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Ramanujan expansions of multiplicative functions

by

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1. Introduction. We propose a proof of the following

THEOREM. *Let $f\colon \mathbf{N}\to \mathbf{C}$ be multiplicative with $f(1)=1$. Let f fulfill the conditions*

(1) $$\sum_p p^{-1}\big(f(p)-1\big) \quad \textit{converges},$$

(2) $$\sum_{\substack{p\\ |f(p)-1|\leqslant 1}} p^{-1}|f(p)-1|^2<\infty,$$

(3) $$\sum_{\substack{p\\ |f(p)-1|>1}} p^{-1}|f(p)-1|<\infty,$$

(4) $$\sum_p\sum_{k=2}^{\infty} p^{-k}|f(p^k)|<\infty.$$

Then the Ramanujan coefficients

$$a_q(f):=\frac{1}{\varphi(q)}\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x} f(n)c_q(n)$$

exist for all q and

$$\sum_{q=1}^{\infty} a_q(f)c_q(n)=f(n) \quad \textit{for all } n.$$

If f fulfills (1) *and the stronger conditions*

(5) $$\sum_p p^{-1}|f(p)-1|^2<\infty,$$

(6) $$\sum_p\sum_{k=2}^{\infty} p^{-k}|f(p^k)|^2<\infty,$$

then

$$\sum_{q=1}^{\infty} \varphi(q)|a_q(f)|^2 = \lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} |f(n)|^2.$$

The first statement is not new. It is to be found in an expository article of H. Delange ([2], Théorème 1).

F. Tuttas ([6]) proved both conclusions under the stronger assumptions:

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} f(n) \quad \text{exists and is } \neq 0, \tag{7}$$

$$\sum_{n\leqslant x} |f(n)|^2 = O(x). \tag{8}$$

These are stronger indeed since by the "first direction" of Elliott's mean value theorem ([3], [1]) (7), (8) imply (1), (5), (6).

The proof we present here rests entirely on the "second direction" of a more recent mean-value theorem by K.-H. Indlekofer and of Elliott's mean-value theorem (see Lemma 1 below).

Notation. If some $g\colon \mathbf{N}\to\mathbf{C}$ is given then we put $\tilde g := g*\mu$ (the convolution of g with the Möbius function μ) and

$$M(g) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} g(n) \quad \text{if this limit exists.}$$

2. Preparatory lemmas. The function $f\colon \mathbf{N}\to\mathbf{C}$ we consider is multiplicative with $f(1) = 1$.

LEMMA 1. *If f fulfills* (1), (2), (3), (4) *then $M(f)$ exists and is given by*

$$M(f) = \prod_p \Big(1-\frac{1}{p}\Big)\Big(1+\sum_{k=1}^{\infty} p^{-k} f(p^k)\Big).$$

If f fulfills (1), (5), (6) *then $M(|f|^2)$ exists and is given by*

$$M(|f|^2) = \prod_p \Big(1-\frac{1}{p}\Big)\Big(1+\sum_{k=1}^{\infty} p^{-k} |f(p^k)|^2\Big).$$

Proof. The first conclusion follows from the fact that (2), (3) are equivalent to the conditions

$$\sum_{\substack{p\\ |f(p)|\leqslant 3/2}} p^{-1}|f(p)-1|^2 < \infty \quad \text{and} \quad \sum_{\substack{p\\ ||f(p)|-1|>1/2}} p^{-1}|f(p)| < \infty$$

combined with the main theorem of [4].

The second conclusion follows from Elliott's mean-value theorem ([3]).

LEMMA 2. *If f fulfills* (3), (4) *then*

$$\sum_p \sum_{k=2}^{\infty} p^{-k} |\tilde f(p^k)| < \infty.$$

Proof. This expression is

$$\leqslant 2\sum_p \sum_{k=2}^{\infty} p^{-k}|f(p^k)| + 2\sum_p p^{-2} + \sum_{\substack{p\\ |f(p)-1|>1}} p^{-2}|f(p)-1|.$$

LEMMA 3. *If f fulfills* (2), (3), (4) *then $M(|\tilde f|) = 0$.*

Proof. If Q denotes the set of all squarefree numbers and S the set of all squarefull numbers one has

$$x^{-1}\sum_{n\leqslant x} |\tilde f(n)| \leqslant \sum_{\substack{s\in S\\ s\leqslant x}} s^{-1}|\tilde f(s)|(x/s)^{-1} \sum_{\substack{q\in Q\\ q\leqslant x/s}} |\tilde f(q)|.$$

Therefore it is sufficient to show that

$$\sum_{s\in S} s^{-1}|\tilde f(s)| < \infty \quad \text{and} \quad \lim_{y\to\infty} y^{-1} \sum_{\substack{q\in Q\\ q\leqslant Y}} |\tilde f(q)| = 0.$$

The former follows from

$$\sum_{s\in S} s^{-1}|\tilde f(s)| \leqslant \exp\Big(\sum_p \sum_{k=2}^{\infty} p^{-k}|\tilde f(p^k)|\Big)$$

and Lemma 2. Now we turn to the second statement.

Let A, B consist of 1 and those squarefree naturals in whose canonical product all primes fulfill $|\tilde f(p)| \leqslant 1$, $|\tilde f(p)| > 1$. Then

$$y^{-1}\sum_{\substack{q\in Q\\ q\leqslant y}} |\tilde f(q)| = \sum_{\substack{b\in B\\ b\leqslant y}} b^{-1}|\tilde f(b)|(y/b)^{-1} \sum_{\substack{a\in A\\ a\leqslant y/b}} |\tilde f(a)|.$$

Therefore it is sufficient to show

$$\sum_{b\in B} b^{-1}|\tilde f(b)| < \infty \quad \text{and} \quad \lim_{z\to\infty} z^{-1} \sum_{\substack{a\in A\\ a\leqslant z}} |\tilde f(a)| = 0.$$

The former follows from

$$\sum_{b\in B} b^{-1}|\tilde f(b)| \leqslant \exp\Big(\sum_{p\in B} p^{-1}|\tilde f(p)|\Big)$$

and (3). Now we turn to the latter statement.

By (2) one has

$$\sum_{a\in A} a^{-1}|\tilde f(a)|^2 \leqslant \exp\Big(\sum_{p\in A} p^{-1}|\tilde f(p)|^2\Big) < \infty.$$

From this we conclude

$$\sum_{\substack{a\in A\\ a\leqslant z}} |\tilde f(a)|^2 = o(z)$$

and this implies what is required.

LEMMA 4. *If f fulfills* (2), (3), (4) *then*

$$M(f) \text{ exists} \Leftrightarrow \sum_{n=1}^{\infty} n^{-1}\tilde f(n) \text{ converges}$$

and both numbers are equal.

This follows from Lemma 3 and

$$x^{-1}\sum_{n\leqslant x} f(n) = \sum_{d\leqslant x} d^{-1}\tilde f(d) + O\Big(x^{-1}\sum_{d\leqslant x}|\tilde f(d)|\Big).$$

LEMMA 5. *Let f fulfill* (1), (2), (3), (4) *and let* $(m, q) = 1$. *Then*

$$S(f;\, m, q) := \sum_{\substack{n=1\\ (n,m)=1\\ n\equiv 0(q)}}^{\infty} n^{-1}\tilde f(n)$$

exists and is given by

$$S(f;\, m, q) = \prod_{p\nmid mq} \eta_0(f;\, p) \prod_{p^k\| q} \eta_k(f;\, p)$$

where we put

$$\eta_k(f;\, p) := \sum_{j=k}^{\infty} p^{-j}\tilde f(p^j) \qquad (k = 0, 1, \ldots).$$

Proof by mathematical induction with respect to the number of different prime factors of q (compare with [5], p. 30). We start with $q = 1$.

Let the multiplicative function f_m be given by

$$f_m(p^k) = \begin{cases} f(p^k) & \text{if } p\nmid m,\\ 1 & \text{else.}\end{cases}$$

Then

$$\tilde f_m(n) = \begin{cases} \tilde f(n) & \text{if } (n, m) = 1,\\ 0 & \text{else.}\end{cases}$$

Since f_m fulfills (1), (2), (3), (4), too Lemmas 1, 4 imply

$$\prod_{p\nmid m} \eta_0(f;\, p) = \prod_p \eta_0(f_m;\, p) = M(f_m) = \sum_{n=1}^{\infty} n^{-1}\tilde f_m(n) = \sum_{\substack{n=1\\ (n,m)=1}}^{\infty} n^{-1} f(n).$$

Now let $q = \bar q r^a$ where $(q, m) = 1$ and r is prime, $r\nmid \bar q$, and $a\in \mathbf{N}$. One has

$$\sum_{\substack{n\leqslant x\\ (n,m)=1\\ n\equiv 0(q)}} n^{-1}\tilde f(n) = \sum_{\substack{n\leqslant x\\ (n,m)=1\\ n\equiv 0(\bar q)}} n^{-1}\tilde f(n) - \sum_{\beta=0}^{a-1} r^{-\beta}\tilde f(r^\beta) \sum_{\substack{h\leqslant xr^{-\beta}\\ (h,mr)=1\\ h\equiv 0(\bar q)}} h^{-1}\tilde f(h).$$

From this and the induction hypothesis we infer that $S(f;\, m, q)$ exists and the recursion formula

$$S(f;\, m, q) = S(f;\, m, \bar q) - S(f;\, mr, \bar q) \sum_{\beta=0}^{a-1} r^{-\beta}\tilde f(r^\beta)$$

from which we deduce the formula for S.

LEMMA 6. *If f fulfills* (1), (2), (3), (4) *and* $(a, q) = 1$ *then*

$$\lim_{x\to\infty} x^{-1}\sum_{n\leqslant x} f(n)\, e\Big(\frac{a}{q}n\Big) = S(f;\, 1, q).$$

This follows from

$$x^{-1}\sum_{n\leqslant x} f(n)\, e\Big(\frac{a}{q}n\Big) = \sum_{\substack{d\leqslant x\\ d\equiv 0(q)}} d^{-1}\tilde f(d) + O\Big(\frac{q}{x}\sum_{d\leqslant x}|\tilde f(d)|\Big)$$

together with Lemmas 5, 3.

3. Proof of the first part of the theorem. From Lemmas 6, 5 and

$$\frac{1}{\varphi(q)}\frac{1}{x}\sum_{n\leqslant x} f(n)\, c_q(n) = \frac{1}{\varphi(q)}\sum_{\substack{a=1\\ (a,q)=1}}^{q} \frac{1}{x}\sum_{n\leqslant x} f(n)\, e\Big(\frac{a}{q}n\Big),$$

we see that $a_q(f)$ exists and is given by

$$a_q(f) = \prod_{p\nmid q} \eta_0(f;\, p) \prod_{p^k\| q} \eta_k(f;\, p).$$

Lemma 3 and (3) imply $\eta_1(f;\, p)\to 0$ for $p\to\infty$. Therefore there is some $p_0 = p_0(f)$ such that $|\eta_0(f;\, p)| \geqslant 1/2$ for all $p \geqslant p_0$. In particular the set $\mathscr{P} := \{p|\ \ \eta_0(f;\, p) = 0\}$ is finite. By P we denote the product of all $p\in\mathscr{P}$.

Then we have $a_q(f) = 0$ for $q \not\equiv 0(P)$. Therefore we have to establish

$$\sum_{\substack{q=1\\ q\equiv 0(P)}}^{\infty} a_q(f)c_q(n) = f(n).$$

We put

$$\eta(p) := \begin{cases} \eta_0(f;p) & \text{if } p \notin \mathscr{P}, \\ 1 & \text{else} \end{cases}$$

and obtain

$$a_q(f) = \prod_p \eta(p) \prod_{p^k \| q} (\eta(p))^{-1}\eta_k(f;p) \quad \text{for } q \equiv 0(P).$$

If for n fixed we define G_n: $N \to C$ by

$$G_n(q) := qc_q(n) \prod_{p^k\|q} (\eta(p))^{-1}\eta_k(f;p),$$

we must show

$$\Big(\prod_p \eta(p)\Big) \sum_{\substack{q=1\\ q\equiv 0(P)}}^{\infty} q^{-1}G_n(q) = f(n).$$

If we define F_n: $N \to C$ by

$$F_n(h) := \sum_{q|h} G_n(q)$$

then one has $\tilde{F}_n = G_n$. Assume that

(*) $\qquad F_n$ fulfills (1), (2), (3), (4).

Then Lemma 5 yields

$$\sum_{\substack{q=1\\ q\equiv 0(P)}}^{\infty} q^{-1}G_n(q) = \prod_{p\nmid P} \eta_0(F_n;p) \prod_{p|P} \eta_1(F_n;p).$$

Therefore

$$\Big(\prod_p \eta(p)\Big) \sum_{\substack{q=1\\ q\equiv 0(P)}}^{\infty} q^{-1}G_n(q) = \prod_{p\nmid P} \eta(p)\eta_0(F_n;p) \prod_{p|P} \eta(p)\eta_1(F_n;p)$$
$$= \prod_p \Big(\eta_0(f;p) + \sum_{j=1}^{\infty} c_{p^j}(n)\eta_j(f;p)\Big).$$

If $\varepsilon = \varepsilon_n(p)$ is given by $p^{\varepsilon}\| n$ one has

$$c_{p^j}(n) = \begin{cases} 0 & \text{for } j > \varepsilon+1, \\ -p^{\varepsilon} & \text{for } j = \varepsilon+1, \\ p^j - p^{j-1} & \text{for } j \leqslant \varepsilon \end{cases} \tag{9}$$

from which it follows that the factors in the product above equal $f(p^{\varepsilon})$.

Thus it remains to verify (*).

If G: $N \to C$ and F: $N \to C$ are given by

$$G(q) := q\mu(q) \prod_{p^k\|q} \frac{\eta_k(f;p)}{\eta(p)} \quad \text{and} \quad F(h) := \sum_{q|h} G(q)$$

then $G_n(q) = G(q)$ for $(q, n) = 1$ and therefore $F_n(h) = F(h)$ for $(h, n) = 1$.

Assume that

(**) $\qquad F$ fulfills (1), (2), (3), (4).

Then F_n fulfills (1), (2), (3) of course and (4) also since $|F_n(p^k)|$ because of (9) can be bounded by a quantity which does not depend on k. Now we prove (**).

We may assume that $p \geqslant p_0$. Then

$$F(p^k) = 1 - p\frac{\eta_1(f;p)}{\eta_0(f;p)} \ \forall k \quad \text{and} \quad |\eta_0(f;p)| \geqslant 1/2.$$

Proof that F fulfills (1). We have

$$-\frac{\tilde{F}(p)}{p} = \frac{\tilde{f}(p)}{p} + r(p) \quad \text{with } r(p) := \eta_0^{-1}\Big(\eta_2 + \frac{\tilde{f}(p)}{p}\eta_1\Big).$$

We have

$$\sum_p |r(p)| \ll \sum_p \big(|\eta_2| + p^{-2}|\tilde{f}(p)|^2\big) < \infty$$

by Lemma 2 and (3).

Proof that F fulfills (2), (3). From $\tilde{F}(p) = -p\eta_0^{-1}\eta_1 = -p(1+\eta_1)^{-1}\eta_1$ we derive

$$|\tilde{F}(p)| \ll p|\eta_1| \leqslant |\tilde{f}(p)| + p|\eta_2|$$

and the implications

$$|\tilde{F}(p)| \leqslant 1 \Rightarrow |\eta_1| \leqslant \frac{1}{p-1} \Rightarrow |\eta_2| \leqslant \frac{1}{p-1} + \frac{|\tilde{f}(p)|}{p},$$

$$|\tilde{F}(p)| > 1 \Rightarrow |\eta_1| > \frac{1}{p+1} \Rightarrow \frac{|\tilde{f}(p)|}{p} + |\eta_2| > \frac{1}{p+1}.$$

Now we have

$$\sum_{\substack{p\\ |\tilde{F}(p)|\leqslant 1}} p^{-1}|\tilde{F}(p)|^2 \ll \sum_{\substack{p\\ |\eta_1|\leqslant 1/(p-1)}} p\,|\eta_1|^2 =: X+Y$$

where

$$X := \sum_{\substack{p\\ |\eta_1|\leqslant 1/(p-1)\\ |\tilde{f}(p)|\leqslant 1}} p\,|\eta_1|^2 \quad \text{and} \quad Y := \sum_{\substack{p\\ |\eta_1|\leqslant 1/(p-1)\\ |\tilde{f}(p)|>1}} p\,|\eta_1|^2 .$$

We have

$$X \ll \sum_{\substack{p\\ |\eta_1|\leqslant 1/(p-1)\\ |\tilde{f}(p)|\leqslant 1}} \big(p^{-1}|\tilde{f}(p)|^2 + p\,|\eta_2|^2\big)$$

$$\leqslant \sum_{\substack{p\\ |\tilde{f}(p)|\leqslant 1}} p^{-1}|\tilde{f}(p)|^2 + \sum_{\substack{p\\ |\eta_2|\leqslant 1/(p-1)+p^{-1}|\tilde{f}(p)|\\ |\tilde{f}(p)|\leqslant 1}} p\,|\eta_2|^2 .$$

The second term is

$$\leqslant \sum_{\substack{p\\ p|\eta_2|\leqslant 3}} p\,|\eta_2|^2 \leqslant 3 \sum_{p} |\eta_2| < \infty$$

by Lemma 2. We have

$$Y \leqslant 2 \sum_{\substack{p\\ |\tilde{f}(p)|>1}} |\eta_1| \leqslant 2\Big(\sum_{\substack{p\\ |\tilde{f}(p)|>1}} p^{-1}|\tilde{f}(p)| + \sum_{p} |\eta_2|\Big) < \infty .$$

Therefore F fulfills (2).

We have

$$\sum_{\substack{p\\ |\tilde{F}(p)|>1}} p^{-1}|\tilde{F}(p)| \ll \sum_{\substack{p\\ p^{-1}|\tilde{f}(p)|+|\eta_2|>1/(p+1)}} p^{-1}|\tilde{f}(p)| + \sum_{p} |\eta_2| .$$

The second term is $<\infty$ by Lemma 2.

The first term equals $X+Y$ where

$$X := \sum_{\substack{p\\ p^{-1}|\tilde{f}(p)|+|\eta_2|>1/(p+1)\\ p^{-1}|\tilde{f}(p)|\leqslant |\eta_2|}} p^{-1}|\tilde{f}(p)| \leqslant \sum_{p} |\eta_2| < \infty$$

and

$$Y := \sum_{\substack{p\\ p^{-1}|\tilde{f}(p)|+|\eta_2|>1/(p+1)\\ p^{-1}|\tilde{f}(p)|>|\eta_2|}} p^{-1}|\tilde{f}(p)| \leqslant \sum_{\substack{p\\ |\tilde{f}(p)|>1/3}} p^{-1}|\tilde{f}(p)|$$

$$\leqslant 3 \sum_{\substack{p\\ |\tilde{f}(p)|\leqslant 1}} p^{-1}|\tilde{f}(p)|^2 + \sum_{\substack{p\\ |\tilde{f}(p)|>1}} p^{-1}|\tilde{f}(p)| < \infty .$$

Therefore F fulfills (3).

Proof that F fulfills (4).

$$\sum_{p}\sum_{k=2}^{\infty} p^{-k}|F(p^k)| \ll \sum_{p} (1+p\,|\eta_1|) \sum_{k=2}^{\infty} p^{-k}$$

$$\ll 1 + \sum_{p} \big(p^{-2}|\tilde{f}(p)| + |\eta_2|\big) < \infty .$$

4. Proof of the second part of the theorem. We have

$$0 \leqslant N^{-1} \sum_{n=1}^{N} \Big| f(n) - \sum_{q=1}^{Q} a_q(f)\,c_q(n) \Big|^2$$

$$= N^{-1} \sum_{n=1}^{N} |f(n)|^2 - 2\,\mathrm{Re}\Big(\sum_{q=1}^{Q} \overline{a_q(f)}\, N^{-1} \sum_{n=1}^{N} f(n)\,c_q(n)\Big) +$$

$$+ \sum_{q,r=1}^{Q} a_q(f)\,\overline{a_r(f)}\, N^{-1} \sum_{n=1}^{N} c_q(n)\,c_r(n) .$$

We first let $N\to\infty$, then $Q\to\infty$ and obtain

$$\sum_{q=1}^{\infty} \varphi(q)\,|a_q(f)|^2 \leqslant M(|f|^2) .$$

Now we have

$$\sum_{h=1}^{\infty} \varphi(h)\,|a_h(f)|^2 = \Big(\prod_{p} |\eta(p)|^2\Big) \sum_{\substack{h=1\\ h\equiv 0(P)}}^{\infty} \varphi(h) \prod_{p^k \| h} \Big|\frac{\eta_k(f;p)}{\eta(p)}\Big|^2 .$$

Each $h\in \boldsymbol{N}$ with $h\equiv 0(P)$ has a unique representation $h = mn$ with $q(m) = P$ and $(n,P) = 1$ where $q(m)$ denotes the product of all primes dividing m. Hence the above equals

$$\Big(\prod_p |\eta(p)|^2\Big)\Big(\sum_{\substack{m=1\\ q(m)=P}}^{\infty} \varphi(m)\prod_{p^k\|m}\Big|\frac{\eta_k(f;p)}{\eta(p)}\Big|^2\Big)\Big(\sum_{\substack{n=1\\ (n,P)=1}}^{\infty}\prod_{p^k\|n}\Big|\frac{\eta_k(f;p)}{\eta(p)}\Big|^2\Big)$$

$$=\Big(\prod_p |\eta(p)|^2\Big)\Big(\prod_{p\in\mathscr{P}}\sum_{k=1}^{\infty}\varphi(p^k)\Big|\frac{\eta_k(f;p)}{\eta(p)}\Big|^2\Big)\Big(\prod_{p\notin\mathscr{P}}\sum_{k=0}^{\infty}\varphi(p^k)\Big|\frac{\eta_k(f;p)}{\eta(p)}\Big|^2\Big)$$

$$=\prod_p\sum_{k=0}^{\infty}\varphi(p^k)|\eta_k(f;p)|^2$$

$$=\prod_p\Big(1+\sum_{j=1}^{\infty}p^{-j}\big(|f(p^j)|^2-|f(p^{j-1})|^2\big)\Big)$$

$$=\prod_p\Big(1-\frac{1}{p}\Big)\Big(1+\sum_{j=1}^{\infty}p^{-j}|f(p^j)|^2\Big) = M(|f|^2)$$

by Lemma 1.

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Sur les fonctions arithmétiques multiplicatives de module $\leqslant 1$

par

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1. Introduction. f étant une fonction arithmétique multiplicative complexe telle que $|f(n)|\leqslant 1$ pour tout $n\in\mathbf{N}^*$, G. Halász a étudié le comportement pour x tendant vers $+\infty$ de la somme $\sum_{n\leqslant x} f(n)$(1). En modifiant légèrement sa formulation, on peut énoncer son résultat principal de la façon suivante.

L'une des deux circonstances suivantes a lieu:

(a) $(1/x)\sum_{n\leqslant x} f(n)$ *tend vers zéro quand x tend vers $+\infty$, autrement dit la fonction f possède une valeur moyenne nulle;*

(b) *Il existe une constante complexe non nulle C, une constante réelle a et une fonction réelle A définie sur l'intervalle $[1, +\infty[$ et satisfaisant à*

$$\lim_{x\to\infty}\{\sup_{x<t\leqslant x^2}|A(t)-A(x)|\} = 0,$$

telles que, quand x tend vers $+\infty$,

$$\frac{1}{x}\sum_{n\leqslant x} f(n) = Cx^{ia}\exp\big(iA(x)\big)+o(1).$$

On voit immédiatement que, dans le cas (b), $|C|$ et a sont bien déterminés par le fait que, si $F(x)=\sum_{n\leqslant x} f(n)$, on a $\lim_{x\to\infty}|(1/x)F(x)| = |C|$ et, pour tout $\lambda>0$, $\lim_{x\to\infty}\frac{F(\lambda x)}{\lambda F(x)} = \lambda^{ia}$. Par contre, la fonction A et la constante C elle-même ne sont pas déterminées: on peut remplacer A par une fonction A_1 quelconque telle que $A_1(x)-A(x)$ tende vers une limite finie θ quand x tend vers $+\infty$, en remplaçant en même temps C par $C_1 = Ce^{-i\theta}$.

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