ются целыми. Как было установлено выше линейная комбинация

$$\eta(\nu) = f_2(-1,\nu) - \frac{1}{q}\operatorname{Im}\big(f_3(-1,\nu)\big) =$$
$$= -\left(\zeta(2) - \frac{3}{q}\zeta(3)\right)\frac{a(-1,\nu)}{2} + \left(2\log 2 - \frac{1}{q}\zeta(2)\right)\frac{\beta(-1,\nu)}{-2} -$$
$$-\varphi(-1,\nu) + \frac{1}{q}\psi(-1,\nu)$$

принимает при $\nu\to\infty$ бесконечное число раз отличные от нуля значения. Поэтому величина

$$2D_\nu n_0\eta(\nu) = \left(\zeta(2) - \frac{3}{q}\zeta(3)\right)n_0K(-1,\nu) +$$
$$+\left(2\log 2 - \frac{1}{q}\zeta(2)\right)n_0L(-1,\nu) - n_0M(-1,\nu) + \frac{n_0}{q}N(-1,\nu)$$

принимает при $\nu\to\infty$ бесконечное число раз отличные от нуля значения из $\boldsymbol{Z}$, что находится в противоречии с (64). Итак, теорема 1 доказана.

Теорема 2 является непосредственным следствием теоремы 1.

Примечание при корректуре. К настоящему времени мной получен такой результат. Пусть $\varrho_1 = \left(5+4\sqrt{2}+2\sqrt{2\sqrt{2}-2}+4\sqrt{2\sqrt{2}+2}\right)e^{-3}$, $\varrho_2 = \varrho_1 e^6$, $\gamma = (\log\varrho_2)/\log\varrho_1$, $\varphi_1(x_1,x_2) = x_1\log 4 + x_2\zeta(2)$, $\varphi_2(x_1,x_2) = x_1\zeta(2) + x_2 3\zeta(3)$, $\|d\|$ – расстояние от d до $\boldsymbol{Z}$. Тогда для любого $\varepsilon > 0$ существует такое $c > 0$, что, если $x_1\in\boldsymbol{Z}$, $x_2\in\boldsymbol{Z}$, $x_1^2+x_2^2 > 0$, то

$$\max_{i=1,2}\left(\|\varphi_i(x_1,x_2)\|\right) \geqslant c\left(\max(|x_1|,|x_2|)\right)^{-\gamma-\varepsilon}.$$

Получены и некоторые другие количественные результаты.

В заключение, но далеко не в последнюю очередь, выражаю глубокую благодарность профессору Ю. И. Журавлёву, оказавшему мне большую административную поддержку.

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On a sequence of (j, ε)-normal approximations to $\pi/4$ and the Brouwer conjecture

by

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1. Introduction. We present a first application of our results concerning the phenomenon of (j,ε)-normality in the rationals ([4], p. 233; see def. Type A, p. 229 and further studies in [5], p. 389) to a specific given convergent sequence of rational approximations $p_n/q_n < 1$ in lowest terms whose limit, in this case as $n\to\infty$, is the interesting number $\pi/4$.

In order to establish the (j,ε)-normal character of the representations of a given sequence of rational numbers in some base g, we need specific information concerning the magnitude of the exponents n_i in the prime decomposition of the denominator $q_n = \prod_{i=1}^{r} p_i^{n_i}$ as n increases without bound. In particular, we need to show, in order to prove (j,ε)-normality, that there is at least one odd prime $p_i|q_n$ such that the exponent n_i of that prime is such that $n_i > z_i + s_i$ where $p_i^{z_i}\|(g^{d_i}-1)$, $d_i|(p_i-1)$, and $p_i^{s_i}\|(d_{i+1},\ldots,d_r)$, i.e. s_i is the maximum exponent to which p_i appears in each least exponent $d_{i+1}, d_{i+2},\ldots,d_r$ for the primes contained in q_n which exceed p_i up to the maximum prime $p_r|q_n$.

Even though there are many convergent sequences of rational approximations p_n/q_n such that $\lim_{n\to\infty} p_n/q_n$ is some real number of interest such as e, π, $\sqrt{2}$, etc., we find that when we consider *given* sequences (It is important to keep in mind here that we are not considering "constructed" cases of sequences but representations based on well-known infinite processes.) that the above arithmetic information concerning behavior of the odd primes contained in q_n as n increases without bound is not known at the present time.

In this paper, we will show that an infinite product representation which can be written as a product of factorials will yield a great deal of arithmetic information concerning the prime decomposition of the successive partial products. For this study, we consider the Wallis infinite product for $\pi/4$ which has the partial product

$$p_n/q_n = \prod_{i=1}^{n} (1-1/(2i+1)^2) \tag{1.1}$$

where $\lim_{n\to\infty} p_n/q_n = \pi/4$.

Since we are investigating questions in relation to $\pi/4$ in base 10 (We could have used π and removed the integral part, but for convenience, we consider $\pi/4 < 1$.), we will show there exists an infinite sub-sequence of n values such that each $p_n/q_n < 1$ is in lowest terms and the primes 2 and 5 are missing from q_n. In addition, we show for the particular sub-sequence of n values that the exponent of any fixed odd prime $q \neq 5$ in q_n becomes arbitrarily large. These results imply that in base $10, p_n/q_n$ over this particular sub-sequence of n values are such that they converge onto $\pi/4$ by means of purely periodic representations, i.e. the expansion of p_n/q_n converging toward $\pi/4$ begins immediately at the decimal point. Furthermore, we also prove that these partial products p_n/q_n for each n value are (j, ε)-normal in base 10. We then apply these results to a conjecture of L. E. J. Brouwer which we promised in ([4], pp. 234–235), i.e. he thought that it was an undecidable proposition to prove that somewhere in π there would occur a fixed chosen set of digits like 0123456789 when π is represented in base 10.

Actually, we can paraphrase Brouwer's conjecture, and say that he believed that it was an undecidable proposition to prove that a given irrational like π was a normal number in the sense of Borel when represented in base 10. This follows at once since, if a real number is normal in any integral base, then any specified block of digits will have its expected limiting relative frequency over the infinite collective in its radix representation. The normality in base 10 of π would assure the occurrence of 0123456789 somewhere in the expansion.

On the other hand, it is surprising to note that even though the preponderance of real numbers are absolutely normal, i.e. normal in every integral base with the non-normal numbers of measure zero, and in addition, with the support of abundant numerical evidence for normality in real numbers such as e, π, $\sqrt{2}$, etc., no demonstrations have been given to date to show that specific given irrationals such as these are normal in any base.

Much of our work to date on the extensive phenomenon of (j, ε)-normality in the rationals has been motivated by an effort to lay a foundation for such a demonstration based upon the (j, ε)-normal properties of rational approximations to irrationals.

We note here that the Brouwer type conjecture is decided ([4], Th. 6, p. 233) for a broad class of (j, ε)-normal Type A rational fractions.

For example, we can state that the block 0123456789 *will occur* somewhere within the set of digits slightly greater than the square root of a period length ([5], p. 377) (even though we cannot compute its first appearance!) of say, $Z/17^n < 1$ as represented in base 10 with $(Z, 17) = 1$ and n sufficiently large (see [4], p. 235).

2. The prime decomposition of p_n/q_n. By filling in the missing prime factors, (1.1) can be written

$$p_n/q_n = 2^{4n}(n!)^3(n+1)!/((2n+1)!)^2 \tag{2.0}$$

where q_n may contain any odd primes $p \leqslant 2n+1$ for $n \geqslant 1$.

If $E(p, n)$ is the exponent of the prime p contained in $n!$, we can write p_n/q_n in lowest terms as

$$p_n/q_n = 2^{G(2,n)}/\prod_{(p)} p^{F(p,n)} \tag{2.1}$$

where

$$F(p, n) = 2E(p, 2n+1) - 3E(p, n) - E(p, n+1) \tag{2.2}$$

or

$$F(p, n) = (1+3S(p, n)+S(p, n+1)-2S(p, 2n+1))/(p-1)$$

and

$$G(2, n) = 4n - F(2, n) \tag{2.3}$$

where $S(p, n)$ is the sum of the coefficients a_i in the p-adic representation of n, i.e. $n(p) = \sum_{(i)} a_i p^i$.

We study the behavior of the exponent $F(p, n)$ from two points of view, (1) we can choose some fixed $n \geqslant 1$ and let p run over all the odd primes $3, 5, \ldots \leqslant 2n+1$; (2) or fix p and let n increase without bound in the successive approximations p_n/q_n.

In essence, this means we can study the complete prime decomposition of a given p_n/q_n, or determine the behavior of the exponent $F(p_0, n)$ of some given fixed odd prime $p_0 | q_n$ as the factor $p_0^{F(p_0,n)}$, when n increases without bound. We find that these results depend intimately on the p-adic form of n. For example, we can show that there exists an infinite sequence of n values such that $F(p_0, n_1) = F(p_0, n_2) = 0, \ldots$ for $n_1 < n_2 < \ldots$, i.e. there exists an infinite sequence of n values such that the chosen odd prime p_0 is missing in the denominators q_n of the successive approximations p_n/q_n. When $F(p, n) = 0$, we call this a *minimum* value of $F(p, n)$. Most important for our (j, ε)-normal considerations, we can also show that there exists a strictly monotonic increasing infinite sequence of n values such that $F(p_0, n)$ is a strictly increasing monotonic infinite sequence of positive integers.

We find that it is convenient to consider $F(p, n)$ in (2.2) as defined over the $p^s(p-1)$ values of n contained in $p^s \leqslant n \leqslant p^{s+1}-1$ for some odd prime p where $s = [\log_p n]$ for any n in the interval. Implicitly, this consideration depends upon the function $E(p, n) = \sum_{i=1}^{s} [n/p^i]$ where $[x]$ is the greatest integer not greater than x with $[n/p^s] = 1$ and $[n/p^{s+1}] = 0$, and we may consider all positive integers $n \geqslant 1$ as contained in the intervals $1 \leqslant n \leqslant p-1$, $p \leqslant n \leqslant p^2-1, \ldots$ for any odd prime p.

In Theorem 1, we obtain precise information on the value of $F(p, n)$ for a fixed p and particular $n = p^s, p^s+1, \ldots, p^{s+1}-1$ within the interval. We find that $F(p, n)$ is a multivalued function which takes on the values $0, 1, \ldots, 2(s+1)$ depending on the digits a_i in the p-adic form of n. We mean *multivalued* in the sense that each of the $2s+3$ nonnegative integers $0, 1, \ldots, 2(s+1)$ are assumed by $F(p, n)$ at least once over the $p^s(p-1)$ values of n in the interval. Necessary for this paper are the zeros or minimums of $F(p, n)$ as well as the *maximums*, i.e. when $F(p, n) = 2(s+1)$. We show in Theorem 3 that these "critical" values occur in the lower half and upper half of the interval $p^s \leqslant n \leqslant p^{s+1}-1$. For example, let $p = 3$ and $s = 2$, then for $n = 3^2, 3^2+1, \ldots, 3^3-1$; we find that $F(3, 9) = F(3, 12) = 0$, $F(3, 11) = 1$, $F(3, 10) = F(3, 18) = F(3, 21) = 2, \ldots, F(3, 13) = F(3, 16) = F(3, 22) = F(3, 25) = 6$ which are the maximums $2(2+1) = 6$ in the interval $3^2 \leqslant n \leqslant 3^3-1$.

In Theorem 2, we prove that all the powers of 2 are in p_n, i.e. $2^{s+2}-2 \leqslant G(2, n) \leqslant 2^{s+3}-(s+5)$ for all n contained in $2^s \leqslant n \leqslant 2^{s+1}-1$ for any $s \geqslant 0$. This result implies that $G(2, n)$ is a positive strictly monotonic increasing function of $n \geqslant 1$.

In all the considerations of the p-adic forms of $n(p) = \sum_{i=0}^{s} a_i p^i$ in $p^s \leqslant n \leqslant p^{s+1}-1$, it is clear by implication that a_s must be $1, 2, \ldots, p-1$, but the a_i for $0 \leqslant i < s$ can be $0, 1, \ldots, p-1$.

THEOREM 1. *Let* $n(p) = \sum_{i=0}^{s} a_i p^i$ *in* $p^s \leqslant n \leqslant p^{s+1}-1$ *where* p *is any odd prime,* $s = [\log_p n]$, $p^w \| (n+1)$, $p^v \| (2n+1)$, $\delta_i = [2n/p^{i+1}] - 2[n/p^{i+1}] \equiv [2n/p^{i+1}] \bmod 2$ *for* $i = 0, 1, \ldots, s$; *and define the function* $k(x) = \sum_{i=x}^{s} \delta_i$ *for any* $0 \leqslant x \leqslant s$, *then the exponent* $F(p, n)$ *of an odd prime* $p \leqslant 2n+1$ *in* $p_n/q_n = 2^{G(2,n)}/\prod_{(p)} p^{F(p,n)}$ *is given by:*

(2.4) Case 1. $w > 0$, $v = 0$, $F(p, n) = 2k(w)+w$ *where* $w \leqslant F(p, n) \leqslant 2(s+1)-w$ *for* $w = 1, 2, \ldots, s+1$ *and* $a_i = p-1$ *for* $i = 0, 1, \ldots, w-1$;

Case 2. $w = 0$, $v > 0$, $F(p, n) = 2(k(v)+v)$ *where* $2v \leqslant F(p, n) \leqslant 2(s+1)$ *for* $v = 1, 2, \ldots, s+1$, *and* $a_i = (p-1)/2$ *for* $i = 0, 1, \ldots, v-1$; *finally*,

Case 3. $w = 0$, $v = 0$, $F(p, n) = 2k(0)$ *where* $0 \leqslant F(p, n) \leqslant 2(s+1)$, $0 \leqslant k(0) \leqslant s+1$, *where* $a_0 \neq (p-1)/2$ *or* $p-1$.

Proof. Write (2.0) as

$$p_n/q_n = 2^{4n}(n+1)/(2n+1)^2 \binom{2n}{n}^2 \tag{2.5}$$

and let $r \geqslant 0$ be defined by $p^r \| \binom{2n}{n}$ where $\binom{2n}{n}$ is the binomial coefficient. Some results concerning the prime factors of $\binom{2n}{n}$ have already appeared (see Erdös [2], p. 516; [3], and a result of E. Kummer in 1852, L. E. Dickson [1], Vol. I, p. 270).

Following the procedure of E. Kummer form $2n = \delta_s p^{s+1} + c_s p^s + \ldots + c_1 p + c_0$ with the "carry-over" system

$$2a_0 = \delta_0 p + c_0, \quad \ldots, \quad \delta_{i-1} + 2a_i = \delta_i p + c_i \quad \text{for } i = 1, 2, \ldots, s \text{ and } \delta_i = 0 \text{ or } 1, \tag{2.6}$$

then

$$r = \sum_{i=0}^{s} \delta_i = k(0),$$

i.e. r is the sum of the carry-overs in forming $n+n$ in p-adic form. We may also write

$$\begin{aligned} r &= E(p, 2n) - 2E(p, n) \\ &= \sum_{i=0}^{s} ([2n/p^{i+1}] - 2[n/p^{i+1}]) = \sum_{i=0}^{s} \delta_i = k(0) \end{aligned} \tag{2.7}$$

where $[2n/p] - 2[n/p] = \delta_0$, $[2n/p^2] - 2[n/p^2] = \delta_1$, $\ldots$, $[2n/p^{s+1}] - 2[n/p^{s+1}] = \delta_s$ are the carry-overs of the Kummer system in (2.6). Therefore, the congruence modulo 2 follows at once. If we consider the particular forms of n implied by cases 1, 2, and 3, we can obtain the total carry-overs in forming $2n$ in each case and obtain $F(p, n)$ from the w and v according to the case. In case 1, $n = a_s p^s + a_{s-1} p^{s-1} + \ldots + a_w p^w + (p-1)p^{w-1} + \ldots + (p-1)$ which implies for $s \geqslant w$, $p^w \| (n+1)$ and $p^v \nmid (2n+1)$, and thus, we have $r = \sum_{i=0}^{s} \delta_i = \sum_{i=w}^{s} \delta_i + w = k(w) + w$ when we form $2n(p)$, i.e. we surely have w carry-overs from the $(p-1)$'s and from there on, a variable number according to the magnitude of the digits a_i for $w \leqslant i \leqslant s$ which is denoted by $k(w) = \sum_{i=w}^{s} \delta_i$. Thus, it follows from (2.5) that the exponent of p is given by $F(p, n) = 2\big(k(w)+w\big) - w = 2k(w)+w$. Similarly, for case 2, $n = a_s p^s + a_{s-1} p^{s-1} + \ldots + a_v p^v + (p-1)p^{v-1}/2 + \ldots \ldots + (p-1)/2$, and thus, $p^v \| (2n+1)$ and $p \nmid (n+1)$ which shows that $r = k(v) = \sum_{i=v}^{s} \delta_i$, i.e. there can only be possible carry-overs δ_i from the vth to the sth place in $2n(p)$. So for case 2, we have from (2.5), $F(p, n) = 2k(v)+2v = 2\big(k(v)+v\big)$.

It is merely convenient to state case 3 in the form given, since each $F(p, n)$ of case 1 for $w = 0$ and case 2 for $v = 0$, reduce to $F(p, n) = 2k(0) = 2\sum_{i=0}^{s} \delta_i$. For the indicated bounds on $F(p, n)$ in each case, these depend only on the sum of the carry-overs δ_i in each $k(x)$, i.e. if all $\delta_i = 1$ for each $i = x, x+1, \ldots, s \Rightarrow k(x) = s+1-x$, or if all $\delta_i = 0 \Rightarrow k(x) = 0$, etc. Finally, the bounds on w and v follow from the possible p-adic structures of $n(p)+1$ and $2n(p)+1$ as indicated above. Theorem 1 is now complete.

Let us discuss a number of important consequences of this result. One can classify all the integers $n(p)$ in $p^s \leqslant n \leqslant p^{s+1}-1$ according to the 3 cases and obtain all the possible values for $F(p, n) \geqslant 0$, i.e. they imply specific restrictions on the digits a_i in the p-adic structure of $n(p)$. In Theorem 3, we give explicitly the p-adic conditions for the occurrence of the maximums, $F(p, n) = 2(s+1)$ and minimums, $F(p, n) = 0$ in the interval. These minimums and maximums in $p^s \leqslant n \leqslant p^{s+1}-1$ for certain n values, necessary for this paper, occur only in case 3 and case 2, resp. If $w = 0$, $v = 0$, $k(0) = 0 \Rightarrow F(p, n) = 0$, these imply digit restrictions on $n(p)$, i.e. case $3 \Rightarrow$ no $(p-1)$'s or $(p-1)/2$'s from the right and no carry-overs in forming $2n(p)$. On the other hand, maximums can occur in cases 2 and 3, i.e $k(s+1) = 0$, $v = s+1$ for case 2 and $k(0) = s+1$ for case 3. In fact, each case reveals the requirements on $n(p)$ so that $F(p, n)$ can take on a strictly monotonic increasing sequence from the indicated minimum $F(p, n) = 0$ to the maximum $F(p, n) = 2(s+1)$, i.e. $F(p, n) = 0, 1, \ldots \ldots, 2(s+1)$. Other conclusions can be drawn in reference to even and odd values for $F(p, n)$ in the interval.

The following corollary gives what we might call the "horizontal" structure of the p_n/q_n in (2.1) for a fixed $n \geqslant 1$ as p runs over the odd primes $3, 5, \ldots \leqslant 2n+1$. It shows that in p_n/q_n as n increases without bound every odd prime p_0 will appear for some $2n+1 = p_0$ and have the exponent 2 as long as $n+1 < p_0 \leqslant 2n+1$ for succeeding n values. Then for some sufficiently large N and all $n > N$ such that $3 \leqslant p_0 \leqslant N+1$, the $F(p, n)$ will begin to run over the values $0, 1, \ldots, 2(s+1)$ according to Theorem 1 and continue the pattern for all $s = [\log_{p_0} n]$. Thus, it is clear that the exponent of any chosen odd prime in q_n will become arbitrarily large for some sufficiently large n.

Corollary of Theorem 1. *Consider all consecutive odd primes $p = 3, 5, \ldots \leqslant 2n+1$ for a given $n \geqslant 1$ divided into 2 classes $3 \leqslant p_i \leqslant n+1$ and $n+1 < p_j \leqslant 2n+1$, then $F(p_j, n) = 2$ and $F(p_i, n)$ will have a value chosen from $0, 1, \ldots, 2(s+1)$ depending on the particular n as stated in Theorem 1 where n is contained in some $p_i^s \leqslant n \leqslant p_i^{s+1}-1$ with $s = [\log_{p_i} n] \geqslant 0$.*

Proof. Using (2.2) in the form

$$F(p, n) = 2\{[(2n+1)/p] + [(2n+1)/p^2] + \ldots\} \tag{2.8}$$

or

$$F(p, n) = -3\{[n/p]+[n/p^2]+\ldots\} - \{[(n+1)/p]+[(n+1)/p^2]+\ldots\}$$

and thus $n+1 < p \leqslant 2n+1$ or $1 \leqslant (2n+1)/p < 2 - 1/p \Rightarrow$

$$[(2n+1)/p] = 1 \quad \text{for such } p \text{ and all } [(2n+1)/p^k] = 0 \text{ for } k > 1.$$

Similarly, $\frac{1}{2} - \frac{1}{2}p \leqslant n/p < 1 - 1/p \Rightarrow [n/p^k] = 0$ for $k \geqslant 1$ and also $[(n+1)/p^k] = 0$. Therefore, we have $F(p_j, n) = 2$ for those $p = p_j$.

If the odd primes are such that $3 \leqslant p_i \leqslant n+1$, then $[n/p^k]$ and $[(n+1)/p^k]$ begin to take on non-zero values. Hence, we are in the content of Theorem 1 which implies the possible values $0, 1, \ldots, 2(s+1)$ for $F(p_i, n)$ according to the particular n values as located in $p_i^s \leqslant n \leqslant p_i^{s+1}-1$ for some $s \geqslant 0$ and choice of $p_i \leqslant n+1$. Thus, we obtain the above corollary.

As an example of this result, let $n = 11$, then

$$3 \leqslant p_i \leqslant 12 < p_j \leqslant 23 \;\Rightarrow\; p_i = 3, 5, 7, 11 \quad \text{and} \quad p_j = 13, 17, 19, 23.$$

Thus, $F(p_j, 11) = 2$ for all p_j and $F(3, 11) = 1$, $F(5, 11) = 4$, $F(7, 11) = 2$, $F(11, 11) = 0$, and therefore $p_n/q_n = p_{11}/q_{11} = 2^{40}/3^1 \cdot 5^4 \cdot 7^2 \cdot 11^0 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2$ where $G(2, n) = 4 \cdot 11 - 4 = 40$. Here $F(2, 11) = 1 + 3S(2, 11) + S(2, 12) - 2S(2, 23) = 4$ where $11 = n(2) = 1011$, $n(2)+1 = 1100$, $2n(2)+1 = 10111$, and therefore, $S(2, 11) = 3$, $S(2, 12) = 2$, $S(2, 23) = 4$, resp.

Also, $n = 11$ is located in $3^2 \leqslant 11 \leqslant 3^3-1$ for $s = 2$, and $5^1 \leqslant 11 \leqslant 5^2-1$, $7^1 \leqslant 11 \leqslant 7^2-1$, $11^1 \leqslant 11 \leqslant 11^2-1$ for $s = 1$. We note that $F(5, 11) = 2(1+1) = 4$ is one of the maximums on the range $5^1 \leqslant 11 \leqslant 5^2-1$ for $s = 1$.

Concerning, $G(2, n)$, we have

Theorem 2. *$G(2, n)$ in $p_n/q_n = 2^{G(2,n)}/\prod_{(p)} p^{F(p,n)}$ is a strictly monotonic positive increasing function of n contained in $2^s \leqslant n \leqslant 2^{s+1}-1$ for $s \geqslant 0$*

where $\Delta G(2,n) = G(2,n+1)-G(2,n)$ *is such that* $3 \leqslant \Delta G(2,n) = 2+ +v \leqslant s+3$ *with* $2^v \| (n+1)$ *or* $(n+2)$ *and* $1 \leqslant v \leqslant s+1$ *for consecutive* n *such that* $2^{s+2}-2 \leqslant G(2,n) \leqslant 2^{s+3}-(s+5)$.

Proof. Form $p_{n+1}/q_{n+1} \div p_n/q_n$, and we obtain from (2.1) and the factorials in (2.0)

$$(2.9)\qquad p_{n+1}q_n/q_{n+1}p_n = 2^{\Delta G(2,n)}/\prod_{(p)} p^{\Delta F(p,n)} = \frac{2^2(n+1)(n+2)}{(2n+3)^2}$$

where $\Delta F(p,n) = F(p,n+1)-F(p,n)$ for any odd prime p.

Hence, we can write

$$(2.10)\qquad 2^{\Delta G(2,n)} \| 2^2(n+1)(n+2)$$

where $\Delta G(2,n)$ is the max power of 2 which divides $2^2(n+1)(n+2)$. Thus, it is clear that $\Delta G(2,n) = 2+v$ where $2^v \| (n+1)$ or $(n+2)$. Since either $(n+1)$ or $(n+2)$ is even but not both, and $(2n+3)$ is always odd, we have $1 \leqslant v \leqslant s+1$ where the upper bound $s+1$ is due to $n = 2^{s+1}-1$, or $n+1 = 2^{s+1}$, but $n+2 = 2^{s+1}+1$ is odd. Hence $3 \leqslant \Delta G(2,n) \leqslant s+3$ which establishes the strictly monotonic positive nature of $G(2,n)$ over consecutive n values. To complete the theorem, we have $4\cdot 2^s - F(2,2^s) \leqslant G(2,n) \leqslant 4(2^{s+1}-1)-F(2,2^{s+1}-1)$ and $S(2,2^s) = 1$, $S(2,2^s+1) = 2$, $S(2,2\cdot 2^s+1) = 2$, i.e. by (2.2) $F(2,2^s) = 2$. A similar calculation shows that $F(2,2^{s+1}-1) = s+1$, and therefore, the upper bound on $G(2,n)$ is $2^{s+3}-4-s-1 = 2^{s+3}-(s+5)$. ■

In passing, we note that (2.9) also implies some interesting additional information concerning $\Delta F(p,n)$ but this is not germane to this paper.

The following result gives the restrictions on the coefficients a_i (or "digits") in the p-adic representation of n so that $F(p,n)$ achieves either its minimum zero or maximum $2(s+1)$ for any $s \geqslant 0$. We require this result in the next section and in (j,ε)-normal considerations.

THEOREM 3. *Consider the* p*-adic form of every* $n \in [p^s, p^{s+1}-1]$ *where* $n(p) = \sum_{i=0}^{s} a_i p^i$, $1 \leqslant a_s \leqslant p-1$, $0 \leqslant a_i \leqslant p-1$ *for* $i = 0,1,\ldots,s-1$ *for any odd prime* p *and* $s \geqslant 0$, *then*

(a) (i) $F(p,n) = 0$ *for a total of* $((p+1)/2)^{s-1}((p-1)/2)^2$ *values of* n *in the lower-half interval* $[p^s, (p^{s+1}-3)/2]$ *for* $s \geqslant 2$, $1 \leqslant a_s \leqslant (p-1)/2$, $0 \leqslant a_i \leqslant (p-1)/2$ *for* $i = 0,1,\ldots,s-1$,

(ii) $F(p,n) = 0$ *for* $s = 1$, $1 \leqslant a_1 \leqslant (p-1)/2$, $0 \leqslant a_0 \leqslant (p-3)/2$,

(iii) $F(p,n) = 0$ *for* $s = 0$ *and* $1 \leqslant a_0 \leqslant (p-3)/2$,

(b) $F(p,n) = 2(s+1)$, *the maximum, for* $((p+1)/2)^s(p-1)/2$ *values of* n *in the upper-half interval* $[(p^{s+1}-1)/2, p^{s+1}-2]$ *with* $(p-1)/2 \leqslant a_i \leqslant p-1$ *for* $i = 1,2,\ldots,s$ *and* $(p-1)/2 \leqslant a_0 \leqslant p-2$ *for* $s \geqslant 0$.

Proof. As we said earlier, $F(p,n) = 0$ only from case 3 in Theorem 1, therefore, $k(0) = \sum_{i=0}^{s} \delta_i = 0$, i.e. no "carry-overs" in forming $2n$, thus $F(p,n) = 0$. It is also clear that we must require that $a_0 \neq (p-1)/2$ since $v = 0$, i.e. $p \leqslant (2n+1)$. The other conditions on the a_i follow at once.

By constructing the least $n = p^s$ such that $F(p,p^s) = 0$ and the largest $n = (p-1)p^s/2 + \ldots + (p-1)/2 - 1 = (p^{s+1}-3)/2$ such that $F(p,(p^{s+1}-3)/2) = 0$, we see that they all fall in the lower half interval.

For the enumeration, in each of the $s-1$ place positions, we can have at most $(p+1)/2$ choices for the a_i for $i = 1,2,\ldots,s-1$ and in the first place and the last $(p-1)/2$ choices, therefore, a total of $((p+1)/2)^{s-1} \times \times ((p-1)/2)^2$ values of n occur in the lower half such that $F(p,n) = 0$.

Part (b) follows by completely analogous arguments using only case 2 and 3 of Theorem 1 assuring that the maximum number of carry-overs occur which leads to the stated restrictions on the digits in the p-adic form of n such that $F(p,n) = 2(s+1)$. ■

3. $F(5,n) = 0$ **and** $F(q,n) = 2(s+1)$**.** In this section, we prove that there exists an infinite sub-sequence of strictly increasing positive integers n such that $F(5,n) = 0$ and $F(q,n) = 2(s+1)$ where $q \neq 5$ and s can increase without bound. Since the powers of 2 are confined to p_n in p_n/q_n and 5 is not contained in q_n over this particular sub-sequence of n-values, we have a sequence of rational approximations to $\pi/4$ for which, the periods begin immediately at the decimal point in the base 10 representation of p_n/q_n.

In 1975, P. Erdös et al. [3] proved that if A, B, p and q are positive integers such that $A/(p-1)+B/(q-1) \geqslant 1$, then there exists infinitely many integers whose base p representation has all digits $\leqslant A$ and whose base q expansion has all digits $\leqslant B$. According to the requirements of Theorem 3, we need to prove there exists infinitely many positive integers such that in base p all digits are $\leqslant A$ and in base q, all digits are $\geqslant B$ where later, we set $A = (p-1)/2$ and $B = (q-1)/2$ with p and q distinct odd primes.

First, an extension of notation, let $(p, \leqslant A)$-G denote that a number in base p is "$(p, \leqslant A)$-good", i.e. all digits are $\leqslant A$, and if $(p, \geqslant A)$-G, then it is "$(p, \geqslant A)$-good", etc. Thus for our purposes, we will show there exists infinitely many positive integers which are $(5, \leqslant 2)$-G and $(q, \geqslant (q-1)/2)$-G in base $q \neq 5$ where q is an odd prime.

To be even more precise according to Theorem 3, this means we can find an infinite sequence of over-lapping intervals, i.e. $I_5 = \{5^{s_1} \leqslant n \leqslant (5^{s_1+1}-3)/2\}$, the lower half and the upper half $I_q = \{(q^{s_2+1}-1)/2 \leqslant n \leqslant q^{s_2+1}-2\}$ for any fixed odd prime $q \neq 5$ where $n \in I_5 \cap I_q$ such that

$F(5, n) = 0$ and $F(q, n) = 2(s_2+1)$. For example, let $s_1 = 3$, $s_2 = 2$, $q = 7$, then $I_5 = \{5^3 \leqslant n \leqslant (5^4-3)/2\} = \{125 \leqslant n \leqslant 311\}$ and $I_7 = \{171 \leqslant n \leqslant 341\}$. We find five "solutions" in $I_5 \cap I_7 = [171, 311]$, i.e. $n = 180$, 185, 186, 276, 285, which have the required digit structure in base 5 and base 7. For instance, $n = 180 = 1210_5 = 345_7$ where n is $(5, \leqslant 2)$-G and $(7, \geqslant 3)$-G for $A = (5-1)/2 = 2$ and $B = (7-1)/2 = 3$, resp. Furthermore, $F(5, 180) = 0$, $F(7, 180) = 2(2+1) = 6$ for $s_2 = 2$ for the powers of the primes 5 and 7 in the denominator of the partial product p_{180}/q_{180} of (2.1).

Even though, we have been assured by D. H. Lehmer (quoting Erdös) and R. Graham (personal correspondance) that the particular proof as given in ([3], pp. 84–86), will go through for the case desired here, i.e. the one change from $(q, \leqslant B)$-G to $(q, \geqslant B)$-G, we present the following independent proof which differs from [3] in a number of salient points.

THEOREM 4. *Let A and B be positive integers satisfying $B/(q-1) \leqslant A/(p-1) \leqslant 1$ where p and q are distinct odd primes, then there exists infinitely many integers whose base p expansion is $(p, \leqslant A)$-G and base q expansion is $(q, \geqslant B)$-G.*

Proof. Since $\log p$ and $\log q$ are incommensurable, there exists infinitely many exponents α and β so that

$$B(q^\beta-1)/(q-1) < Rp^\alpha < q^\beta-1 \tag{3.0}$$

where $R = 1, 2, \ldots, p-1$. In other words, the base q expansion of Rp^α has either all digits $\geqslant B$ or has a digit $> B$ preceding any digit $< B$ where B is some fixed choice of $1, 2, \ldots, p-1$. For example, we have $3(7^3-1)/6 < 2\cdot 5^3 = 505_7 < 7^3-1$ with $p = 5$, $\alpha = 3$, $R = 2$, $q = 7$, $\beta = 3$, and $B = 3$.

Our proof follows the basic ideas in that of [3], but with some alterations. We shall define the so-called *tail* T for N_q (the *tail* is essentially an unacceptable sequence of digits in N_q or we could say, "$(q, \geqslant B)$-NG"), i.e. we have

$$N = a_n p^n + \ldots + a_m p^m = b_r q^r + \ldots + b_i q^i + T \tag{3.1}$$

where the tail $T = b_{i-1}q^{i-1} + \ldots + b_1 q + b_0$ is such that $b_{i-1} \leqslant B-1$ and all the b_k in $N-T$ are acceptable, i.e. $b_k \geqslant B$ for $k = i, i+1, \ldots, r$. In [3], digits which are $= B$ can be part of a tail, i.e. $B(q^i-1)/(q-1)+1 \leqslant T \leqslant q^i-1$, ([3], p. 85, top). For example according to this choice, if $q = 7$, $B = 3$, and $i = 2$, then we have $434, 435, \ldots, 466$ where the tails are $34, 35, \ldots, 66$ or $25 \leqslant T \leqslant 48 \Rightarrow 34_7 \leqslant T \leqslant 66_7$ which contain the acceptable digit 3 in the tails.

We seek those N_q which are $(q, \geqslant B)$-G, and we might proceed by analogy with tails such that $0 \leqslant T \leqslant B(q^i-1)/(q-1)$, for our case, i.e. for the case above, in $400, 401, \ldots, 430, 431, 432$; tails $T = 30, 31, \ldots, 32$

would be unacceptable according to this analogous choice. However, we want to keep *all* acceptable digits $\geqslant B = 3$, so for tails, we would choose $00, 01, \ldots, 26$ and keep 430, 431, 432, with tails $T = 0, 1, 2$.

With this in mind, we define our tails

$$0 \leqslant T \leqslant (B-1)q^{i-1} + (q-1)q^{i-2} + \ldots + (q-1) = Bq^{i-1}-1 \tag{3.2}$$

and we seek those $U < p^m$ in $N^* = (N+U)_q$ such that

$$B(q^i-1)/(q-1) - T \leqslant U \leqslant q^i-1-T. \tag{3.3}$$

This infers that $N^* = (N+U)_q$ is such that

$$\begin{aligned} b_r q^r + \ldots + b_i q^i + Bq^{i-1} + \ldots + B \\ \leqslant N^* \leqslant b_r q^r + \ldots + b_i q^i + (q-1)q^{i-1} + \ldots + (q-1). \end{aligned} \tag{3.4}$$

As our first "modification" (similar to the requirement in ([3], p. 84), in "proof of lemma"), we require that

$$T = b_{i-1}q^{i-1} + \ldots + b_1 q + b_0 < q^i < S = p^m - \frac{A(p^m-1)}{p-1} < p^m. \tag{3.5}$$

Essentially, this means determine the tail by (3.2) with $T < q^i < S$ for our initial choice of N. Clearly this means that (3.1) is such that $N \equiv T \bmod q^i$ and $N \equiv 0 \bmod p^m$, i.e. our initial $N = p^m q^i C$ where $(C, pq) = 1$. In this direction, it is not difficult to prove the useful result:

LEMMA. *Let $B(q^{r+1}-1)/(q-1) < a_n p^n < q^{r+1}-1$ where $a_n = 1, 2, \ldots, A$, then for all $N = p^m q^i C \in \left[B\left(\frac{q^{r+1}-1}{q-1}\right), q^{r+1}-1\right]$, we have*

$$q^i < p^m \leqslant p^n \quad \textit{where} \quad 0 \leqslant i \leqslant r = \left[\frac{n \log p}{\log q}\right].$$

This characterizes "starting values" or initial choices for N which will guarantee $(p, \leqslant A)$-G and $(q, \geqslant B)$-G numbers for some N^* when using the U-values contained in (3.6) to follow.

Using the "Fact" (see [3], p. 85) that there exists at least one $(p, \leqslant A)$-G number in the range $\left[x, \left(\frac{p-1}{A}\right)x\right]$, we require that

$$x = B\left(\frac{q^i-1}{q-1}\right) - T \leqslant U \leqslant \left(\frac{p-1}{A}\right)\left(B\left(\frac{q^i-1}{q-1}\right) - T\right) \leqslant q^i-1-T. \tag{3.6}$$

In (3.6), we have

$$\left(\frac{p-1}{A}\right)\left(B\left(\frac{q^i-1}{q-1}\right) - T\right) \leqslant \left(\frac{p-1}{A}\right)\left(\frac{B}{q-1}\right)(q^i-1) < p^m-1$$

which holds if $q^i < p^m$ as assured by (3.5) and

$$\left(\frac{p-1}{A}\right)\left(\frac{B}{q-1}\right) \leqslant 1 \quad \text{or} \quad \frac{B}{q-1} \leqslant \frac{A}{p-1}.$$

Finally, we want

$$\left(\frac{p-1}{A}\right)\left(B\left(\frac{q^i-1}{q-1}\right)-T\right) \leqslant q^i-1-T$$

in (3.6) which can be written

$$\frac{\big(B/(q-1)-A/(p-1)\big)(q^i-1)}{1-A/(p-1)} \leqslant T \leqslant Bq^{i-1}-1. \tag{3.7}$$

Since $B/(q-1) \leqslant A/(p-1) \Rightarrow B/(q-1)-A/(p-1) \leqslant 0$, it is clear that (3.6) will surely hold for $0 \leqslant T \leqslant B(q^{i-1}-1)$ as required in (3.2) if we further require that $1-A/(p-1) > 0$. For the case $A = p-1$, returning to (3.6), we have for any T, $B(q^i-1)/(q-1)-T \leqslant q^i-1-T$ if $B/(q-1) \leqslant 1$. Therefore, all requirements are satisfied if we assume that $B/(q-1) \leqslant A/(p-1) \leqslant 1$. Theorem 4 follows by a finite number of modifications. ∎

In the following theorem, we establish the (j, ε)-normality of the p_n/q_n in base $g = 2^\alpha \cdot p^\beta$ in such a way that the representation in this base has no non-periodic part, i.e. the expansion of the Wallis partial product representation of $\pi/4$ begins directly at the "decimal point" or g^{-1}, i.e. is "purely periodic". If $\alpha = 1$, $\beta = 1$, and $p = 5$, then we have the expansion in base 10 and can make our statements about the Brouwer conjecture which concerns the base 10 representation of $\pi/4$.

THEOREM 5. *There exists an infinite sequences of positive integers n such that the associated partial Wallis infinite products* $p_n/q_n = \prod_{i=1}^{n}\big(1-1/(2i+1)^2\big)$ *are (j, ε)-normal and purely periodic in base $g = 2^\alpha \cdot p^\beta$ where p is any fixed odd prime.*

Proof. The fraction $p_n/q_n < 1$ in lowest terms is (j, ε)-normal in base $g = 2^\alpha \cdot p^\beta$ (i.e. it is a Type A fraction (see [4], Def. Type A, p. 229 and Th. 6, p. 233)), if we can show that there exists at least one odd prime $q \mid q_n$ such that its exponent $F(q, n) > Z(q)+s(q)$ where $Z(q) \geqslant 1$ is fixed for some choice of odd prime $q \neq p$ and g such that $q^{Z(q)} \| (g^{d(q)}-1)$, and $q^{s(q)} \| \big(d(p_j),\ d(p_{j+1}), \ldots, d(p_M)\big)$ where $q < p_j < p_{j+1} < \ldots < p_M \leqslant 2n+1$ contained in q_n. (The notation $b^y \| (a, b, c, \ldots, T)$ denotes the maximum exponent of b which divides $a, b, c, \ldots, T$, an extension of the usual symbol.) Consider the p_n/q_n defined over the sequence of n-values which we showed in Theorem 4 such that $F(p, n) = 0$ and $F(q, n) = 2(s+1)$ where $q^s \leqslant n \leqslant q^{s+1}-1$. For these n, the odd prime factor $p^{F(p,n)} = p^0 = 1$ in q_n and q attains to the maximum $q^{2(s+1)}$ in q_n.

Since at most $d(p_M) = p_M-1 = 2q^{s(q)} \leqslant n+1$ which implies $q^{s(q)} \leqslant n$, and noting that $q^s \leqslant n \leqslant q^{s+1}-1$, we see that at most $s(q) = s$, or $0 \leqslant s(q) \leqslant s$. Therefore, we can write the requirement for (j, ε)-normality here as $Z(q)+s(q) \leqslant s+Z(q) \leqslant F(q, n) = 2(s+1)$ or $Z(q) \leqslant s+2$ which will always hold for any fixed $Z(q) \geqslant 1$ by some sufficiently large initial s_0, i.e. choice of n_0, such that subsequently $s \geqslant s_0$ for successively larger $n \geqslant n_0$. This implies that for some initial p_{n_0}/q_{n_0} we have (j, ε)-normality and subsequently for all larger n-values in the infinite subsequence of $p_n/q_n \to \pi/4$ such that $F(p, n) = 0$ and $F(q, n) = 2(s+1)$. ∎

If we set $\alpha = 1$, $\beta = 1$, and $p = 5$, then the results here show that there exists an infinite convergent subsequence of p_n/q_n for those n contained in $q^s \leqslant n \leqslant q^{s+1}-1$ for some fixed $q \neq 5$, as s increases without bound such that for each n, $F(5, n) = 0$ and $F(q, n) = 2(s+1)$ in the denominator of each q_n where $s = [\log_q n]$.

In other words, this means we have obtained a convergent sequence of (j, ε)-normal Type A rational fractions such that their representation in base 10 converges toward $\pi/4$ beginning immediately at the decimal point. Since we have proved in ([5], pp. 377–378, Th. 2 & 4), that sets of digits within the period of a (j, ε)-normal Type A rational fraction slightly greater than the square root of the period length, i.e. $\big(\omega(q_n)\big)^{1/2+\varepsilon}$ are (j, ε)-normal, it follows that the block 0123456789 *will occur* within such a set of digits beginning at the decimal point, i.e. within the approximation to $\pi/4$.

Unfortunately, at the present time, we cannot say whether the block 0123456789 is in the "stable" portion of the approximation to $\pi/4$, i.e. the part of the period which does not change as n increases, or the portion of the period which changes as n increases without bound.

The methods presented here can also be applied to an infinite product representation of an algebraic irrational. For example, in the infinite product

$$\cos\theta = \prod_{k=1}^{\infty}\big(1-4\theta^2/(2k-1)^2\pi^2\big) \tag{3.8}$$

set $\theta = \pi/4$, hence $4\theta^2/\pi^2 = 1/2^2$. By filling in factorials in the successive products, we may show that

$$p_n/q_n = (4n-1)!\,[(n-1)!]^2/2^{2n}[(2n-1)!]^3 \tag{3.9}$$

where $\lim_{n\to\infty} p_n/q_n = \sqrt{2}$ for $n \geqslant 1$. If $p_n/q_n = \prod_{(p)} p^{F(p,n)}$, we would have the "exponent functions" $F(p, n)$ to study

$$F(p, n) = E(p, 4n-1)+2E(p, n-1)-3E(p, 2n-1) \tag{3.10}$$

when p is an odd prime, and

$$(3.11)\qquad F(2, n) = E(2, 4n-1)+2E(2, n-1)-3E(2, 2n-1)-2n$$

for $p = 2$. From (3.11), one can see that p_n is odd and q_n is even, i.e. the powers of 2 are strictly monotonic increasing in the denominator q_n. Also, the associated p-adic sum functions based on $S(p, n)$ can be used.

It is interesting to speculate on what distinguishes the algebraic from the transcendental irrationals based on the infinite product representation which leads to the related exponent functions $F(p, n)$ based on the factorial functional forms. These clearly determine the behavior of the primes in the prime decomposition of the successive rational approximations p_n/q_n to the limit of the sequence, i.e. in general, a rational, or an algebraic or transcendental irrational.

Perhaps, a Diophantine result, such as the Thue–Siegel–Roth theorem could specify restrictions on the exponential function $F(p, n)$ in the factorial representation of the associated partial products so that the limit number of an infinite product is algebraic or transcendental.

Soon, we shall present results concerning some of the conjectured ([4], p. 230) (j, ε)-normal sets in the periods of certain Type B rational fractions (see Def. 1, p. 229), i.e. those positive integers $m = \prod_{i=1}^{r} p_i^{n_i}$ where $g^{d_i} \equiv 1 \bmod p_i^{z_i}$, $d_i \mid (p_i - 1)$, $p_i^{s_i} \| (d_{i+1}, \ldots, d_r)$, and $n_i \leqslant z_i + s_i$. This will include cases which involve nth power residues.

Finally, there appears to be two possible avenues for further work on the question studied in this paper which, of course, relates ultimately to proving that given irrationals are normal numbers. Find a convergent sequence of Type A rational fractions, i.e. (j, ε)-normal, which has a rate of convergence such that the stable set of digits exceeds the square root of the period length, or improve the bounds on the set of digits within a period which are (j, ε)-normal such that these sets are within the stable portion of the approximations.

If such sequences can be determined, we could prove by the procedures in this paper that the limit number of such a convergent sequence of rational approximations is a normal number.

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