar{\varphi}_p^2 = 1$. This is clearly impossible since each $\psi_p$ is odd. Thus $h_\psi$ must be primitive.

We remark that character twists can be used to prove that the exact level of $h_\psi$ is $4r^2$ (in most cases). We start from the assumption that the exact level of $h_\psi$ is a square dividing $4r^2$ (see Lemma 13 of [9] for motivation). As no new results are obtained, the arguments will only be sketched.

Proposition 3.2. *If $r$ is square-free then the exact level of $h_\psi$ is $4r^2$.*

Proof. Without loss of generality, we need only consider the case where $2\nmid r$. If $4r^2$ is not the exact level of $h_\psi$ then by assumption the exact level must divide $4r^2/q^2$ for some prime $q|2r$. But in this case the character of $h_\psi$, $\left(\frac{-1}{*}\right)\psi$, has conductor $4r$ and so is not definable mod $4r^2/q^2$ (i.e., $4r\nmid 4r^2/q^2$) so the exact level must be $4r^2$.



For an arbitrary character $\psi$, we decompose it into $p$th components as before: $\psi = \prod_{p|r}\psi_p$, where each $\psi_p$ has conductor $r_p$. For simplicity we shall consider only the case where $2\nmid r$.

Each $\psi_p$ can be further decomposed as $\psi_p = \varepsilon_p\varphi_p^2$ where

$$\varepsilon_p = \begin{cases} 1 & \text{if} \quad \psi_p \text{ is even,} \\ \psi_p & \text{if} \quad r_p = p \text{ and } \psi_p \text{ is odd,} \\ \text{any odd character mod } p & \text{if} \quad p^2|r_p \text{ and } \psi_p \text{ is odd} \end{cases}$$

and

$$\varphi_p = \begin{cases} 1 & \text{if} \quad r_p = p \text{ and } \psi_p \text{ is odd,} \\ \text{a primitive character mod } r_p & \text{if} \quad p^2|r_p \text{ or } \psi_p \text{ is even.} \end{cases}$$

If we put $\varepsilon = \prod_{p|r}\varepsilon_p$ and $\varphi = \prod_{p|r}\varphi_p$ and let $r_\varepsilon$ be the conductor of $\varepsilon$, then by Proposition 3.2 the exact level of $h_\varepsilon$ is $4r_\varepsilon^2$ and Theorem 2.5 yields that $h_\varepsilon$ is a cuspidal newform of that level.

We now make one final restriction: if $\psi_p$ is even we require $p^2|r_p$. Then using induction on the primes dividing $r$ one verifies using either Theorem 6.6 or 6.10 of [10] that $h_\varepsilon^{\varphi_p} = h_{\varepsilon\varphi_p^2}$ has exact level $4t^2$ where $t$ is the conductor of $\varepsilon\varphi_p^2$ and hence is a cuspidal newform of that level. One continues twisting by each $\varphi_p$ in succession to obtain

Theorem 3.3. *Assume $2\nmid r$ and either $\psi_p$ is odd or $p^2|r_p$ for each prime $p|r$. Then $h_\psi$ has exact level $4r^2$ and hence is a cuspidal newform of that level.*

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# Oscillatory properties of $M(x) = \sum_{n\le x}\mu(n)$, III

by

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**1.** In part I [6] of this series we proved that if $\zeta(\varrho_0) = \zeta(\beta_0+i\gamma_0) = 0$, then for $Y > e^{|\gamma_0|+4}$

$$(1.1)\qquad \max_{Y/(100\log Y)\le x\le Y}|M(x)| \ge \frac{1}{Y}\int\limits_{Y/(100\log Y)}^{Y}|M(x)|\,dx \ge \frac{Y^{\beta_0}}{6|\varrho_0|^3}.$$

This implies by easy calculation that for $Y > 2$

$$(1.2)\qquad \max_{Y/(100\log Y)\le x\le Y}|M(x)| \ge \frac{1}{Y}\int\limits_{Y/(100\log Y)}^{Y}|M(x)|\,dx > \frac{\sqrt{Y}}{17000}.$$

In part II [7] we showed that $M(x)$ changes sign in every interval of the form

$$(1.3)\qquad [Y\exp(-3\log_2^{3/2}Y),\ Y]$$

for $Y > c_1$, where $\log_\nu Y$ denotes the $\nu$ times iterated logarithmic function, and $c_1, c_2, \ldots$ denote explicitly calculable positive absolute constants. Concerning these problems, it is natural to ask how large are the oscillations of $M(x)$ in positive and negative directions and what kind of estimates can be proved for $\max_{x\le Y}M(x)$ and $\min_{x\le Y}M(x)$.

The first results in this field are due to S. Knapowski. By the application of Turán's method he proved in [4] that the Riemann hypothesis implies for $Y > c_2$ the inequality

$$(1.4)\qquad \max_{x\le Y}M(x) \ge \max_{A(Y)\le x\le Y}M(x) \ge \sqrt{Y}\exp\left(-15\frac{\log Y}{\log_2 Y}\log_3 Y\right)$$

and the corresponding inequality for $\min_{x\le Y}M(x)$, where

$$(1.5)\qquad A(Y) = Y\exp\left(-c_3\frac{\log Y}{\log_2 Y}\log_3 Y\right).$$