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Distribution of coefficients of Eisenstein series in residue classes

by

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1. Let $k \geqslant 3$ be an odd integer and denote by $F_{k+1}(z)$ the Eisenstein series of weight $k+1$, i.e.

$$F_{k+1}(z) = \zeta(k+1) + \frac{(2\pi i)^{1+k}}{k!} \sum_{n=1}^{\infty} \sigma_k(n) e^{2\pi i n z}$$

(for z in the upper half-plane) where

$$\sigma_k(n) = \sum_{d|n} d^k.$$

The function σ_k is integer-valued and multiplicative and its arithmetical properties were investigated in [8], [9], [11]. In this paper we shall consider the distribution of values of σ_k (also for even $k \neq 2$) in residue classes (mod N) prime to N, with N being a given integer, and primarily we shall be interested in those values of N for which this distribution is uniform, i.e. for which σ_k is weakly uniformly distributed (mod N) (WUD mod N). Since σ_k is polynomial-like, i.e. for any fixed $m \geqslant 1$ and prime p we have

$$\sigma_k(p^m) = V_{k,m}(p)$$

where $V_{k,m}(x)$ is a polynomial with rational integral coefficients it is possible to utilize a method given in [2] which permits to decide for a given value of N whether a given polynomial-like multiplicative function f is WUD (mod N) or not. In certain cases this method permits to determine the set $M(f)$ of all such N's for which f is WUD (mod N), as it happened for the divisor function $d(n)$ or Euler's function $\varphi(n)$, which were considered in [2]. However this method does not lead to an algorithm giving $M(f)$ for arbitrary functions and in particular the treatment of σ_k presented certain difficulties. The first case ($k = 1$) was settled in [10] and recently the set $M(\sigma_2)$ was determined ([6]). In the general case it was proved by Fomenko ([1]) that $M(\sigma_k)$ contains all sufficiently large primes and the same result follows also from the main theorem in [4] which implies that if f is multiplicative and for all primes p one has $f(p) = V(p)$ with a non-constant polynomial $V(x)$ which is not of the form $cW^k(x)$ with a polynomial $W(x)$, a constant c and $k \geqslant 2$, then there is an effectively determined finite set E of primes with the property that $M(f)$ contains all integers which are not divisible by any member of E.

In this paper we shall establish the existence of an effective algorithm determining $M(f)$ for a class of polynomial-like multiplicative functions and then shall prove that all functions σ_k with $k \geqslant 3$ are contained in this class. As an example of an application of this algorithm we shall compute the set $M(\sigma_3)$.

2. Let f be a multiplicative integer-valued function, which is polynomial-like, i.e. for $j = 1, 2, \ldots$ and all primes p one has

$$f(p^j) = V_j(p)$$

where $V_1, V_2, \ldots$ are polynomials over Z. For $N \geqslant 2$ put

$$R_j(N, f) = \{V_j(x)\colon (xV_j(x), N) = 1\} \quad (j = 1, 2, \ldots)$$

and denote by $M(N, f)$ the smallest value of $j \geqslant 1$ for which the set $R_j(N, f)$ is non-void, provided such a value exists. In sequel, when regarding a fixed function f, we shall suppress the letter f and write simply $R_j(N)$ and $M(N)$. Throughout it will be assumed that f satisfies the following condition:

(A_N) Not all sets $R_1(N, f), R_2(N, f), \ldots$ are empty.

If a function f satisfies this condition for all N, then we shall say that f satisfies the condition (A). Note that if f does not satisfy (A_N) then $f(n)$ can be co-prime with N only if every prime divisor of n divides also N, thus the number of $n \leqslant x$ with $(f(n), N) = 1$ is $O(\log^t x)$ with t being the number of distinct prime factors of N.

We shall also utilize the following restrictive condition on f:

(B) None of the polynomials $V_1, V_2, \ldots$ is of the form cW^r where c is a constant, W a polynomial over Z and $r \geqslant 2$ an integer.

The following necessary and sufficient condition for f to be WUD (mod N) was established in [2]:

PROPOSITION 1. *If f is multiplicative, polynomial-like and integer-valued, $N \geqslant 2$ and the condition* (A_N) *is satisfied, then f is* WUD (mod N) *if and only if for every non-principal character χ* (mod N) *which equals unity on the set $R_M(N, f)$ (where $M = M(N, f)$) there exists a prime $p \leqslant 2^M$ such that*

$$1 + \sum_{k=1}^{\infty} \chi\big(f(p^k)\big) p^{-k/M} = 0. \tag{1}$$

This proposition implies in particular that if $R_M(N, f)$ generates the multiplicative group $G(N)$ of residue classes (mod N) prime to N, then f is WUD (mod N), which result in case $M = 1$ goes back to E. Wirsing [12].

We shall say that f is regularly WUD (mod N) provided the set $R_M(N, f)$ generates $G(N)$ with $M = M(N, f)$. This definition is applicable only to multiplicative and polynomial-like integer-valued functions which satisfy (A_N) but it can be also extended to cover those functions f for which WUD (mod N) and Dirichlet-WUD (mod N) (see [7] and [2]) coincide, provided the analogue of the condition (A_N) holds.

If f is WUD (mod N) but $R_M(N, f)$ does not generate $G(N)$ (which may happen, as the example $f = \sigma_2$, $N = 40$ (see [6]) shows) then we shall say that f is irregularly WUD (mod N). A function f is called *regular* if for no N it can be irregularly WUD (mod N). From [2], [10] and [6] it follows that the functions $\varphi(n)$, $d(n)$ and $\sigma(n)$ are regular, whereas $\sigma_2(n)$ is not. One sees also easily that the class of regular functions contains all completely multiplicative polynomial-like integer-valued functions. It would be interesting to have an intrinsic characterization of this class.

For a given function f denote by $M_0(f)$ the set of all those integers N for which f is regularly WUD (mod N). Denote also by $T(f)$ the value $\sup\{M(N, f)\colon N \geqslant 2\}$ (which may be infinite). If $T(f)$ is finite then we shall say that f satisfies the condition (C).

Using the Theorem II of [5] we are now able to give a description of the shape the set $M_0(f)$ may have:

PROPOSITION 2. *Let f be a polynomial-like integer-valued multiplicative function satisfying the condition* (A) *and assume that for $j = 1, 2, \ldots, T(f)$ the polynomial $V_j(x)$ is not of the form $cW^k(x)$ with a constant c, a polynomial $W(x)$ and $k \geqslant 2$. (This is clearly satisfied if f satisfies the condition* (B).) *Then there exist integers $D_1, D_2, \ldots, D_T$ (with $T = T(f)$) and finite sets $X_1, X_2, \ldots, X_T$ of integers such that if we denote by $S(X)$ the set of all positive integers which are not divisible by any element of X then*

$$M_0(f) = \bigcup_{k=1}^{T} \{N\colon (N, D_j) \neq 1 \text{ for } j = 1, 2, \ldots, k-1;\ (N, D_k) = 1,\ N \in S(X_k)\}.$$

Moreover for each fixed k the integer D_k and the set X_k can be effectively determined.

Proof. Since for coprime a, b the set $R_m(ab, f)$ equals the product of $R_m(a, f)$ and $R_m(b, f)$ due to the Chinese Remainder Theorem and for prime p the set $R_m(p^j, f)$ $(j \geqslant 1)$ is non-empty if and only if $R_m(p, f)$ is non-empty we obtain that $R_m(N, f)$ is non-empty if and only if N has no prime factor p such that $R_m(p, f)$ is empty. Now $R_m(p, f)$ will be empty if and only if the polynomial $xV_m(x)$ has all its values divisible by p and this can happen only for finitely many primes p which all may be effectively found. Denoting by D_m their product we see that $M(N,f) = k$ holds if and only if for $j = 1, 2, \ldots, k-1$ one has $(D_j, N) \neq 1$ and $(N, D_k) = 1$. From this and Theorem II of [5] our assertion follows. ∎

COROLLARY. *If f is a polynomial-like, integer-valued, multiplicative and regular function satisfying the conditions* (A), (B) *and* (C) *then the set $M(f)$ can be found effectively and has the form described in the proposition.*

Proof. It suffices to observe that due to the regularity of f we have the equality $M(f) = M_0(f)$. ∎

3. Before we turn to the study of $\sigma_k(n)$ we prove a result which is useful in establishing the regularity for many multiplicative functions:

PROPOSITION 3. *Let $a_1, a_2, \ldots$ be a sequence of integers, let M, N be two positive integers and let χ be a character* (mod N) *of prime order q with the property that the sequence $\chi(a_j)$ $(j = 1, 2, \ldots)$ is periodic. Assume moreover that for $k = 1, 2, \ldots, q-1$ there exists a prime $p = p(k)$ such that the sum*

$$1+\sum_{j=1}^{\infty}\chi^k(a_j)p^{-j/M}$$

vanishes. Then $p(k) = 2$ for $k = 1, 2, \ldots, q-1$, the character χ is real (i.e. $q = 2$) and

$$\chi(a_j) = \begin{cases} -1 & \text{if } M|j, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Under the assumption $q = 2$ this was proved in [6] (Lemma 4) hence we may assume that q is an odd prime and try to reach a contradiction. The same lemma shows that p does not depend on k and equals q. Let X be one of the characters χ^k $(1 \leqslant k \leqslant q-1)$ and write $y = q^{1/M}$. Denoting the period of $\chi(a_j)$ by T (which may be always assumed to exceed M) and adding the occurring geometrical series we arrive at

$$y^T+\sum_{r=1}^{T-1}X(a_{T-r})y^r+X(a_T)-1 = 0. \tag{2}$$

If $X(a_T)$ would be equal to 0 or 1 then y would be an algebraic unit which it is not and it follows that we may write

$$X(a_T) = \zeta_q^s$$

with $1 \leqslant s \leqslant q-1$ and a primitive qth root of unity ζ_q. Since with a suitable unit ε we have

$$q = \prod_{j=1}^{q-1}(1-\zeta_q^j) = \varepsilon^{-1}(1-\zeta_q^s)^{q-1}$$

we get

$$\varepsilon_q = \big(1-X(a_T)\big)^{q-1} = \Big(y^T+\sum_{r=1}^{T-1}X(a_{T-r})y^r\Big)^{q-1}.$$

If now j denotes the smallest index for which $X(a_{T-j})$ does not vanish then it does not depend of X but only of χ and we may write

$$\varepsilon y^M = \varepsilon q = \Big(y^T+\sum_{r=j+1}^{T-1}X(a_{T-r})y^r+X(a_{T-j})y^j\Big)^{q-1}$$
$$= y^{j(q-1)}\Big(y^{T-j}+\sum_{r=j+1}^{T-1}X(a_{T-j})y^{r-j}+X(a_{T-j})\Big)^{q-1}.$$

Since the bracketed term cannot be divisible by a prime ideal of the ring Z_K dividing y (with $K = Q(\zeta_q, y)$) we obtain

$$M = j(q-1). \tag{3}$$

Applying now (2) to $X = \chi, \chi^2, \ldots, \chi^{q-1}$ and adding the obtained equalities we arrive at

$$0 = (q-1)y^T+(q-1)\sum_{\substack{1\leqslant r\leqslant T-1\\ \chi(a_{T-r})=1}}y^r-\sum_{\substack{1\leqslant r\leqslant T-1\\ \chi(a_{T-r})\neq 0,1}}y^r-q.$$

Reducing (mod qZ_K) and using $y^M = q$ we obtain finally

$$0 \equiv y^T+\sum_{\substack{1\leqslant r\leqslant T-1\\ (a_{T-r},N)=1}}y^r \equiv y^T+y^j\Big(1+\sum_{\substack{j<r\leqslant T-1\\ (a_{T-r},N)=1}}y^{r-j}\Big) \pmod{y^M}.$$

Since the right-hand side is divisible by y^j but not by y^{1+j} we obtain now $j \geqslant M$ which in view of $q \geqslant 3$ contradicts (3). ∎

COROLLARY. *Let f be a polynomial-like multiplicative integer-valued function and $N \geqslant 2$ an integer such that the condition (A_N) is satisfied and for every prime $p \leqslant 2^M$ (with $M = M(N,f)$) the sequence $f(p^j)$ mod N $(j = 1, 2, \ldots)$ is periodic.*

If f is irregularly WUD (mod N) *then N is even, the set $R_M(N)$ generates a subgroup of index 2 in $G(N)$ and the only non-principal character* (mod N) *which equals unity on $R_M(N)$ satisfies for $j = 1, 2, \ldots$*

$$\chi\big(f(2^j)\big) = \begin{cases} -1 & \text{if } M|j, \\ 0 & \text{if } M\nmid j.\end{cases}$$

Proof. Since f is irregularly WUD (mod N) the Proposition 1 implies that for every non-principal character χ (mod N) which trivializes on $R_M(N)$ there is a prime $p \leqslant 2^M$ for which the equality (1) holds. If d is the order of χ and q is any prime divisor of d then $\psi = \chi^{d/q}$ is a character of order q and applying to it the Proposition 4 we get $p = q = 2$ and

$$\chi(f(2^j)) = \begin{cases} -1 & \text{if } M \mid j, \\ 0 & \text{if } M \nmid j. \end{cases}$$

If N were odd, then $1 = \chi(f(2^M)) = -1$ a contradiction, thus N must be even. Since $\chi(f(2^j))$ vanishes for all j not divisible by M hence

$$1 + \sum_{n=1}^{\infty} \chi(f(2^{nM}))2^{-n} = 0$$

and we see that for all $j \equiv 0 \pmod M$ the equality $\chi(f(2^j)) = -1$ holds. But for χ we can take any non-principal character equal to unity on $R_M(N)$ and the last equality shows that there can be only one of them. This implies $d = 2$ and all assertions become evident. ∎

4. Now we state and prove our results concerning σ_k.

THEOREM I. *For $k \geqslant 3$ the function σ_k is regular and satisfies the conditions* (A), (B) *and* (C).

COROLLARY. *For every fixed $k \geqslant 3$ the set $M(\sigma_k)$ can be effectively determined and has the form given in Proposition* 2.

THEOREM II. *$M(\sigma_3)$ consists of all odd integers not divisible by* 7 *and all even integers not divisible by* 3.

Theorem II has a computer-free proof. Using a computer it is easy to find $M(\sigma_k)$ for larger values of k however this seems to be not very interesting, since the structure of $M(\sigma_k)$ does not seem to follow any regular pattern, except that given in Proposition 2. Note also that from Theorem I one can deduce that if k is an odd prime then σ_k is WUD $(\mathrm{mod}\, N)$ for all odd integers N with exception of multiples of $2k+1$ provided this number is a prime congruent to $7 \pmod 8$ and of multiples of $3(2k+1)$ in case when $2k+1$ is a prime $\equiv 3 \pmod 8$. It was conjectured by F. Rayner on the basis of a computer experiment (letter of 8th October, 1981) that if k is an odd prime and $2k+1$ is composed, then σ_k is WUD $(\mathrm{mod}\, N)$ if and only if $6 \nmid N$, however our methods seems to be insufficient to deal with this question. (Added in proof: A further computer search made by F. Rayner revealed that this fails for $k = 43$ in which case there is no WUD (mod 2066).)

Proof of Theorem I. First we shall compute the value of $T(\sigma_k)$:

LEMMA 1. *For odd k one has $M(N, \sigma_k) = 1$ if N is odd and $M(N, \sigma_k) = 2$ if N is even. Thus $T(\sigma_k) = 2$ for k odd. If however k is even, then $T(\sigma_k)$ equals $Q-1$, where $Q = Q(k)$ is the minimal prime with the property that k is not divisible by $Q-1$.*

Proof. If k is odd there is no problem: for odd N the set $R_1(N, \sigma_k)$ contains $2 = 1+1^k$ and for N even the set $R_1(N, \sigma_k)$ is empty, but $R_2(N, \sigma_k)$ contains $-1 = 1+(-1)^k+(-1)^{2k}$. Now let k be even. Since $Q-1$ does not divide k, the primitive root $g \pmod Q$ satisfies $g^k \not\equiv 1 \pmod Q$ and if the set $R_{Q-1}(N, \sigma_k)$ would be empty, then for a suitable prime divisor p of N we would have $p \mid Q = 1+1^k+1^{2k}+ \ldots +1^{(Q-1)k}$ (hence $p = Q$) and for all $x \not\equiv 0 \pmod p$ also

$$\frac{x^{Qk}-1}{x^k-1} \equiv 0 \pmod p,$$

thus in particular $g^{Qk} \equiv 1 \pmod Q$. But this leads to $Q-1 \mid kQ$ and since $(Q, Q-1) = 1$ the divisibility of k by $Q-1$ results, a contradiction. This proves that $R_{Q-1}(N, \sigma_k)$ is non-empty for all choices of N and implies the inequality

$$T(\sigma_k) \leqslant Q-1.$$

To prove the converse inequality observe first that for N even all sets $R_j(N, \sigma_k)$ with odd j are empty and thus it suffices to show that for every even j smaller than $Q-1$ one can find a prime $p_j \neq 2$ such that $R_j(p_j, \sigma_k)$ is empty since then for the number $N = 2p_2p_4 \ldots p_{Q-3}$ we would have $R_j(N, \sigma_k) = \emptyset$ for $j = 1, 2, \ldots, Q-2$. We can for p_j take any prime divisor of $1+j$, since then, by the definition of Q we have $p_j - 1 \mid k$ and $p_j \mid 1+j$ and this easily implies that the set $R_j(p_j, \sigma_k)$ is void. ∎

From this lemma it follows immediately that σ_k satisfies the conditions (A) and (C). Since the truth of the condition (B) is for σ_k evident it suffices to establish the regularity. For this purpose we shall utilize Proposition 3 but first we have to convince ourselves that it is applicable in this situation.

LEMMA 2. *Let k be a positive integer and let $N = \prod_{p \mid N} p^{a_p}$ be an integer satisfying the condition $a_p \leqslant 2k$ for all primes p dividing N. Then for every prime q the sequence $\sigma_k(q^j) \bmod N$ $(j = 1, 2, \ldots)$ is periodic.*

Proof. It suffices to consider the case $N = p^a$ with prime p and $a \leqslant 2k$. Put

$$a_j = \sigma_k(q^j) = (q^{k(1+j)}-1)/(q^k-1)$$

and note that for $T \geqslant 1$

$$a_{j+T} - a_j = q^{k(j+1)}(q^{kT}-1)/(q^k-1).$$

If $p \neq q$ and $q^k \not\equiv 1 \pmod p$ then taking for T the order of $q^k \pmod{p^a}$ we obtain the periodicity of $a_j \pmod{p^a}$ with period T. If $p \neq q$ and p divides q^k-1, then define b by $p^b \| q^k-1$ and let T be the order of $q^k \pmod{p^{a+b}}$.

Then again $a_j \pmod{p^a}$ is periodic with period T. Finally in the case $p = q$ observe that $a_{j+1} - a_j$ is divisible by $p^{k(j+1)}$ and since $j \geqslant 1$ and by assumption $a \leqslant 2k$ it follows that

$$p^a \mid p^{2k} \mid p^{k(j+1)} \mid a_{j+1} - a_j$$

and thus the sequence $a_j \pmod{p^a}$ has period 1. ∎

Assume now that N is an integer such that σ_k is irregularly WUD (mod N); write

$$N = \prod_{p|N} p^{a_p}$$

and define

$$b_p = \begin{cases} \min(a_p, 2) & \text{if } p \text{ is odd,} \\ \min(a_p, 3) & \text{if } p = 2, \end{cases}$$

and $N_0 = \prod_{p|N} p^{b_p}$. Then obviously σ_k is WUD (mod N_0) and moreover it must be irregularly WUD (mod N_0). In fact, if X is a non-principal character (mod N) which equals unity on $R_M(N, \sigma_k)$ then a suitable power of X has its conductor not divisible by 16 nor by a cube of an odd prime and thus is induced by a character (mod N_0) which is non-principal and trivializes on $R_M(N_0, \sigma_k)$. (This argument is valid obviously for arbitrary polynomial-like functions.)

Denoting this character by χ we obtain from Lemma 2 and the corollary to Proposition 3 that χ is real and so is induced by a character (mod N_1), where $N_1 = 2^{b_2} \prod_{p|N} p$. This character we shall also denote by χ. Since the previous argument shows that σ_k is irregularly WUD (mod N_1) we obtain from the corollary to Proposition 3 that for $j = 1, 2, \ldots$ the equality

$$\chi\big(\sigma_k(2^j)\big) = \begin{cases} -1 & \text{if } M \mid j, \\ 0 & \text{if } M \nmid j \end{cases} \tag{4}$$

holds with $M = M(N, \sigma_k)$. Observe that $M \neq 1$ since by the same corollary N is even and obviously in this case $M(N, \sigma_k) \neq 1$.

Now we prove that (4) leads to a contradiction for $k \geqslant 3$. Write

$$a_j = \sigma_k(2^j) = (2^{k(j+1)} - 1)/(2^k - 1) \quad (j = 1, 2, \ldots)$$

and

$$N_1 = 2^{b_2} AB$$

where A is composed of all prime divisors of N_1 which divide $2^k - 1$ and B is the maximal odd divisor of N_1 prime to $2^k - 1$. The character χ can be written as

$$\chi = \chi_2 \cdot \chi_A \cdot \chi_B$$

where χ_2 is a character (mod 2^{b_2}), χ_A a character (mod A) and χ_B a character (mod B), all of them real. Since $k \geqslant 3$ we have $a_j \equiv 1 \pmod 8$ and in view of $b_2 \leqslant 3$ we obtain

$$\chi_2(a_j) = 1. \tag{5}$$

(This is the only place when the assumption $k \geqslant 3$ is used.) Denote by r the order of 2 (mod B) and let $j = nrM$ be any multiple of rM. Then on one hand we have

$$\chi(a_j) = -1$$

but on the other hand in view of

$$2^{k(j+1)} - 1 \equiv 2^k - 1 \pmod B$$

we obtain $a_j \equiv 1 \pmod B$, thus $\chi_B(a_j) = 1$ and hence $\chi_A(a_j) = -1$ must hold for all $j \equiv 0 \pmod{Mr}$. However A divides $2^k - 1$, thus $\chi_A(a_j) = \chi_A(1+j)$ and we obtain

$$1 = \chi_A(1+rM)\chi_A(1+rM) = \chi_A\big(1 + rM(2+rM)\big) = -1,$$

a contradiction. Thus σ_k is for $k \geqslant 3$ regular and the theorem follows. ∎

The corollary follows directly from the theorem and the corollary to Proposition 2.

5. Proof of Theorem II. Note first that if f is a polynomial-like multiplicative function which is not WUD (mod N) then it cannot be WUD (mod N_0) where

$$N_0 = \Big(N, 8 \prod_{p|N} p^2\Big).$$

This follows from the fact that if χ is a non-trivial character (mod N) trivial on a subgroup H of $G(N)$ then a certain power of it is trivial on H (mod N_0) without being equal to the principal character (mod N_0). We may thus assume in sequel that $N = N_0$.

Observe also that if f is not WUD (mod N) then for every prime p with $p^{a_p} \| N$ there is a character χ_p (mod p^{a_p}) which is constant on $R_M(p^{a_p}, f)$ (where $M = M(N, f)$) and for at least one prime p the character χ_p is non-principal. Lemma 4 of [5] provides an upper bound for such primes which in the case of σ_k leads to

$$p \leqslant Mk(Mk+1) \tag{6}$$

provided $k \geqslant 3$.

In the case $k = 3$ Lemma 1 gives $M = 1$ for N odd and $M = 2$ for N even. For N odd (6) implies $p \in \{3, 5, 7, 11\}$ however since 5 and 11 are congruent to 2 (mod 3) the sets $R_1(p)$ and $R_1(p^2)$ are too large in these cases to admit the existence of a non-principal character constant on them. Moreover $R_1(3) = \{2\}$, $R_1(3^2) = \{2\}$ and $R_1(7) = \{2\}$ and in view of $2^3 \equiv 1 \pmod 7$ we see that for odd N σ_3 is WUD (mod N) except when N is divisible by 7. For N even (6) gives $2 \leqslant p \leqslant 41$ and after discarding all primes $p \geqslant 5$, $p \equiv 2 \pmod 3$ by the same reason as above we are left with the set $\{2, 3, 7, 13, 19, 31, 37\}$. Here $R_1(3) = \{1\}$ and a dull check shows that in no other case χ_p can be non-principal. This establishes our assertion about σ_3. ∎

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