On a result of Littlewood concerning prime numbers II

by

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1. Introduction. Let

\[ \psi(x) = \sum_{n \leq x} A(n) , \]

where \( A(n) \) is equal to \( \log p \) if \( n \) is a prime or prime power, and zero otherwise. Littlewood proved in [4], assuming the Riemann hypothesis,

\[ \psi(x) - x = - \sum_{\beta < y} \frac{x^\beta}{\beta} + O(x^{3/2} \log x) \]

uniformly for \( x \geq 3, y \geq x^{3/2} \). Here \( \beta = \beta + iy \) is a non-trivial zero of the Riemann zeta-function. Without the Riemann hypothesis, an earlier result of Landau [3] implies (2) holds uniformly for \( x \geq 3, y \geq x^{3/2} \log x \).

In [2] I showed how Littlewood's method can be improved to prove (2) holds uniformly for \( x \geq 3, y \geq x^{3/2} \log x \), on the Riemann hypothesis. The purpose of this note is to improve slightly on Landau's result:

**Theorem.** Let \( x \geq 3, y \geq 3 \). Then there is an absolute constant \( c \) such that

\[ \left| \psi(x) - x + \sum_{\beta < y} \frac{x^\beta}{\beta} \right| < c \left( \frac{x \log x \log \log x}{y} + \frac{x \log y}{y} + \log x \right) . \]

Landau's result has the first term on the right of (3) replaced by \( (x \log x)/y \). Taking \( y \geq x^{3/2} \log x \log x \) we obtain (2), which nearly proves Littlewood's result unconditionally. Equation (3) usually is applied in the range \( 3 \leq y \leq x \), where we obtain

\[ \psi(x) - x = - \sum_{\beta < y} \frac{x^\beta}{\beta} + O \left( \frac{x \log x \log \log x}{y} \right) . \]

I would like to thank J. E. Landau for a helpful discussion.

2. Proof of the theorem. The theorem follows from the following two lemmas.

\[ 1 \quad \text{Acta Arithmetica} \, 46, \, 1 \]
Lemma 1. Let \( x \geq 3, \ y \geq 3 \). Then for \( \eta = 1 + 1/\log x \),
\[
\psi(x) - x = - \sum_{\eta < \nu \leq x} \nu^\eta + O\left( \frac{x \log^2 x}{\nu} \right) + O\left( \frac{x \log y}{y} \right) +
\]
\[
+ O(\log x) + O\left( \frac{x^\eta}{\nu} \sum_{\nu < \eta < \nu + 1} A(\nu) \right).
\]

This is proved in [1] or [3]. For \( \eta \leq n \leq 2 \eta \) we have \( n^\eta \approx \psi \) and \( x^\eta \approx x \) (\( f \ll g \) means \( f \ll g \)). The theorem follows immediately from Lemma 1 and the following result:

Lemma 2. Let \( x \) and \( \psi(x) \) be given by \( \sum_{n \leq x} A(n)/\log(x/n) \). Then we have
\[
\psi(x) \ll \log x \log\log x.
\]
The same result holds for \( \psi(x) = \sum_{n < x} A(n)/\log(n/x) \).

Proof. We start with the elementary inequality
\[
x^\eta \ll \log(1 + x) \ll x \quad \text{for} \quad 0 < x \ll 1.
\]
Since \( \log(x/n) = \log(1 + (x/n - 1)) \), we have
\[
\sum_{\nu \leq x} \frac{n A(n)}{x - n} \ll \psi(x) \ll \sum_{\nu \leq x} \frac{n A(n)}{x - n},
\]
and hence
\[
The last sum we use a sieve estimate which was not available when Landau proved his result. Let \( \pi(x) \) denote the number of primes less than \( x \).

\[
\pi(x) - \pi(x - y) \ll y / \log y, \quad 1 < y \ll x;
\]
this is proved with \( c = 2 \) in [5]. Denote by \( P(x, y) \) the number of primes and prime powers in the interval \( [x, y) \). Since the number of prime powers in \( [x - y, x) \) is \( \ll y^\nu \), we have
\[
P(x, y) \ll y / \log(y + 1), \quad 1 < y \ll x.
\]
Now consider \( \sum_{\nu \leq x} 1/(x - n) \). We can replace \( x \) by the nearest even integer \( 2k \) with an error \( \ll 1 \). Now
\[
\sum_{\nu \leq x} \frac{1}{2k - n} = \sum_{n \leq x} \frac{1}{\nu} \{P(2k, \nu) - P(2k, \nu - 1)}
\]

\[\sum_{n \leq x} \frac{1}{2k - n} = \sum_{n \leq x} \frac{1}{\nu} \{P(2k, \nu) - P(2k, \nu - 1)}
\]

The lemma now follows. The same argument proves the result for \( \psi(x) \).

References


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