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Correction to the paper "Structure theorems for radical extensions of fields", Acta Arith. 38(1980), pp. 111–115

by

WILLIAM YSLAS VÉLEZ (Tucson, Ariz.)

We shall assume the notation and conventions of [1].

The method of proof used in Theorem 2 of [1] is not correct. There we asserted that if $M_1 \cap M_2 = M$, $L = M_1M_2$, then $N \to M_2N$ defines an injection from the lattice of intermediate fields of M_1 over M to the lattice of intermediate fields of L over M_2. To see that this is false let $2^{1/35}$ denote the real root of $x^{35}-2$ and ζ_{35} a primitive 35th root of unity. Let $M_1 = Q(2^{1/35}, \zeta_5, \zeta_7+\zeta_7^{-1})$, $M_2 = Q(\zeta_{35}2^{1/35})$. It is easy to see that $M_1 \cap M_2 = M = Q(\zeta_5 2^{1/5})$ and $[M_1:Q] = 35\cdot 4\cdot 3$, $[M_2:Q] = 35$, $L = Q(\zeta_{35}, 2^{1/35})$. Let $N_1 = M(2^{1/7})$, $N_2 = M(2^{1/7}, \zeta_7+\zeta_7^{-1})$. Thus $[N_1:Q] = 35$, $[N_2:Q] = 35\cdot 3$, so $N_1 \neq N_2$ and $N_1M_2 \subset N_2M_2$. Now $N_1M_2 = Q(\zeta_{35}2^{1/35}, \zeta_5 2^{1/5}, 2^{1/7})$, so $N_1M_2 = M_2(\zeta_7)$ and $\zeta_7+\zeta_7^{-1} \in N_1M_2$, thus $N_1M_2 = N_2M_2$, so $N \to M_2N$ does not preserve the aforementioned injection.

In the following we provide a correct proof of Theorem 2.

THEOREM 2. *Let* $F(a) \supset K \supset F$, $K \cap F(\zeta_n) = F(\theta)$, $t = \min\{i: i\,|\,m$ *and* $a^i \in K\}$, $r = \max\{i: i\,|\,m$ *and* $F(a^i) \supset K\}$. *Then* $K = F(\theta, a^t)$ *iff* $(s, t) = (s, r)$.

Proof. The following result will be used often in the proof: Let L_1, L_2 be fields such that L_1 is finite and normal over $L_1 \cap L_2$, then $[L_1L_2:L_2] = [L_1: L_1 \cap L_2]$, and thus $[L_2: L_1 \cap L_2] = [L_1L_2: L_1]$. See [2], page 196 for a proof.

Since $F(\zeta_n)$ is normal over $F(a^t) \cap F(\zeta_n)$, we have that

$$[F(a^t): F(a^t) \cap F(\zeta_n)] = [F(a^t, \zeta_n): F(\zeta_n)] = [F(a^t, \theta): F(\theta)]$$

since $F(\theta)$ is a subfield of $F(\zeta_n)$ containing $F(a^t) \cap F(\zeta_n)$. Hence, $K = F(\theta, a^t)$ if $[K: F(\theta)] = [F(a^t): F(a^t) \cap F(\zeta_n)]$. Thus, the theorem will be proven provided we can verify this last equality.

It is easy to see that $F(a^k)\cdot F(a^l) = F(a^{(k,l)})$. To study the intersection of such fields we need the following claim:

Claim A: *If $l|s$ and $F(a^l)$ is normal over $F(a^k)\cap F(a^l)$, then*

$$F(a^k)\cap F(a^l) = F(a^{[k,l]}) \quad and \quad [F(a^k): F(a^{[k,l]})] = [k, l]/k.$$

Proof of Claim A: Since $l|s$, $(l, k)|s$ we have by Theorem 1 of [1] that $l = [F(a): F(a^l)]$ and $(l, k) = [F(a): F(a^{(l,k)})]$, thus $[F(a^{(l,k)}): F(a^l)] = l/(l, k)$. Clearly, $F(a^{[k,l]})$ is contained in both $F(a^l)$ and $F(a^k)$, thus $[F(a^k): F(a^k)\cap F(a^l)] \leqslant [k, l]/k$. However, since $F(a^l)$ is normal over $F(a^l)\cap F(a^k)$ by assumption, we have that

$$[F(a^{(l,k)}): F(a^l)] = l/(l, k) = [F(a^k): F(a^k)\cap F(a^l)] \leqslant [k, l]/k.$$

But $l/(l, k) = [k, l]/k$, so $[F(a^k): F(a^k)\cap F(a^l)] = [l, k]/k$, thus $F(a^k)\cap \cap F(a^l) = F(a^{[l,k]})$ and the claim is proven.

Since $F(\zeta_n) = F(a^s)$ and $K\cdot F(\zeta_n) \supset F(\zeta_n)$, we have that $K\cdot F(\zeta_n) = F(a^w)$, where $w = [F(a): K\cdot F(\zeta_n)]$ and $w|s$, by Theorem 1 of [1]. By definition, $F(a^r) \supset K$, so $F(a^w) \supset F(a^w)\cap F(a^r) \supset K$ and since $F(a^w)/K$ is normal, we have that $F(a^w)/F(a^w)\cap F(a^r)$ is normal, thus by Claim A, $F(a^w)\cap F(a^r) = F(a^{[w,r]}) \supset K$. However, r was maximal with this property, hence $r = [r, w]$, so $w|r$ and we have that $F(a^w) \supset F(a^r) \supset K$, thus $F(a^w) = F(a^r)\cdot F(\zeta_n) = F(a^{(r,s)})$, since $F(\zeta_n) = F(a^s)$, and we have that $w = (r, s)$.

With $l = s$ and $k = t$, we have by Claim A that $F(a^t)\cap F(\zeta_n) = F(a^{[t,s]})$ and $[F(a^t): F(a^{[t,s]})] = [t, s]/t = s/(t, s)$. We also have that

$$[K: K\cap F(\zeta_n)] = [K\cdot F(\zeta_n): F(\zeta_n)] = s/w = s/(r, s).$$

Thus

$$[K: K\cap F(\zeta_n)] = [F(a^t): F(a^t)\cap F(\zeta_n)] \quad \text{iff} \quad (t, s) = (r, s),$$

and the theorem is proven.

Remark: Note that in the proof only the fact that $F(\zeta_n)$ is of the form $F(a^s)$ and that $F(\zeta_n)/F$ is normal is used and $F(\zeta_n)/F$ abelian was not necessary. Thus, if we replace $F(\zeta_n)$ by a field $F(a^q)$, $q|s$, such that $F(a^q)/F$ is normal and replace s by q, then the result is still valid.

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[1] M. J. Norris and W. Yslas Vélez, *Structure theorems for radical extensions of fields*, Acta Arith. 38(1980), pp. 111–115.
[2] S. Lang, *Algebra*, Addison–Wesley Publishing Co., Reading, Mass., 1969.

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