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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

On adele rings of arithmetically equivalent fields

by

KEIICHI KOMATSU (Tokyo)

Let Q be the rational number field, k an algebraic number field, k_A the adele ring of k and $\zeta_k(s)$ the Dedekind zeta-function of k. For a prime number p, we denote by Q_p the p-adic number field. The word isomorphism for topological rings means a topological isomorphism. In this paper we shall show the following:

THEOREM. *For every positive integer r there are $r+1$ non-isomorphic algebraic number fields $k_0, k_1, \ldots, k_r$ such that their adele rings are isomorphic and their Dedekind zeta-functions coincide.*

Namely, let $m_1, \ldots, m_r$ be squarefree integers $\neq \pm 1, \pm 2$ such that m_i does not divide $\prod_{j\neq i}^{r} m_j$ and $m_i \equiv 2 \pmod{16}$. Then we can take

$$k_0 = Q(\sqrt[8]{m_1}, \ldots, \sqrt[8]{m_r}),$$

$$k_i = Q(\sqrt[8]{16m_1}, \ldots, \sqrt[8]{16m_i}, \sqrt[8]{m_{i+1}}, \ldots, \sqrt[8]{m_r}) \quad \text{for} \quad i = 1, 2, \ldots, r.$$

LEMMA 1 (cf. [3], Lemma 7). *Let k be an algebraic number field, V_k the set of places of k and W_k the set of non-zero prime ideals of k. We adopt similar notations for an algebraic number field k'. Then the following conditions are equivalent:*

(1) *k_A and k'_A are isomorphic.*

(2) *There exists a bijection Φ of V_k onto $V_{k'}$ such that $k_\mathfrak{p}$ and $k'_{\Phi(\mathfrak{p})}$ are isomorphic for every $\mathfrak{p} \in V_k$.*

(3) *There exists a bijection Ψ of W_k onto $W_{k'}$ such that $k_\mathfrak{p}$ and $k'_{\Psi(\mathfrak{p})}$ are isomorphic for every $\mathfrak{p} \in W_k$.*

(4) *The tensor product $k \otimes_Q Q_p$ is isomorphic to $k' \otimes_Q Q_p$ for every prime number p.*

LEMMA 2 (cf. [1], p. 362, [5]). *Let L be a finite Galois extension of Q and $G = G(L/Q)$ the Galois group of L over Q. Let H and H' be subgroups of G. For every element σ of G, let $C(\sigma) = \{\tau^{-1}\sigma\tau \mid \tau \in G\}$. Let k and k' be subfields* *of L corresponding to the subgroups H and H' of G, respectively. Then the following conditions are equivalent:*

(1) *For every element σ of G, the cardinality of $C(\sigma)\cap H$ is equal to the cardinality of $C(\sigma)\cap H'$.*

(2) *For every prime number p, the collection of degrees of the factors of p in k is identical with the collection of degrees of the factors of p in k'.*

(3) *The zeta-functions $\zeta_k(s)$ and $\zeta_{k'}(s)$ are the same.*

Algebraic number fields are said to be arithmetically equivalent, when their zeta-functions coincide. The following lemma follows from Lemma 2:

LEMMA 3 (cf. [2], [4]). *Notations being as in the above theorem, the fields $k_0, k_1, \ldots, k_r$ are not isomorphic to each other and we have $\zeta_{k_0}(s) = \zeta_{k_i}(s)$ for $i = 0, \ldots, r$.*

Now we show the following:

LEMMA 4. *Let k and k' be algebraic number fields and p a prime number such that the tensor product $k\otimes_{Q}Q_p$ is isomorphic to $k'\otimes_{Q}Q_p$. Let F be an algebraic number field such that $(Fk:k) = (Fk':k') = (F:Q)$. Then we have*

$$(kF)\otimes_{Q}Q_p \cong (k'F)\otimes_{Q}Q_p.$$

Proof. Let θ be an algebraic number such that $F = Q(\theta)$, $f(x)$ the minimal polynomial of θ over Q, $\mathfrak{p}$ a prime ideal of k which lies above p, $k_{\mathfrak{p}}[x]$ the polynomial ring in one variable and $(f(x))$ the ideal of $k_{\mathfrak{p}}[x]$ generated by $f(x)$. Then we have

$$(kF)\otimes_{Q}Q_p \cong \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}\in V_k}} (kF)\otimes_k k_{\mathfrak{p}} \cong \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}\in V_k}} \big(k_{\mathfrak{p}}[x]/(f(x))\big) \cong \prod_{\substack{\mathfrak{p}'|p\\ \mathfrak{p}'\in V_{k'}}} \big(k'_{\mathfrak{p}'}[x]/(f(x))\big)$$
$$\cong \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}'\in V_{k'}}} (k'F)\otimes_{k'} k'_{\mathfrak{p}'} \cong (k'F)\otimes_{Q}Q_p.$$

Proof of Theorem. Since we have

$$k_0 = Q(\sqrt[8]{m_1}, \sqrt[8]{m_1m_2}, \ldots, \sqrt[8]{m_1m_i}, \sqrt[8]{m_{i+1}}, \ldots, \sqrt[8]{m_r})$$

and

$$k_i = Q(\sqrt[8]{16m_1}, \sqrt[8]{m_1m_2}, \ldots, \sqrt[8]{m_1m_i}, \sqrt[8]{m_{i+1}}, \ldots, \sqrt[8]{m_r}),$$

it is sufficient to prove $k_{0A} \cong k_{1A}$. If a prime number p is unramified in k_0/Q, then we have $k_0\otimes_{Q}Q_p \cong k_1\otimes_{Q}Q_p$ from Lemma 3. Now we assume that p is ramified and that $p \neq 2$. If $p \equiv 1, 7 \pmod 8$, then $k_0\otimes_{Q}Q_p \cong k_1\otimes_{Q}Q_p$ follows from that Q_p contains $\sqrt{2}$. If $p \equiv 3 \pmod 8$, then $k_0\otimes_{Q}Q_p \cong k_1\otimes_{Q}Q_p$ follows from that Q_p contains $\sqrt{-2}$. If $p \equiv 5 \pmod 8$, then $k_0\otimes_{Q}Q_p \cong k_1\otimes_{Q}Q_p$ follows from that Q_p contains $\sqrt{-1}$. Suppose that $p = 2$. We assume that $m_1 \equiv 2 \pmod{16}$. We should notice that 2 is totally ramified in $Q(\sqrt[8]{m_1})/Q$ and $Q(\sqrt[8]{16m_1})/Q$. We see that $Q_2(\sqrt[8]{m_1})$ contains $\sqrt{2}$. Hence we have $k_0\otimes_{Q}Q_2 \cong k_1\otimes_{Q}Q_2$ from Lemma 4. Hence we have $k_{0A} \cong k_{1A}$ from Lemma 1.

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DEPARTMENT OF MATHEMATICS
TOKYO UNIVERSITY OF AGRICULTURE AND TECHNOLOGY
Fuchu, Tokyo, Japan

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