ACTA ARITHMETICA
XLII (1983)

Remarks on the arithmetic properties of the values of hypergeometric functions

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1. Introduction. The purpose of the present paper is to apply the classical method of Siegel [10] to a consideration of the arithmetic properties of the values of certain hypergeometric functions

$$F(\alpha,\beta,\gamma;z)=\sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)}z^n \qquad (1)$$

$$(\gamma\neq 0,\,-1,\ldots)$$

satisfying

$$z(1-z)y''+\big(\gamma-(1+\alpha+\beta)z\big)y'-\alpha\beta y=0. \qquad (2)$$

We shall prove the following theorems.

THEOREM 1. *If α, β, γ are rational numbers, $\gamma\neq 0,-1,-2,\ldots$, then the functions $F(\alpha,\beta,\gamma;z)$ and $F'(\alpha,\beta,\gamma;z)$ belong to Galochkin's* [8] *class of G-functions* (for definition see § 3).

THEOREM 2. *Let $\alpha_i,\beta_i,\gamma_i$ $(i=1,\ldots,m)$ be rational numbers satisfying $\gamma_i\neq 0,-1,\ldots;$ $\alpha_i,\beta_i,\gamma_i-\alpha_i,\gamma_i-\beta_i\notin\mathbf{Z}$ $(i=1,\ldots,m)$; $\alpha_i-\alpha_j$, $\beta_i-\beta_j$, $\alpha_i-\beta_j\notin\mathbf{Z}$ $(i\neq j,\ i,j=1,\ldots,m)$; none of the numbers $\alpha_i+\beta_i-(\alpha_j+\beta_j)$ $(i\neq j,\ i,j=1,\ldots,m)$ is an even integer. Let x_0,x_i,y_i $(i=1,\ldots,m)$ be integers, not all zero, and let us denote $h_i=\max\{1,|x_i|,|y_i|\}$ $(i=1,\ldots,m)$, $H=\prod_{i=1}^{m}h_i^2$. Let ε, $0<\varepsilon<1$, be given. There then exist positive constants λ, C, depending only on ε, m, p and the functions $F(\alpha_i,\beta_i,\gamma_i;z)$ $(i=1,\ldots$ $\ldots,m)$, such that*

$$\Big|x_0+\sum_{i=1}^{m}\big(x_iF(\alpha_i,\beta_i,\gamma_i;p/q)+y_iF'(\alpha_i,\beta_i,\gamma_i;p/q)\big)\Big|>q^{-\lambda}H^{-1-\varepsilon}$$

for any rational number $p/q\neq 0$ satisfying $q>C$. In particular, the numbers 1, $F(\alpha_i,\beta_i,\gamma_i;p/q)$ and $F'(\alpha_i,\beta_i,\gamma_i;p/q)$ $(i=1,\ldots,m)$ are linearly independent over $\mathbf{Q}$ for all $q>C$.

THEOREM 3. *Let α, β and γ $(\neq 0, -1, \ldots)$ be rational numbers such that $F(\alpha, \beta, \gamma; z)$ is not an algebraic function and α, β, $\gamma-\alpha$, $\gamma-\beta \notin \boldsymbol{Z}$. Let $P(x_1, x_2) \not\equiv 0$ be a polynomial in $\boldsymbol{Z}[x_1, x_2]$ of degree $\leqslant N$ and height $\leqslant H$. There then exist positive constants c, λ, depending only on α, β and γ, and a positive constant C, depending only on α, β, γ, N, θ, h and τ, such that*

$$\left|P\left(F(\alpha, \beta, \gamma; \theta), F'(\alpha, \beta, \gamma; \theta)\right)\right| > CH^{-\lambda\tau(\log h/\log\log h)^{1/2}}$$

for any algebraic number θ of degree $\leqslant \tau$ and height $h(\theta) \leqslant h \geqslant e^e$ satisfying

$$\log h \geqslant (\max\{2, N\})^4 \log\log h,$$

$$0 < |\theta| < e^{-c\tau N(\log h)^{3/4}(\log\log h)^{1/4}}.$$

This theorem implies, in particular, the linear independence (over $\boldsymbol{Q}$) of the numbers 1, $F(\alpha, \beta, \gamma; \theta)$, $F'(\alpha, \beta, \gamma; \theta)$ for all algebraic numbers θ of degree $\leqslant \tau$ and height $\leqslant h \geqslant e^e$ satisfying $\log h > 16 \log\log h$, $0 < |\theta| < e^{-c\tau(\log h)^{3/4}(\log\log h)^{1/4}}$.

THEOREM 4. *Let $\boldsymbol{K}$ denote an algebraic number field of degree $\varkappa$ over $\boldsymbol{Q}$. Assume that α, β and γ satisfy the conditions of Theorem 1, and let $L_1(x) = a_1 + b_1 x \not\equiv 0$, $L_2(x) = a_2 + b_2 x \not\equiv 0$ be linear forms with integer coefficients in $\boldsymbol{K}$ satisfying $\max\{\overline{|a_1|}, \overline{|a_2|}, \overline{|b_1|}, \overline{|b_2|}\} \leqslant H$* (for any algebraic number α the notation $\overline{|\alpha|}$ denotes the maximum of the absolute values of the conjugates of α). *There then exist positive constants $\bar{c}$, $\bar{\lambda}$, depending only on α, β and γ, and a positive constant $\bar{C}$, depending only on $\boldsymbol{K}$, α, β, γ, θ and h, such that*

$$\max\left\{\left|L_1\left(F(\alpha, \beta, \gamma; \theta)\right)\right|, \left|L_2\left(F'(\alpha, \beta, \gamma; \theta)\right)\right|\right\} > \bar{C}H^{-\bar{\lambda}\varkappa(\log h/\log\log h)^{1/2}}$$

for any $\theta \in \boldsymbol{K}$ of height $h(\theta) \leqslant h \geqslant e^e$ satisfying

$$\log h \geqslant 16 \log\log h, \quad 0 < |\theta| < e^{-\bar{c}\varkappa(\log h \log\log h)^{1/2}}.$$

In particular, at least one of the numbers $F(\alpha, \beta, \gamma; \theta)$ and $F'(\alpha, \beta, \gamma; \theta)$ does not belong to $\boldsymbol{K}$ for any $\theta \in \boldsymbol{K}$ satisfying the above conditions.

Siegel [10] already mentions that his method can be used to obtain results like the above, in fact he gives an explicit result for the function $F(1/2, 1/2, 1; z)$, i.e. that the number $F(1/2, 1/2, 1; p/q)$ is irrational for all rationals p/q satisfying

$$0 < |p/q| < c10^{-\sqrt{\log|q|}},$$

where c is a positive constant. In this special case our Theorem 4 does not give a result as strong as that of Siegel. We note, however, that in the case $\alpha = \beta = 1/2$, $\gamma = 1$ we have Theorems 3 and 4 without the term $\log\log h$ in the bounds (see [9]).

Hypergeometric series have subsequently been considered in a number of papers (see [2], [3], [4], [5], [6], [9]). Chudnovsky [4], [5] (p. 64) presents results on the arithmetic properties of the values of $F(1/2, 1/2, 1; z)$, while in [6] he uses Padé approximations to obtain important irrationality measures e.g. for the numbers $F(1/2, 1/2, 1; 1/14)$ and $F(3/2, 3/2, 1; 1/14)$. It should further be noted that Bombieri's [1] new important p-adic considerations in connection with Siegel's method can evidently be applied to obtain results closely analogous to our Theorems 3 and 4.

The proof of Theorem 1 requires certain information on the divisibility properties of the coefficients of the polynomials p_n and q_n in the expression

$$\frac{1}{n!}\left(z(1-z)\right)^n F^{(n)}(\alpha, \beta, \gamma; z) = p_n(z) F'(\alpha, \beta, \gamma; z) + q_n(z) F(\alpha, \beta, \gamma; z)$$

obtained from (2). This kind of result, the need of which was already pointed out by Siegel [10], is obtained in the main lemma in Section 2. The proof of the theorems is then completed in Section 3.

2. Main lemma. First we prove,

LEMMA 1. *Let $\delta \in \boldsymbol{Q}$ and let K be the denominator of δ. For any $n \in \boldsymbol{N}$ let $L_n = K^n \prod_{p|K} p^{[n/p-1]}$. Then $L_n \binom{\delta}{n}$ is an integer.*

Proof. Notice that

$$\binom{\delta}{n} = \frac{\delta(\delta-1)\ldots(\delta-n+1)}{n!} = K^{-n}\frac{K\delta(K\delta-K)\ldots(K\delta-(n-1)K)}{n!}.$$

Let p be a prime not dividing K. The number of factors p in $n!$ equals $[n/p]+[n/p^2]+\ldots$ Notice that $p^a | K\delta - lK$, $p^a | K\delta - mK$ if and only if $p^a | m-l$, $p^a | K\delta - lK$. Furthermore, at least one of the numbers $K\delta$, $K\delta - K$, $\ldots$, $K\delta-(p^a-1)K$ is divisible by p^a. Consequently, among the numbers $K\delta$, $\ldots$, $K\delta-(n-1)K$ there exist at least $[n/p]$ divisible by p, at least $[n/p^2]$ divisible by p^2, etc. Hence the product $K\delta \ldots (K\delta-(n-1)K)$ contains at least $[n/p]+[n/p^2]+\ldots$ factors p. Thus we see that $\binom{\delta}{n}$ has only prime divisors of K in its denominator.

Let $p | K$. Since $n!$ contains at most $[n/p]+[n/p^2]+\ldots \leqslant [n/p-1]$ factors p we see that

$$K^n \prod_{p|K} p^{[n/p-1]} \binom{\delta}{n}$$

is integral, as asserted.

In the following computations α, β and γ denote rational numbers with $\gamma \neq 0, -1, -2, \ldots$ It follows from (2) that

$$(3) \qquad z(1-z)F^{(n+2)} = [(2n+\alpha+\beta+1)z - n - \gamma]F^{(n+1)} + (n+\alpha)(n+\beta)F^{(n)}.$$

This implies

$$\frac{1}{n!}\big(z(1-z)\big)^n F^{(n)} = p_n F' + q_n F,$$

where p_n, q_n are polynomials in z with rational coefficients. The following lemma gives a bound for the denominators of these rational coefficients.

MAIN LEMMA. *Let K be the common denominator of α, β, γ and let L_n be as in Lemma 1. Suppose that $\alpha \neq 0, -1, -2, \ldots$ For each $n = 1, 2, \ldots$ the coefficients of p_n, q_n are rational numbers whose denominators divide $L_{n-1}^2[1, \ldots \ldots, n-1]Kn$, where $[1, \ldots, n]$ denotes the least common multiple of $1, \ldots, n$.*

Notation. The fact that a rational number ϱ has denominator d will be denoted by $\varrho \in \mathbf{Z}/d$.

Proof of the Main Lemma. Write $V_n = (n!)^{-1}\big(z(1-z)\big)^n F^{(n)}$. Then, by (3),

$$(n+2)(n+1)V_{n+2} = [(2n+\alpha+\beta+1)z-n-\gamma](n+1)V_{n+1} + (n+\alpha)(n+\beta)z(1-z)V_n.$$

Substitute $V_n = \alpha(\alpha+1)\ldots(\alpha+n-1)U_n/n!$. Then

$$(4)\qquad (n+\alpha+1)U_{n+2} = [(2n+\alpha+\beta+1)z-n-\gamma]U_{n+1} + (n+\beta)z(1-z)U_n.$$

Let $U(t) = \sum_{n=0}^{\infty} U_n t^n$ be the generating function of U_n. We shall derive an expression for $U(t)$. It follows from (4) that

$$[t-(2z-1)t^2+z(z-1)t^3](U-U_0)' - \big[1-\alpha+\big((\alpha+\beta-1)z+1-\gamma\big)t + \beta z(1-z)t^2\big](U-U_0) = \alpha U_1 t + \beta z(1-z)t^2 U_0.$$

We solve this differential equation for $U-U_0$ by standard methods. After division by $t-(2z-1)t^2+z(z-1)t^3$ we obtain

$$(U-U_0)' + \left[\frac{\alpha-1}{t} - z\,\frac{\beta-\gamma+1}{1-zt} + (1-z)\,\frac{\gamma-\alpha}{1+(1-z)t}\right](U-U_0)$$

$$= \alpha U_1\left[\frac{z}{1-zt} - \frac{z-1}{1+(1-z)t}\right] + \beta z(1-z)U_0\left[\frac{1}{1-zt} - \frac{1}{1+(1-z)t}\right].$$

Solution of this differential equation yields

$$U-U_0 = G_1 U_1 + G_0 U_0,$$

where

$$G_1 = \alpha P^{-1}\int\left[\frac{z}{1-zt} - \frac{z-1}{1+(1-z)t}\right]P\,dt,$$

$$G_0 = \beta z(z-1)P^{-1}\int\left[\frac{1}{1-zt} - \frac{1}{1+(1-z)t}\right]P\,dt$$

and

$$P = t^{\alpha-1}(1-zt)^{\beta-\gamma+1}\big(1+(1-z)t\big)^{\gamma-\alpha}.$$

Let K be the common denominator of α, β, γ. If $\delta \in \mathbf{Q}$ and $\delta \in \mathbf{Z}/K$ we know by Lemma 1 that $\binom{\delta}{n} \in \mathbf{Z}/L_n$. Therefore the nth coefficient of the powerseries expansions in t of the functions $(1-tz)^{\beta-\gamma+1}$ and $\big(1+(1-z)t\big)^{\gamma-\alpha}$ is a polynomial of degree $\leqslant n$ in z with coefficients in $\mathbf{Z}/L_n$. The same holds for the product of these two expansions and its inverse. It is now straightforward to see that the nth coefficient of G_0, G_1 is a polynomial of degree $\leqslant n+2$ in z having coefficients of the shape $\sum_{k=0}^{n-1} a_k(\alpha+k)^{-1}$, where $a_k \in \mathbf{Z}/L_{n-1}K$. The terms $(\alpha+k)^{-1}$ arise from the integration in the expression for G_0, G_1. This implies that

$$U_n = \tilde{p}_n U_1 + \tilde{q}_n U_0,$$

where $\tilde{p}_n$ and $\tilde{q}_n$ are polynomials in z having coefficients of the shape $\sum_{k=0}^{n-1} a_k(\alpha+k)^{-1}$, $a_k \in \mathbf{Z}/L_{n-1}K$. Finally, we have

$$p_n = \frac{(\alpha+1)\ldots(\alpha+n-1)}{n!}\,z(1-z)\tilde{p}_n, \qquad q_n = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!}\,\tilde{q}_n.$$

Let us consider the numbers of the shape

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!}\sum_{k=0}^{n-1}\frac{a_k}{\alpha+k}.$$

Notice that

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!(\alpha+k)} = \frac{k!(n-k-1)!}{n!}\,\frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!}\,\frac{(\alpha+k+1)\ldots(\alpha+n-1)}{(n-k-1)!}$$

$$= \frac{1}{n\binom{n-1}{k}}\binom{\alpha+k-1}{k}\binom{\alpha+n-1}{n-k-1}.$$

By Lemma 1, the product of the last two binomial coefficients is in $\mathbf{Z}/L_k L_{n-k-1} \subset \mathbf{Z}/L_{n-1}$. Thus we conclude that our numbers have denominators dividing $\left[\binom{n-1}{1}, \ldots, \binom{n-1}{n-1}\right]L_{n-1}^2 Kn$. It is a well-known fact that if a prime power p^b divides $\binom{n}{k}$, then $p^b \leqslant n$. This implies that $\left[\binom{n-1}{1}, \ldots\right.$

$\ldots, \binom{n-1}{n-1}\big]$ divides $[1, \ldots, n-1]$ which in turn implies that the coefficients of p_n and q_n have denominators dividing $L^2_{n-1}[1, \ldots, n-1]Kn$. This proves Main Lemma.

3. Proof of the theorems. Let

(5) $$g_1(z), \ldots, g_s(z)$$

denote analytic functions satisfying

(6) $$y'_i = Q_{i0}(z) + \sum_{j=1}^{s} Q_{ij}(z) y_j \quad (i = 1, \ldots, s),$$

where all $Q_{ij} \in \boldsymbol{K}(z)$, $\boldsymbol{K}$ denoting an algebraic number field of degree $\varkappa$ over $\boldsymbol{Q}$. From this, it follows, for all $l = 0, 1, \ldots$, that

$$g_i^{(l)} = Q_{i0l}(z) + \sum_{j=1}^{s} Q_{ijl}(z) g_j \quad (i = 1, \ldots, s),$$

where all $Q_{ijl}(z) \in \boldsymbol{K}(z)$. The functions (5) are said to belong to Galochkin's [8] class

$$G(\boldsymbol{K}, \gamma_1, Q_1, \gamma_2, Q_2, \gamma_3, Q_3), \quad Q_1 > 0;\ \gamma_1, \gamma_2, \gamma_3, Q_2, Q_3 \geqslant 1,$$

if these functions are of the form

$$g_i(z) = \sum_{\nu=0}^{\infty} a_{i\nu} z^{\nu} \quad (i = 1, \ldots, s),$$

where all $a_{i\nu} \in \boldsymbol{K}$, and the following conditions are satisfied:

(i) $\overline{|a_{i\nu}|} \leqslant \gamma_1 Q_1^{\nu}$ $(i = 1, \ldots, s;\ \nu = 0, 1, \ldots)$;

(ii) *there exist a sequence* $\{b_n\}$ *of natural numbers such that* $b_n \leqslant \gamma_2 Q_2^n$ $(n = 0, 1, \ldots)$ *and all the numbers* $a_{i\nu} b_n$ $(i = 1, \ldots, s;\ \nu = 0, 1, \ldots, n)$ *are integers in* $\boldsymbol{K}$;

(iii) *there exist a sequence* $\{d_n\}$ *of rational numbers and a polynomial* $T(z)$ *with integer coefficients in* $\boldsymbol{K}$, *not all zero, such that* $d_n \leqslant \gamma_3 Q_3^n$ $(n = 0, 1, \ldots)$ *and all the functions*

$$\frac{d_n}{l!} \big(T(z)\big)^l Q_{ijl} \quad (i = 1, \ldots, s;\ j = 0, \ldots, s;\ l = 0, \ldots, n)$$

are polynomials with integer coefficients in $\boldsymbol{K}$.

To prove Theorem 1 we show that the functions $F(\alpha, \beta, \gamma; z)$ and $F'(\alpha, \beta, \gamma; z)$ belong to some class $G(\boldsymbol{Q}, \gamma_1, Q_1, \gamma_2, Q_2, \gamma_3, Q_3)$. The conditions (i) and (ii) follow from [2], Lemma 3 (see [9], § 2).

In order to prove condition (iii) we note that by the Main Lemma it is possible to choose for the sequence $\{d_n\}$ the sequence

$$d_n = L_n^2 [1, \ldots, n+1]^2 K,$$

which clearly satisfies $d_n \leqslant \gamma_3 Q_3^n$ $(n = 0, 1, \ldots)$ for some constants γ_3, Q_3, since $[1, \ldots, n] \leqslant e^{1.04n}$. In the case $\alpha \in \{0, -1, -2, \ldots\}$ F and F' are polynomials and thus condition (iii) is obviously satisfied. Theorem 1 is thus proved.

Theorem 2, which improves [2], Theorem 1, is now an immediate corollary of Theorem 1, [2], Lemma 2, and [12], Corollary 1.

In the proof of Theorem 3 we shall need the following result giving the conditions for the algebraic independence of $F(\alpha, \beta, \gamma; z)$ and $F'(\alpha, \beta, \gamma; z)$.

THEOREM 5. *Let* α, β *and* γ $(\neq 0, -1, \ldots)$ *be rationals. If* $F(\alpha, \beta, \gamma; z)$ *and* $F'(\alpha, \beta, \gamma; z)$ *are algebraically dependent over* $\boldsymbol{C}(z)$, *then either* (2) *has only algebraic solutions or at least one of the numbers* α, β, $\gamma - \alpha$, $\gamma - \beta$ *is integral.*

Proof. If F and F' are algebraically dependent then it follows by Siegel [11], pp. 60–62, that there exists a solution $w \not\equiv 0$ of (2) such that w'/w is an algebraic function. In order to study the analytic behaviour of w throughout the complex plane we continue w analytically along closed loops in $\boldsymbol{C} \setminus \{0, 1\}$ beginning and ending in a point $z_0 \in \boldsymbol{C}$ different from 0 and 1. After traversing such a loop the function will in general change into a different branch w_1. We now distinguish three cases.

I. There exist two other branches w_1, w_2 such that w'/w, w_1'/w_1 and w_2'/w_2 are mutually different. Then the difference $w'/w - w_1'/w_1 = (w'w_1 - ww_1')/(ww_1)$ and $w'w_1 - ww_1'$, which is a non-zero multiple of the Wronskian $z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}$, are algebraic. Therefore ww_1 is algebraic. Analogously ww_2 and $w_1 w_2$ are also algebraic, which implies that (2) has only algebraic solutions.

II. There are exactly two branches w, w_1 such that $w'/w \neq w_1'/w_1$. Let $\Gamma_0, \Gamma_1, \Gamma_\infty$ be simple loops enclosing $z = 0$, $z = 1$, $z = \infty$ respectively. Suppose $\Gamma_\infty \Gamma_0 \Gamma_1 \sim 1$, that is, the path $\Gamma_\infty \Gamma_0 \Gamma_1$ can be contracted to z_0 in $\boldsymbol{C} \setminus \{0, 1\}$. After traversing such a loop Γ_i two things may happen, we have either 1) a substitution $w \to \lambda w_1$, $w_1 \to \mu w$ or 2) a substitution $w \to \lambda w$, $w_1 \to \mu w_1$ $(\lambda, \mu \in \boldsymbol{C})$. Because of $\Gamma_\infty \Gamma_0 \Gamma_1 \sim 1$, the possibility 1) occurs exactly twice. Let us assume that Γ_0 and Γ_1 are the loops under consideration. Denote by S_i the substitution that w_1/w undergoes after traversing Γ_i $(i = 0, 1)$. Clearly S_i $(i = 0, 1)$ has order two. Since $\Gamma_0 \Gamma_1 \sim \Gamma_\infty^{-1}$, the functions w, w_1 change into θw, $\theta_1 w_1$ after traversing $\Gamma_0 \Gamma_1$, and since (2) has rational exponents it follows that θ and θ_1 are roots of unity, and thus $S_0 S_1$ has finite order. A group generated by S_0, S_1 such that $S_0^2 = S_1^2 = 1$ and such that $S_0 S_1$ has finite order is necessarily finite and thus w_1/w must be algebraic. By the argument in I we know that ww_1 is also algebraic. Thus w and w_1 are both algebraic and (2) has only algebraic solutions.

III. Every branch of w is a multiple of w. Then w'/w as a single-valued algebraic function must be rational. This happens only in the case at least one of the numbers $\alpha, \beta, \gamma-\alpha, \gamma-\beta$ is an integer (see [7], Chapter II).

Because of Theorem 5, Theorems 3 and 4 are immediate corollaries of the following Theorems A and B, which are proved in [13], and which slightly improve [9], Theorems 4 and 6.

THEOREM A. *Assume that the functions* (5) *satisfying* (6) *are algebraically independent over* $C(z)$ *and belong to the class* $G(\boldsymbol{K}, \gamma_1, Q_1, \gamma_2, Q_2, \gamma_3, Q_3)$, *and let* $P(x_1, \ldots, x_s) \not\equiv 0$ *be a polynomial with integer coefficients in* $\boldsymbol{K}$ *satisfying* $\deg P \leqslant N$, $\overline{|\text{coeff} P|} \leqslant H$. *There then exist positive constants* c, λ, *depending only on* $g_1(z), \ldots, g_s(z)$ *and* s, *and a positive constant* C, *depending only on* $\boldsymbol{K}, g_1(z), \ldots, g_s(z), s, N, \theta, h$ *and* τ, *such that*

$$|P(g_1(\theta), \ldots, g_s(\theta))| > CH^{-\lambda\tau(\log h/\log\log h)^{1/2}}$$

for any algebraic number θ *of height* $h(\theta) \leqslant h \geqslant e^e$ *satisfying* $[\boldsymbol{K}(\theta):\boldsymbol{Q}] \leqslant \tau$ $(\geqslant \varkappa)$ *and*

$$\theta T(\theta) \neq 0, \quad \log h \geqslant (\max\{2, N\})^{2s} \log\log h,$$

$$0 < |\theta| < e^{-c\tau N(\log h)^{(2s-1)/2s}(\log\log h)^{1/2s}}.$$

THEOREM B. *Let the functions* (5) *satisfy the conditions of Theorem* A. *Let* $L_i(x) = a_i + b_i x \not\equiv 0$ $(i = 1, \ldots, s)$ *be linear forms with integer coefficients in* $\boldsymbol{K}$ *satisfying* $\max\limits_{1\leqslant i\leqslant s}\{\overline{|a_i|}, \overline{|b_i|}\} \leqslant H$. *There then exist positive constants* $\bar{c}$, $\bar{\lambda}$, *depending only on* $g_1(z), \ldots, g_s(z)$ *and* s, *and a positive constant* $\bar{C}$, *depending only on* $\boldsymbol{K}, g_1(z), \ldots, g_s(z), s, \theta, h$ *and* τ, *such that*

$$\max_{1\leqslant i\leqslant s}\{|L_i(g_i(\theta))|\} > \bar{C} e^{-\bar{\lambda}\tau(\log h/\log\log h)^{1/2}}$$

for any algebraic number θ *of height* $h(\theta) \leqslant h \geqslant e^e$ *satisfying* $[\boldsymbol{K}(\theta):\boldsymbol{Q}] \leqslant \tau$ *and*

$$\theta T(\theta) \neq 0, \quad \log h \geqslant 2^{2s} \log\log h,$$

$$0 < |\theta| < e^{-\bar{c}\tau(\log h \log\log h)^{1/2}}.$$

References

[1] E. Bombieri, *On G-functions*, In: *Recent Progress in Analytic Number Theory*, Vol. II, H. Halberstam and C. Hooley (eds.), Academic Press, 1981.

[2] V. G. Chirsky, M. S. Nurmagomedov, *On arithmetical properties of values of hypergeometrical functions* (Russian), Vestnik Moscov. Univ. Ser. I, Mat. Meh. 28 (2) (1973), pp. 38–45.

[3] V. G. Chirsky, *On the arithmetic properties of the values of hypergeometric series with irrational parameters* (Russian), ibid. 33 (5) (1978), pp. 3–8.

[4] G. V. Chudnovsky, *Arithmetical properties of values of analytical functions*, Lecture given (by M. Waldschmidt and D. Brownawell) at the meeting "Diophantische Approximationen" at Oberwolfach in March 1977.

[5] — *Transcendental Numbers*, Lecture Notes in Mathematics 751 (1979), pp. 45–69.

[6] — *Un système explicite d'approximants de Padé pour les fonctions hypergéométriques généralisées, avec applications à l'arithmétique*, C.R. Acad. Sci. Paris, 288, Serie A, (1979), pp. 1001–1004.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, Toronto, London 1953.

[8] A. I. Galochkin, *Lower bounds for polynomials of the values of the class of analytic functions* (Russian), Mat. Sbornik 95 (137) (3) (1974), pp. 396–417.

[9] T. Matala-aho, K. Väänänen, *On the arithmetic properties of certain values of one Gauss hypergeometric function*, Acta Univ. Oulu. A 112, Math. 24 (1981).

[10] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuß, Akad. Wiss., Phys.-mat. Kl. No. 1 (1929).

[11] — *Transcendental Numbers*, Princeton University Press, Princeton 1949.

[12] K. Väänänen, *On linear forms of a certain class of G-functions and p-adic G-functions*, Acta Arith. 36 (1980), pp. 273–295.

[13] — *On a class of G-functions*, Mathematics, University of Oulu 1/81 (1981).

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Received on 15. 5. 1981
and in revised form on 16. 2. 1982 (1253)