Now if n can be represented in the form

(5.14) $$n = n_1 n_2, \quad f(n_1), f(n_2) \in G_1,$$

then in view of (5.13) we have $ef(n_1) \mathbin{\tilde{\supset}} f(n_1)$, thus

(5.15) $$f_4(n) = ef(n_1)f(n_2) = f(n_1)f(n_2) = f(n).$$

By the remark made after the proof of Lemma (5.7), a.a. numbers admit a representation of type (5.14), consequently $f(n) = f_4(n)$ holds for a.a. natural numbers n. Since f_4 has been proved to be stable, so is f. ∎

Proof of Theorem 2. First we apply Lemma (5.3). If f has a deconcentrated distribution, we are finished. If it does not, we get a function f_0 satisfying requirements (i) and (ii) of the lemma. To obtain that f is stable it is sufficient to show, according to Lemma (2.14), that all the functions f_0/ϱ_a are stable. The stability of f_0 is asserted in Lemma (5.4) and the same lemma will be applied to the functions f_0/ϱ_a; we have only to show that these functions also satisfy the condition (ii) of Lemma (5.3). To this end we show that if $G' = G/\varrho_a$ and φ: $G \to G'$ is the natural homomorphism, then the image of a quasiassociate class under φ will be contained a quasiassociate class of G', i.e.

$$x \mathbin{\tilde{\circ}} y \Rightarrow \varphi(x) \mathbin{\tilde{\circ}} \varphi(y).$$

Write $\varphi(x) = x'$, $\varphi(y) = y'$. We have to prove that for arbitrary elements $u' = \varphi(u)$ and $v' = \varphi(v)$ of G' the equivalence

$$u'x' = v'x' \Leftrightarrow u'y' = v'y'$$

holds. But by the definition of ϱ_a we have

$$u'x' = v'x' \Leftrightarrow aux = avx \Leftrightarrow auy = avy \Leftrightarrow u'y' = v'y',$$

and this concludes the proof. ∎

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On an average of primes in short intervals

by

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Introduction and statement of the result. The distribution of prime numbers in short intervals is a classical topic in analytic number theory. It mainly consists of establishing asymptotic formulae or estimates for $\pi(x+F(x))-\pi(x)$, where $F(x)$ is a monotonically increasing function.

The best unconditional results concerning asymptotic formulae were obtained by Huxley [5], those concerning estimates from below were obtained by Iwaniec–Jutila [7] and Heath-Brown–Iwaniec [4], while under the Riemann Hypothesis there are results of Cramér [2] and Selberg [10]. Moreover Gallagher [3] obtained interesting results assuming a certain uniform version of the prime r-tuple conjecture of Hardy–Littlewood.

In this paper we define the functions

(1) $$\pi_{2k}(x, h) = \sum_{\substack{p_1,\ldots,p_{2k} \\ p_{\min}\leqslant x \\ 0\leqslant p_{\max}-p_{\min}\leqslant h}} 1, \quad h = h(x),$$

where $p_{\min} = \min(p_1, \ldots, p_{2k})$ and $p_{\max} = \max(p_1, \ldots, p_{2k})$, and we obtain the following

Theorem. *Assuming the Riemann Hypothesis, for every integer $k \geqslant 1$ we have*

(2) $$\pi_{2k}(x, h) \sim 2kh^{2k-1}x\log^{-2k}x,$$

provided $h/f_{2k}(x) \to \infty$, where

$$f_{2k}(x) = x^{(k-1)/(2k-1)}\log^{2k/(2k-1)}x\log\log^{1/(2k-1)-\varepsilon_k}x, \quad \varepsilon_k = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

Remarks. In the sequel the Riemann Hypothesis is always assumed, unless the contrary is explicitly stated.

(a) Evidently

$$\pi_{2k}(x, h) = \sum_{\substack{p_1\neq p_2\neq\ldots\neq p_{2k} \\ p_{\min}\leqslant x \\ 0<p_{\max}-p_{\min}\leqslant h}} 1 + \sideset{}{'}\sum_{\substack{p_1,\ldots,p_{2k} \\ p_{\min}\leqslant x \\ 0\leqslant p_{\max}-p_{\min}\leqslant h}} 1,$$

where $'$ indicates that the sum is extended to all the $2k$-tuples in which at least two primes are equal. Therefore

$$\sum'_{\substack{p_1,\dots,p_{2k}\\ p_{\min}\leqslant x\\ 0\leqslant p_{\max}-p_{\min}\leqslant h}} 1 \ll_k \sum_{\substack{p_1,\dots,p_{2k}\\ p_{\min}\leqslant x\\ 0\leqslant p_{\max}-p_{\min}\leqslant h\\ p_1=p_2}} 1 = \sum_{\substack{p_2,\dots,p_{2k}\\ p_{\min}\leqslant x\\ 0\leqslant p_{\max}-p_{\min}\leqslant h}} 1 = \pi_{2k-1}(x,h).$$

Using Hölder's inequality with $p = 2k/(2k-1)$, $q = 2k$ in (8) (see the proof of the theorem) we can easily see that

$$\pi_{2k-1}(x,h) \sim xh^{2k-2}\log^{-(2k-1)}x$$

provided $h/f_{2k}(x)\to\infty$, whence the main term of $\pi_{2k}(x,h)$ is contained in

$$\sum_{\substack{p_1\neq p_2\neq\dots\neq p_{2k}\\ p_{\min}\leqslant x\\ 0<p_{\max}-p_{\min}\leqslant h}} 1.$$

(b) From (1) and remark (a) we get straightforwardly

$$\pi_{2k}(x,h) \sim 2k\sum_{p\leqslant x}\big(\pi(p+h)-\pi(p)\big)^{2k-1}$$

provided $h/f_{2k}(x)\to\infty$, and therefore (2) can be viewed as an asymptotic formula for an average of primes in short intervals.

(c) Again from (1) and remark (a) it follows that

$$\pi_{2k}(x,h) \sim 2k \sum_{\substack{u_1,\dots,u_{2k-1}\\ 0<u_j\leqslant h\\ j=1,\dots,2k-1}} \pi(x,u_1,\dots,u_{2k-1}), \tag{3}$$

where $\pi(x,u_1,\dots,u_s) = \#\{p\leqslant x\colon p+u_j = p',\ j=1,\dots,s\}$, and therefore (2) can be also viewed as an asymptotic formula for an average of $2k$-tuples of "twin-primes".

In this respect it is worth considering Theorem 1 of Lavrik [8] where is given the asymptotic formula for $\pi(x,u_1,\dots,u_s)$ for every s-tuple of distinct integers $u_1,\dots,u_s$ with $1\leqslant u_j\leqslant x\log^{-(s+c)}x$, except for at most $x^s\log^{-s(s+c)-M}x$ s-tuples, where $M>0$ and $c>1$ are arbitrary constants. Although Lavrik's theorem holds unconditionally, its possible exceptions have an order of magnitude by far larger than that of the terms actually considered in (3).

(d) It is worth noting that, setting

$$S_m(x,h) = \sum_{\substack{p,p'\\ p\leqslant x\\ 0<p'-p\leqslant h}} (p'-p)^m,$$

we have

$$S_{m+1}(x,h) = \int_0^h S_m(x,u)\,du - hS_m(x,h), \tag{4}$$

hence from (1) and (4) we have also, by an inductive argument,

$$S_m(x,h) \sim 2h^{m+1}x/(m+1)\log^2 x$$

provided $h\log^{-2}x\to\infty$.

(e) As far as unconditional results are concerned, we remark that using the density theorems of Ingham [6] and Huxley [5] in Lemmas 5 and 6 of Saffari–Vaughan [9], we easily obtain

$$\pi_2(x,h) \sim 2hx\log^{-2}x, \quad \text{provided } hx^{-1/6-\varepsilon}\to\infty,$$

while from Theorem 2 and (33) of Bombieri–Davenport [1] we get

$$\pi_2(x,h) \leqslant \big(8+o(1)\big)xh\log^{-2}x, \quad \text{provided } h\to\infty.$$

In a following paper we shall give unconditional bounds for the least order of magnitude for $h(x)$ admissible in order to have (2).

Proof of the theorem. We define

$$\psi_{2k}(x,h) = \sum_{\substack{n_1,\dots,n_{2k}\\ n_{\min}\leqslant x\\ 0\leqslant n_{\max}-n_{\min}\leqslant h}} \Lambda(n_1)\dots\Lambda(n_{2k})$$

where $\Lambda(n)$ is the von Mangoldt function and $n_{\min} = \min(n_1,\dots,n_{2k})$ $n_{\max} = \max(n_1,\dots,n_{2k})$.

By partial summation it is easily seen that (2) is equivalent to $\psi_{2k}(x,h)\sim 2kh^{2k-1}x$, thus we shall prove this formula.

We only consider the case $h\ll x^{1/2+\varepsilon}$, since otherwise the theorem follows from a result of Cramér [2].

We have

$$\int_0^x \big(\psi(t+h)-\psi(t)\big)^{2k}dt = \int_0^x \Big(\sum_{t<n\leqslant t+h}\Lambda(n)\Big)^{2k}dt = \sum_{\substack{n_j\leqslant x+h\\ j=1,\dots,2k}} \Lambda(n_1)\dots\Lambda(n_{2k})\,d(n_1,\dots,n_{2k})$$

where

$$d(n_1,\dots,n_{2k}) = \operatorname{meas}\{t\in[0,x]\colon t<n_1,\dots,n_{2k}\leqslant t+h\}.$$

Hence,

$$(5)\qquad \int_0^x \big(\psi(t+h)-\psi(t)\big)^{2k}dt = \sum_{\substack{n_1,\ldots,n_{2k}\\ n_{\min}\leqslant x\\ 0\leqslant n_{\max}-n_{\min}\leqslant h}} \Lambda(n_1)\ldots\Lambda(n_{2k})(h-n_{\max}+n_{\min})+ \sum_{\substack{n_1,\ldots,n_{2k}\\ x<n_{\min}\leqslant x+h\\ 0\leqslant n_{\max}-n_{\min}\leqslant h}} \Lambda(n_1)\ldots\Lambda(n_{2k})(x+h-n_{\max}).$$

From (5) we obtain

$$(6)\qquad \int_0^x \big(\psi(t+h)-\psi(t)\big)^{2k}dt = \int_0^h \psi_{2k}(x,u)\,du+O(h^{2k+1}\log^{2k}x).$$

We let as usual $\psi(x) = x+R(x)$, whence

$$(7)\qquad \int_0^x \big(\psi(t+h)-\psi(t)\big)^{2k}dt = xh^{2k}+\int_0^x \big(R(t+h)-R(t)\big)^{2k}dt+ +O\Big(h\int_0^x |R(t+h)-R(t)|^{2k-1}dt+h^{2k-1}\int_0^x |R(t+h)-R(t)|\,dt\Big).$$

Suppose now that h is such that

$$(8)\qquad I_{2k}(x,h) = \int_0^x \big(R(t+h)-R(t)\big)^{2k}dt = o(xh^{2k});$$

then from (6) and (7), using Hölder's inequality, we get

$$(9)\qquad \int_0^h \psi_{2k}(x,u)\,du = xh^{2k}\big(1+o(1)\big).$$

Now the theorem follows from (9), using a well-known tauberian argument.

In order to estimate $I_{2k}(x,h)$ we consider

$$J_{2k}(x,\theta) = \int_x^{2x} |\psi(t+\theta t)-\psi(t)-\theta t|^{2k}dt$$

and use the well-known explicit formula

$$\psi(x) = x-\sum_{\substack{\varrho\\ |\gamma|\leqslant T}} x^{\varrho}/\varrho+O(T^{-1}x\log^2 x),\qquad T\leqslant x.$$

We follow the method of Saffari–Vaughan [9] in writing

$$J_{2k}(x,\theta)\ll \int_1^2 \Big(\int_{xv/2}^{2xv} |\psi(t+\theta t)-\psi(t)-\theta t|^{2k}dt\Big)\,dv.$$

Hence

$$(10)\qquad J_{2k}(x,\theta)\ll \sum_{\substack{\varrho_1,\ldots,\varrho_{2k}\\ |\gamma_j|\leqslant 1/\theta\\ j=1,\ldots,2k}} \frac{\big(|C_{\varrho_1}(\theta)|^{2k}+\ldots+|C_{\varrho_{2k}}(\theta)|^{2k}\big)x^{k+1}}{(1+|\gamma_1+\ldots+\gamma_k-\ldots-\gamma_{2k}|)^2}+ +\log\theta T\sum_{m=\left[\frac{\log(1/\theta)}{\log 2}\right]}^{\left[\frac{\log T}{\log 2}\right]+1}\ \sum_{\substack{\varrho_1,\ldots,\varrho_{2k}\\ 2^m<|\gamma_j|\leqslant 2^{m+1}\\ j=1,\ldots,2k}} \frac{\big(|C_{\varrho_1}(\theta)|^{2k}+\ldots+|C_{\varrho_{2k}}(\theta)|^{2k}\big)x^{k+1}}{(1+|\gamma_1+\ldots+\gamma_k-\gamma_{k+1}-\ldots-\gamma_{2k}|)^2}+ +x^{2k+1}T^{-2k}\log^{4k}x,$$

where $C_{\varrho}(\theta) = \big((1+\theta)^{\varrho}-1\big)/\varrho$.

Let

$$N_{2k}(y) = \#\{(\varrho_1,\ldots,\varrho_{2k}):\ \zeta(\varrho_j)=0,\ |\gamma_1+\ldots+\gamma_k-\gamma_{k+1}-\ldots-\gamma_{2k}|\leqslant 1,\ |\gamma_j|\leqslant y,\ j=1,\ldots,2k\}.$$

Then clearly

$$N_{2k}(y)\ll y^{2k-1}\log^{2k}y$$

and

$$(11)\qquad J_{2k}(x,\theta)\ll x^{k+1}\theta\log^{2k}(1/\theta)+x^{k+1}\log\theta T\sum_{m=\left[\frac{\log(1/\theta)}{\log 2}\right]}^{\left[\frac{\log T}{\log 2}\right]+1} 2^{-m}\log^{2k}2^m+ +x^{2k+1}T^{-2k}\log^{4k}x \ll x^{k+1}\theta\log^{2k}(1/\theta)\log\theta T+x^{2k+1}T^{-2k}\log^{4k}x.$$

The estimate for $I_{2k}(x,h)$ is obtained from (11) as in Saffari–Vaughan [9]; we have

$$(12)\quad I_{2k}(x,h)\ll (x/h)\int_{h/3x}^{3h/x}\Big(\int_x^{3x} |\psi(t+\theta t)-\psi(t)-\theta t|^{2k}dt\Big)\,d\theta \ll (x/h)\big(x^{k+1}(h/x)^2\log^{2k}(x/h)\log(hT/x)\big)+x^{2k+1}T^{-2k}\log^{4k}x \ll hx^k\log^{2k}x\log(hT/x)+x^{2k+1}T^{-2k}\log^{4k}x.$$

Let now T be such that

$$Th/x\log^2 x\to\infty.$$

Then from (12) we get

$$I_{2k}(x,h)\ll hx^k\log^{2k}x\log\log x,$$

hence

$$I_{2k}(x,h) = o(h^{2k}x)$$

provided $h/f_{2k}(x)\to\infty$, $k>1$.

Finally we remark that there is no difference between the case $k = 1$ and the general one but for the fact that there is no need to subdivide the zeros with $|\gamma| > 1/\theta$ in (10).

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On a conjecture of D. H. Lehmer

by

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1. Introduction. K. Mahler assigned the measure

$$M(\theta) = \prod_{i=1}^{d} \max\{1, |\theta_i|\}$$

to the algebraic integer θ of degree d with conjugates $\theta_1, \ldots, \theta_d$. D. H. Lehmer conjectured that there is a constant $c > 1$ so that $M(\theta) < c$ implies that θ is a root of unity. While Lehmer's conjecture remains unproved, there has been significant progress in giving lower bounds depending on the degree d for $M(\theta)$. Recently E. Dobrowolski [1] has shown that there exists a positive constant $c > 0$ such that $M(\theta) < 1 + c(\log\log d/\log d)^3$ implies that θ is a root of unity.

In this note we follow Dobrowolski's ideas and obtain a somewhat simpler proof of his result coupled with an improvement on the constant.

THEOREM. *If $c < 2$ then for all sufficiently large d the inequality*

$$M(\theta) < 1 + c(\log\log d/\log d)^3$$

implies that the algebraic integer θ of degree d is a root of unity.

Our main tool is an estimation of a Vandermonde determinant which is constructed so as to have a large integral divisor. If $M(\theta)$ is too small, this Vandermonde vanishes, proving that θ is a root of an algebraic integer of lower degree.

2. Proof of theorem. Suppose n is a positive integer and a is a complex number. Define the (column) vectors

$$v_0(a) = (1, a, a^2, \ldots, a^{n-1})^t$$

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