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# On a result of Littlewood concerning prime numbers II

by

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**1. Introduction.** Let

$$\psi(x) = \sum_{n\leqslant x} \Lambda(n), \tag{1}$$

where $\Lambda(n)$ is equal to $\log p$ if $n$ is a prime or prime power, and zero otherwise. Littlewood proved in [4], assuming the Riemann hypothesis,

$$\psi(x) - x = -\sum_{|\gamma|<y} \frac{x^{\varrho}}{\varrho} + O(x^{1/2}\log x) \tag{2}$$

uniformly for $x \geqslant 5$, $y \geqslant x^{1/2}$. Here $\varrho = \beta + i\gamma$ is a non-trivial zero of the Riemann zeta-function. Without the Riemann hypothesis, an earlier result of Landau [3] implies (2) holds uniformly for $x \geqslant 5$, $y \geqslant x^{1/2}\log x$. In [2] I showed how Littlewood's method can be improved to prove (2) holds uniformly for $x \geqslant 5$, $y \geqslant x^{1/2}/\log x$, on the Riemann hypothesis. The purpose of this note is to improve slightly on Landau's result:

THEOREM. *Let $x \geqslant 3$, $y \geqslant 3$. Then there is an absolute constant $c$ such that*

$$\left|\psi(x) - x + \sum_{|\gamma|<y} \frac{x^{\varrho}}{\varrho}\right| < c\left\{\frac{x\log x\log\log x}{y} + \frac{x\log y}{y} + \log x\right\}. \tag{3}$$

Landau's result has the first term on the right of (3) replaced by $(x\log^2 x)/y$. Taking $y \geqslant x^{1/2}\log\log x$ we obtain (2), which nearly proves Littlewood's result unconditionally. Equation (3) usually is applied in the range $3 \leqslant y \leqslant x$, where we obtain

$$\psi(x) - x = -\sum_{|\gamma|<y} \frac{x^{\varrho}}{\varrho} + O\left(\frac{x\log x\log\log x}{y}\right). \tag{4}$$

I would like to thank L. Ein for a helpful discussion.

**2. Proof of the theorem.** The theorem follows from the following two lemmas.

LEMMA 1. *Let* $x \geqslant 3$, $y \geqslant 3$. *Then for* $\eta = 1+1/\log x$,

$$\psi(x) - x = -\sum_{|\gamma|<y} \frac{x^\varrho}{\varrho} + O\left(\frac{x\log x}{y}\right) + O\left(\frac{x\log y}{y}\right) + $$
$$+ O(\log x) + O\left(\frac{x^\eta}{y}\left\{\sum_{\substack{n \\ 1\leqslant|x-n|\leqslant x/2}} \frac{\Lambda(n)}{n^\eta|\log(x/n)|}\right\}\right).$$

This is proved in [1] or [3]. For $x/2 \leqslant n \leqslant 2x$ we have $n^\eta \cong x$ and $x^\eta \cong x$ ($f \cong g$ means $f \ll g$ and $f \gg g$). The theorem follows immediately from Lemma 1 and the following result:

LEMMA 2. *Let* $x \geqslant 3$ *and* $S(x) = \sum_{x/2\leqslant n\leqslant x-1} \Lambda(n)/\log(x/n)$. *Then we have*

$$S(x) \ll x\log x\log\log x. \tag{5}$$

*The same result holds for* $\bar{S}(x) = \sum_{x+1\leqslant n\leqslant 2x} \Lambda(n)/\log(n/x)$.

Proof. We start with the elementary inequality

$$x/2 \leqslant \log(1+x) \leqslant x \quad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{6}$$

Since $\log(x/n) = \log\{1+(x-n)/n\}$, we have

$$\sum_{x/2\leqslant n\leqslant x-1} \frac{n\Lambda(n)}{x-n} \leqslant S(x) \leqslant 2\sum_{x/2\leqslant n\leqslant x-1} \frac{n\Lambda(n)}{x-n}, \tag{7}$$

and hence

$$S(x) < 2x\log x \sum_{x/2\leqslant p^m\leqslant x-1} 1/(x-p^m). \tag{8}$$

To estimate this last sum we use a sieve estimate which was not available when Landau proved his result. Let $\pi(x)$ denote the number of primes less than or equal to $x$. Then

$$\pi(x) - \pi(x-y) < cy/\log y, \quad 1 < y \leqslant x; \tag{9}$$

this is proved with $c = 2$ in [5]. Denote by $P(x, y)$ the number of primes and prime powers in the interval $[x-y, x]$. Since the number of prime powers in $[x-y, x]$ is $\ll y^{1/2}$, we have

$$P(x, y) \ll y/\log(y+1), \quad 1 \leqslant y \leqslant x. \tag{10}$$

Now consider $\sum_{x/2\leqslant p^m\leqslant x-1} 1/(x-p^m)$. We can replace $x$ by the nearest even integer $2k$ with an error $\ll 1$. Now

$$\sum_{k\leqslant p^m\leqslant 2k-1} \frac{1}{2k-p^m} = \sum_{n=1}^{k} \frac{1}{n}\{P(2k, n) - P(2k, n-1)\}$$



$$= \sum_{n=1}^{k-1} P(2k, n)\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{k}P(2k, k) - P(2k, 0)$$
$$\ll \sum_{n=1}^{k-1} \frac{1}{n\log(n+1)} + \frac{1}{\log k} \ll \log\log k \ll \log\log x.$$

The lemma now follows. The same argument proves the result for $\bar{S}(x)$.

**References**

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