Remarks on the arithmetic properties of the values of hypergeometric functions

by

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1. Introduction. The purpose of the present paper is to apply the classical method of Siegel [10] to a consideration of the arithmetic properties of the values of certain hypergeometric functions

\[ F(a, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{a(a+1) \ldots (a+n-1) \beta(\beta+1) \ldots (\beta+n-1)}{n! \gamma(\gamma+1) \ldots (\gamma+n-1)} z^n \]

\( (\gamma \neq 0, -1, \ldots) \)

satisfying

\[ z(1-z)y'' + (y - (1 + a + \beta)z)y' - ay = 0. \]

We shall prove the following theorems.

**Theorem 1.** If \( a, \beta, \gamma \) are rational numbers, \( \gamma \neq 0, -1, -2, \ldots \), then the functions \( F(a, \beta, \gamma; z) \) and \( F'(a, \beta, \gamma; z) \) belong to Gallokhin's \([3]\) class of \( G \)-functions (for definition see § 3).

**Theorem 2.** Let \( a_i, \beta_i, \gamma_i \ (i = 1, \ldots, m) \) be rational numbers satisfying

\( \gamma_i \neq 0, -1, \ldots; a_i, \beta_i, \gamma_i - a_i, \gamma_i - \beta_i \notin \mathbb{Z} \ (i = 1, \ldots, m); \quad a_i - a_j, \beta_i - \beta_j, \gamma_i - \beta_i \notin \mathbb{Z} \ (i \neq j; \ i, j = 1, \ldots, m); \) none of the numbers \( a_i + \beta_i - (a_i + \beta_i) \ (i \neq j; \ i, j = 1, \ldots, m) \) is an even integer. Let \( x_i, y_i, z_i \ (i = 1, \ldots, m) \) be integers, not all zero, and let us denote \( h_i = \max \{ |x_i|, |y_i|, |z_i| \} \ (i = 1, \ldots, m) \), \( H = \prod_{i=1}^{m} h_i \). Let \( 0 < \epsilon < 1 \) be given. There exist positive constants \( \lambda, C, \) depending only on \( \epsilon, m, p \) and the functions \( F(a_i, \beta_i, \gamma_i; z) \ (i = 1, \ldots, m) \), such that

\[ |x_i + \sum_{i=1}^{m} \left( a_i F(a_i, \beta_i, \gamma_i; z/q) + y_i F'(a_i, \beta_i, \gamma_i; z/q) \right) | \geq \lambda q^{-1} H^{-1-\epsilon} \]

for any rational number \( p/q \neq 0 \) satisfying \( q > C \). In particular, the numbers \( 1, F(a_i, \beta_i, \gamma_i; p/q) \) and \( F'(a_i, \beta_i, \gamma_i; p/q) \ (i = 1, \ldots, m) \) are linearly independent over \( \mathbb{Q} \) for all \( q > C \).
Theorem 3. Let \( a, \beta \) and \( \gamma \) \((\neq 0, -1, \ldots)\) be rational numbers such that \( F(a, \beta, \gamma; z) \) is not an algebraic function and \( a, \beta, \gamma \neq 0 \). Let \( F(x_1, x_2) \equiv 0 \) be a polynomial in \( \mathbb{Z}[x_1, x_2] \) of degree \( \leq N \) and height \( \leq H \). There then exist positive constants \( c, \delta, \) depending only on \( a, \beta, \gamma, N, H, \delta \) and \( \varepsilon \) such that
\[
|F(a, \beta, \gamma; 0)| = O(H^{-1/2}(\log \log H)^{1/2})
\]
for any algebraic number \( 0 \) of degree \( \leq \tau \) and height \( h(0) \leq H \geq \varepsilon \) satisfying
\[
\log h \geq \left( \max \left\{ 2, N \right\} \right) \log \log h,
\]
which is then satisfied for \( \varepsilon \) of order \( \min \{ \varepsilon, 1 \} \).

This theorem implies, in particular, the linear independence (over \( \mathbb{Q} \)) of the numbers \( F(a, \beta, \gamma; 0), F'(a, \beta, \gamma; 0) \) for all algebraic numbers \( 0 \) of degree \( \leq \tau \) and height \( H \geq \varepsilon \) satisfying \( \log h \geq 16 \log \log h, \log \log h > 0 \), which is obtained from (3). This kind of result, the need of which was already pointed out by Siegel [10], is obtained in the main lemma in Section 2. The proof of the theorems is then completed in Section 3.

2. Main lemma. First we prove.

**Lemma 1.** Let \( \delta \in \mathbb{Q} \) and let \( K \) be the denominator of \( \delta \). For any \( n \in \mathbb{N} \), let \( L_n = K^n \prod_{p \in \mathbb{Z}^+} p^{n(p-1)} \). Then \( L_n \frac{\delta}{n} \) is an integer.

**Proof.** Notice that:
\[
L_n \frac{\delta}{n} = K^n \frac{\delta(n-1)}{n} \prod_{p \in \mathbb{Z}^+} p^{n(p-1)}
\]
for any \( n \in \mathbb{N} \). Let \( p \) be a prime not dividing \( K \). The number of factors \( p \) in \( n! \) equals \( [n/p] + [n/p^2] + \ldots \). Notice that \( p^a K^{\delta} - mK \) if and only if \( p^a m - b \) is divisible by \( K \). Furthermore, at least one of the numbers \( K, K\delta - mK \) divisible by \( p \). Consequently, among the products \( K, K\delta - (n-1)K \) are divisible by \( p \). Hence the product \( K\delta - (n-1)K \) contains at least \( [n/p] + [n/p^2] + \ldots \) factors \( p \). Thus we see that \( L_n \frac{\delta}{n} \) has only prime divisors of \( K \) in its denominator.

Let \( p \mid K \). Since \( n! \) contains at most \( [n/p] + [n/p^2] + \ldots \leq [n/p - 1] \) factors \( p \) we see that
\[
K^n \prod_{p \in \mathbb{Z}^+} p^{n(p-1)} \frac{\delta}{n}
\]
is integral as asserted.

In the following computations \( a, \beta \) and \( \gamma \) denote rational numbers with \( \gamma \neq 0, -1, -2, \ldots \). It follows from (2) that
\[
F(1/z, 1/z, 1/z) = (2\beta + a + \beta + 1)z - \gamma \frac{(n+1) + (n+a)(n+\beta)}{E^n}.
\]

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This implies
\[ \frac{1}{n!} \left( (1 - z)^n \mathcal{F}(n) \right) = \frac{p_n}{q_n} F^r = p_n F^r + q_n F^r, \]
where \( p_n \), \( q_n \) are polynomials in \( z \) with rational coefficients. The following lemma gives a bound for the denominators of these rational coefficients.

**Main Lemma.** Let \( K \) be the common denominator of \( a, b, \gamma \) and let \( \eta_n \) be as in Lemma 1. Suppose that \( \delta \neq 0, -1, -2, \ldots \). For each \( n = 1, 2, \ldots \), the coefficients of \( p_n, q_n \) are rational numbers whose denominators divide \( s \eta_n \), where \( \eta_n \) denotes the least common multiple of \( 1, \ldots, n \).

**Notation.** The fact that a rational number \( \eta \) has denominator \( d \) will be denoted \( \eta \in \mathbb{Z}/d \).

**Proof of the Main Lemma.** Write \( V_n = (n+1)^{-1}(1-z)^{n+1} \mathcal{F}(n) \).

Then, by (3),
\[ (n+1)(n+2) V_{n+2} = \left( [2n+1 \alpha + \beta + 1] + (n+1\alpha + \gamma) V_{n+1} \right) + \left( n+1 \beta + \gamma \right) V_n. \]

Substitute \( V_n = a(n+1) \ldots (n+1\alpha + \gamma) U_{n+1}/n! \). Then
\[ (n+1)(n+2) U_{n+2} = \left( [(2n+1 \alpha + \beta + 1) + (n+1\alpha + \gamma)] U_{n+1} + \right. \]
\[ \left. + (n+1 \beta + \gamma) \right) V_n. \]

Let \( U(t) = e^{\frac{\alpha}{n!} t \mathcal{F}(n)} \) be the generating function of \( U_n \). We shall derive an expression for \( U(t) \). It follows from (4) that
\[ U(t) = \sum U_n e^{t \mathcal{F}(n)} \]

We solve this differential equation for \( U - U_0 \) by standard methods. After division by \( t - (2s-1) t^2 + (s-1) t^3 \) we obtain
\[ \left( U - U_0 \right) + \left( \frac{a-\gamma+1}{1-s} \right) U_0 = \frac{1}{1-s} \left( a \mathcal{F}(n) + \beta \mathcal{F}(n-1) \right). \]

Solution of this differential equation yields
\[ \mathcal{F}(n) = \mathcal{F}_1 U_1 + \mathcal{F}_2 U_2, \]
where
\[ \mathcal{F}_1 = a e^{ \int \mathcal{F}(n) dt}, \]
\[ \mathcal{F}_2 = \beta e^{ \int \mathcal{F}(n) dt}. \]

and
\[ \mathcal{F} = e^{\alpha t - \gamma t} \left( 1 + \mathcal{F}(n) \right). \]

Let \( \mathcal{K} \) be the common denominator of \( a, b, \gamma \). If \( \delta \in \mathbb{Q} \) and \( \delta \in \mathbb{Z}/\mathcal{K} \) we know by Lemma 1 that \( \left[ \frac{\delta}{\eta} \right] \in \mathbb{Z}/\mathcal{K} \). Therefore the \( \eta \)th coefficient of the power series expansions in \( t \) of the functions \( (1-z)^{n+1} \) and \( (1+1-z) \beta \) is a polynomial of degree \( \leq n \) in \( z \) with coefficients in \( \mathbb{Z}/\mathcal{K} \). The same holds for the product of these two expansions and its inverse. It is now straightforward to see that the \( \eta \)th coefficient of \( \mathcal{G}_1 \), \( \mathcal{G}_2 \) is a polynomial of degree \( \leq n + 2 \) in \( z \) having coefficients of the shape \( \sum a_k (a+k)^{-1} \), where \( a_k \in \mathbb{Z}/\mathcal{K} \). Finally, we have
\[ \frac{a(n+1) \ldots (n+1\alpha + \gamma) U_{n+1}}{n!} - \frac{1}{1-s} \left( a \mathcal{F}(n) + \beta \mathcal{F}(n-1) \right). \]

Let us consider the numbers of the shape
\[ \left( a(n+1) \ldots (n+1\alpha + \gamma) \right) \frac{1}{n!} \sum_{k=\alpha}^{n} \frac{a_k}{a+k}. \]
Notice that
\[ \frac{a(n+1) \ldots (n+1\alpha + \gamma)}{n!} (a+k) \]
\[ = \frac{k!(n-k-1)!}{a! (a+k-1)!} \frac{a(n+1) \ldots (n+k-1)}{n!} \frac{(a+k)(a+k-1) \ldots (a+1)}{(a+k-1)!} \]
\[ = \frac{1}{n!} \left( \frac{(a+k-1)}{a+k} \right)^{(a+n-1)} \]

By Lemma 1, the product of the last two binomial coefficients is in \( \mathbb{Z}/\mathcal{K} \). Thus we conclude that our numbers have denominators dividing \( \left( \frac{n+1!}{1!}, \ldots, \frac{n+1!}{1!}, \ldots \right) \mathcal{K} \). It is a well-known fact that if a prime power \( p^b \) divides \( \left( \frac{n+1!}{1!}, \ldots \right) \mathcal{K} \), then \( p^b \leq n \). This implies that \( \left( \frac{n+1!}{1!}, \ldots \right) \mathcal{K} \).
In order to prove condition (iii) we note that by the Main Lemma it is possible to choose for the sequence \( \{d_n\} \) the sequence
\[
d_n = \frac{T_n}{n} [1, \ldots, n+1] K \alpha, \]
which clearly satisfies \( d_n \leq \gamma_n Q_n^\alpha (n = 0, 1, \ldots) \) for some constants \( \gamma_n, Q_n \), since \( [1, \ldots, n] \leq \gamma_n^\alpha \). In the case \( n \in \{0, -1, -2, \ldots\} \) \( F \) and \( F' \) are polynomials and thus condition (iii) is obviously satisfied. Theorem 1 is thus proved.

Theorem 2, which improves [2], Theorem 1, is now an immediate corollary of Theorem 1, [2], Lemma 2, and [12], Corollary 1.

In the proof of Theorem 3 we shall need the following result giving the conditions for the algebraic independence of \( F(a, \beta, \gamma; z) \) and \( F'(a, \beta, \gamma; z) \).

**Theorem 5.** Let \( a, \beta, \gamma \) (\( \neq 0, -1, \ldots \)) be rationals. If \( F(a, \beta, \gamma; z) \) and \( F'(a, \beta, \gamma; z) \) are algebraically independent over \( C(z) \), then either (2) has only algebraic solutions or at least one of the numbers \( a, \beta, \gamma - a, \gamma - \beta \) is algebraic.

**Proof.** If \( F \) and \( F' \) are algebraically dependent then it follows by Siegel [11], pp. 60–62, that there exists a solution \( w \neq 0 \) of (2) such that \( \omega'/w \) is an algebraic function. In order to study the analytic behaviour of \( w \) throughout the complex plane we continue \( w \) analytically along closed loops in \( C \setminus \{0, 1\} \) beginning and ending in a point \( z \) of \( C \) different from 0 and 1. After traversing such a loop the function will in general change into a different branch \( w_z \). We now distinguish three cases.

I. There exist two other branches \( w_1, w_2 \) such that \( w'/w = w_1'/w_1 \) and \( w'/w_2 = w_2'/w_2 \) are mutually different. Then the difference \( w_1'/w - w_2'/w_2 \) is a non-zero multiple of the Wronskian \( s' (1 - 1/z + 1/z) \), which is algebraic. Therefore \( w \) is algebraic.

II. There are exactly two branches \( w, w \), such that \( w'/w \neq w'/w \). Let \( T_{\beta}, T_{\gamma}, T_{\alpha} \) be simple loops enclosing \( z = 0, z = 1, z = \infty \) respectively. Suppose \( T_{\beta} \cap T_{\gamma} \cap T_{\alpha} \approx 1 \), that is, the path \( T_{\beta} \cap T_{\gamma} \cap T_{\alpha} \) can be contracted to \( z \) in \( C \setminus \{0, 1\} \). After traversing such a loop \( T \), two things may happen, we have either (1) a substitution \( w \to w_0 \), \( w_0 \to w_2 \), or (2) a substitution \( w \to w_0 \), \( w_0 \to \mu \), \( \mu \in C \). Because of \( T_{\beta} \cap T_{\gamma} \cap T_{\alpha} \approx 1 \), the possibility (1) occurs exactly twice. Let us assume that \( T_{\beta} \) and \( T_{\gamma} \) are the loops under consideration. Denote by \( S \) the substitution that \( w \to w \) undergoes after traversing \( T_{\alpha} (i = 0, 1) \). Clearly \( S \) \((i = 0, 1) \) has order two. Since \( T_{\beta} \cap T_{\gamma} \approx T_{\alpha} \), the functions \( w_0, w_1 \) change into \( w_0, w_1 \), after traversing \( T_{\beta} \cap T_{\gamma} \), and since (2) has rational exponents it follows that \( w_0 \), \( w_1 \) are roots of unity, and thus \( S_0 S_1 \) has finite order. A group generated by \( S_0, S_1 \) such
that $S_2^2 = S_2^2 = 1$ and such that $S_2 S_1$ has finite order is necessarily finite and thus $w_1 w_0$ must be algebraic. By the argument in I we know that $w_1 w_0$ is also algebraic. Thus $w_0$ and $w_0$ are both algebraic and (2) has only algebraic solutions.

III. Every branch of $w$ is a multiple of $w$. Then $w'/w$ is a single-valued algebraic function must be rational. This happens only in the case in which at least one of the numbers $a, b, c, d$ is an integer (see [7], Chapter II).

Because of Theorem 6, Theorems 3 and 4 are immediate corollaries of the following Theorems A and B, which are proved in [13], and which slightly improve [9], Theorems 4 and 6.

**Theorem A.** Assume that the functions (5) satisfying (6) are algebraically independent over $\mathbb{C}(z)$ and belong to the class $\mathcal{F}(K, g_1, g_2, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7)$, and let $P(x_1, \ldots, x_n) \neq 0$ be a polynomial with integer coefficients in $K$ satisfying $\deg P \leq N$, $|\text{coeff} P| \leq H$. There then exist positive constants $\delta, \lambda, \tau$, depending only on $g_1(\theta), \ldots, g_7(\theta)$ and $\varepsilon$, and a positive constant $C$, depending only on $K, g_1(\theta), \ldots, g_7(\theta), \varepsilon, N, \theta, \delta$, and $\tau$, such that

$$|P(g_1(\theta), \ldots, g_7(\theta))| > C^{-1}(\log(\log|\theta|))^n$$

for any algebraic number $\theta$ of height $h(\theta) \leq \delta \varepsilon$ satisfying $[K(\theta): Q] \leq \tau$.

Then $\theta(\theta) \neq 0$, $\log h \geq (\max \{2, \delta N\})^{1/2} \log \log h$, $0 < |\theta| < e^{-\varepsilon(\log(\log h))^2}$.

**Theorem B.** Let the functions (5) satisfy the conditions of Theorem A. Let $L_1(x) = a_1 + b_1 x \neq 0 (i = 1, \ldots, s)$ be linear forms with integer coefficients in $K$ satisfying $\max \{|a_i|, |b_i|\} < H$. There then exist positive constants $\delta, \lambda, \tau$, depending only on $g_1(\theta), \ldots, g_7(\theta)$ and $\varepsilon$, and a positive constant $C$, depending only on $K, g_1(\theta), \ldots, g_7(\theta), \varepsilon, N, \theta, \delta$, and $\tau$, such that

$$\max_{1 \leq i \leq s} |P_i(g_1(\theta), \ldots, g_7(\theta))| > C^{-1}(\log(\log h))^n$$

for any algebraic number $\theta$ of height $h(\theta) \leq \delta \varepsilon$ satisfying $[K(\theta): Q] \leq \tau$.

Then $\theta(\theta) \neq 0$, $\log h \geq 2^{1/2} \log \log h$, $0 < |\theta| < e^{-\varepsilon(\log(\log h))^2}$.

### References


[12] — *In a class of G-functions, Mathematics*, University of Oulu 1/81 (1941).