(Use of Theorem 4 of [1] does not give significant improvements.) Taking $s_1 = 21$, $s_2 = 26$, we see that condition (14) is satisfied, so that

$$G(8) \leqslant 2(26)+21 = 73.$$

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References

[1] H. Davenport, *On sums of positive integral k-th powers*, Amer. J. Math. 64 (1942), pp. 189–198.
[2] L. K. Hua, *Additive Primzahltheorie*, Leipzig 1959.
[3] K. Thanigasalam, *On Waring's problem*, Acta Arith. 38 (1979), pp. 141–155.

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Semigroup-valued multiplicative functions

by

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Concepts and notations. The following concepts and notations will be used throughout the paper without any further reference.

$\boldsymbol{N}$: the set of natural numbers.

arithmetical function: any function whose domain is $\boldsymbol{N}$.

G-valued function: any function whose values lie in G.

multiplicative function: an arithmetical function f, on whose range an operation, written multiplicatively, is defined, and which satisfies $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$.

For a G-valued multiplicative function, where G is a semigroup, we shall use the term *G-multiplicative function.*

An arithmetical function is *completely multiplicative* if it satisfies the above functional equation for all, not necessarily coprime pairs of integers.

An arithmetical function f is *strongly multiplicative* if it is multiplicative and satisfies $f(p^k) = f(p)$ for every prime p and natural number k.

p always stands for a prime number.

$\sum_p$, $\prod_p$ denote sum, resp. product over primes.

The *asymptotic density* of a set A of natural numbers is defined by

$$d(A) = \lim_{x\to\infty} x^{-1} |A \cap [1, x]|$$

if this limit exists.

The *logarithmic density* is defined by

$$\mathrm{dl}(A) = \lim_{x\to\infty} (\log x)^{-1} \sum_{a\leqslant x, a\in A} a^{-1}$$

if it exists. Taking the limit superior in the above formula we obtain the *upper logarithmic density*, denoted by $\mathrm{dl}\sup A$.

$G < G_1$ among groups denotes that G is a subgroup of G_1.

1. Introduction. In a sense this paper is a continuation to my paper *General multiplicative functions*, [3], to which I shall refer as GMF, but the reader is not supposed to have read it, every concept and result needed will be restated.

In GMF the local distribution of group-valued multiplicative functions has been investigated. The main result asserts that for every Abelian group G, $g \in G$ and G-multiplicative f the sequence $f^{-1}(g) = \{n: f(n) = g\}$ has an asymptotic density (Theorem (1.4)), and it was pointed out that this does not hold for semigroup-valued functions in general (Theorem (2.2)). The counterexample was based on Besicovitch's construction [1] of a sequence A which is a multiplicative ideal (i.e. if it contains a number, then it contains all its multiples as well) and does not possess asymptotic density. For such a sequence A a commutative semigroup G and a G-multiplicative function f can be constructed so that $A = f^{-1}(g)$ for some $g \in G$.

Davenport and Erdös [2] proved, however, that these sequences always possess logarithmic density. This result inspired the following theorem (and plays a crucial role in its proof).

(1.1) Definition. An arithmetical function f is *stable*, if the set $f^{-1}(g)$ possesses an asymptotic density for every $g \in \operatorname{im} f$. If these sets possess logarithmic densities, f is called *logarithmically stable.*

Theorem 1. *Let G be a commutative semigroup. Every G-multiplicative function is logarithmically stable.*

This will be proved in Section 3.

The above theorem can be strengthened to the asymptotic stability for some special semigroups.

(1.2) Definition. We call a commutative semigroup G *stabilizing*, if every G-multiplicative function is stable.

A result in this direction has been already stated in GMF without proof (Theorem (2.3)). It has the following consequence. Call a semigroup G *almost group*, if $G = G_1 \cup G_2$, where G_1 is a group and G_2 is finite. (This includes e.g. groups, zero-extensions of groups and arbitrary finite semigroups.)

(1.3) Corollary. *Any semigroup which is the direct product of finitely many almost groups is stabilizing.*

The class of stabilizing semigroups is obviously descending, i.e. together with a given semigroup it contains all its subsemigroups, while the class occurring in the above corollary is not, neither is the class given in Theorem (2.3) of GMF; on the other hand, they are closed under direct multiplication. In Section 4 we shall specify a wider class of stabilizing semigroups which will be both descending and closed under multiplication.

(1.4) Problem. *Is the class of stabilizing semigroups closed under direct multiplication?*

2. Preliminaries. Here we state some definitions and auxiliary results which will be needed in the sequel.

We shall confine ourselves to commutative semigroups with unity e and multiplicative functions f satisfying $f(1) = e$. Note that this is automatically fulfilled if G is replaced by its subsemigroup generated by the values of f.

(2.1) Lemma. (Theorem (1.4) of GMF.) *Abelian groups are stabilizing.*

(2.2) Definition. Let G be an Abelian group. Call a G-multiplicative function f *concentrated*, if there is a finite subgroup G_1 of G such that

$$\sum_{f(p)\notin G_1} 1/p < \infty,$$

and *deconcentrated* otherwise.

(2.3) Lemma. (Theorem (3.10) of GMF.) *Let G be an Abelian group and f G-multiplicative. Then either*

(a) *f is deconcentrated, $d(f^{-1}(g)) = 0$ for all $g \in G$, or*

(b) *f is concentrated, $d(f^{-1}(g)) > 0$ for every $g \in \operatorname{im} f$ and*

$$\sum_{g\in G} d(f^{-1}(g)) = 1.$$

For a semigroup G, $g \in G$ and $K \subset G$ write

$$g^{-1}K = \{h: h \in G, gh \in K\}.$$

(2.4) Definition. A class $\mathfrak{K}$ of subsets of a semigroup G is called *divisible*, if $K \in \mathfrak{K}$, $g \in G$ imply

$$g^{-1}K \in \mathfrak{K}.$$

(2.5) Lemma. (Theorem (7.3) of GMF.) *Let G be a semigroup, f, f_0 G-multiplicative functions and $\mathfrak{K}$ a divisible class of subsets of G. Suppose moreover that*

$$\sum_{f(p)\neq f_0(p)} 1/p < \infty$$

(i.e. f and f_0 are essentially the same) and that either $f(2^k) = f_0(2^k)$ for all k or $f_0(2^k) = f_0(2)^k$ for all k. If the sequences $f_0^{-1}(K)$ have densities for all $K \in \mathfrak{K}$, then so do the sequences $f^{-1}(K)$. The same assertion holds for logarithmic density.

In GMF this was stated and proved only for asymptotic density, but the same proof works for logarithmic density as well.

(2.6) Lemma. (Statement (1.8) of GMF.) *Let $S \subset \mathbf{N}$. The following three conditions are equivalent.*

(a) *There exists an Abelian group G, a $g \in G$ and a completely G-multiplicative f such that*

$$S = f^{-1}(g).$$

(b) *There exists a subgroup Q_0 of the multiplicative group $\boldsymbol{Q}$ of positive rational numbers and a $q \in \boldsymbol{Q}$ such that*

$$S = qQ_0 \cap \boldsymbol{N} \tag{2.7}$$

(that is, S consists of the natural numbers contained in a coset of a subgroup of the multiplicative group of positive rational numbers).

(c) *For arbitrary $s_1, \ldots, s_{n+1}, t_1, \ldots, t_n \in S$*

$$\prod t_j \Big| \prod s_i \Rightarrow \Big(\prod s_i / \prod t_j \Big) \in S. \tag{2.8}$$

(2.9) Lemma (Davenport–Erdös [2]). *Let $Q = \{q_1, q_2, \ldots\}$ be a sequence of numbers and let R be the set of natural numbers which are divisible by some element of Q. The set R has a logarithmic density; moreover, denoting by d_k the (asymptotic) density of those numbers which are divisible by at least one of $q_1, \ldots, q_k$, we have*

$$\mathrm{dl}\, R = \lim d_k.$$

(2.10) Corollary. *If R_k is the set of those numbers which are divisible by some element of Q but not by any of $q_1, \ldots, q_k$, then we have*

$$\mathrm{dl}\, R_k \to 0.$$

Naturally, with the notation of the previous lemma we have

$$\mathrm{dl}\, R_k = \mathrm{dl}\, R - d_k.$$

(2.11) Definition. Let G be a semigroup, ϱ a congruence relation on G and f a G-multiplicative function. By the *factor-function* f/ϱ we mean the G/ϱ-multiplicative function defined by $f/\varrho = \varphi \circ f$, where $\varphi\colon G \to G/\varrho$ is the canonical homomorphism. Specially if G is an Abelian group and $G_1 < G$, then we shall write f/G_1.

(2.5) will generally be applied to the smallest divisible class containing all the one-element sets, which is

$$\mathfrak{G} = \{a^{-1}\{b\}\colon a, b \in G\}. \tag{2.12}$$

Let ϱ_a be the congruence relation defined by

$$x \varrho_a y \Leftrightarrow ax = ay. \tag{2.13}$$

Observe that $f(n) \in a^{-1}\{b\}$ is equivalent to

$$f/\varrho_a(n) = \varphi(b_1),$$

where $\varphi\colon G \to G/\varrho_a$ is the natural homomorphism and b_1 is any solution of $ab_1 = b$. Thus the existence of the density of $f^{-1}(A)$ for all $A \in \mathfrak{G}$ is equivalent to the stability of all the factor-functions f/ϱ_a. Combining these considerations and Lemma (2.5) we obtain the following assertion.

(2.14) Lemma. *Let G be a commutative semigroup and let the functions f and f_0 satisfy the requirements of Lemma (2.5). If all the functions f_0/ϱ_a, $a \in G$, are stable, resp. logarithmically stable, then so is f.*

3. The logarithmic stability. Here we prove Theorem 1. Assume first that f is completely G-multiplicative. Let $g \in G$ and $S = f^{-1}(g)$. S evidently satisfies

$$n_1, n_2, n_1 m \in S \Rightarrow n_2 m \in S. \tag{3.1}$$

Now define

$$T = \{t \in \boldsymbol{N}\colon \text{there is an } n \in S \text{ for which } nt \in S\}, \tag{3.2}$$

the "quotient sequence" of S.

(3.3) Lemma. *If $s \in S$ and $t \in T$, then $st \in S$.*

Proof. There is an $n \in S$ such that $nt \in S$. Now apply (3.1) with $n_1 = n$, $n_2 = s$, $m = t$. ∎

(3.4) Lemma. (a) *If $t_1, t_2 \in T$, then $t_1 t_2 \in T$.*

(b) *If t_1, $t_2 \in T$ and $t_1 | t_2$, then $t_2/t_1 \in T$.*

Proof. (a) Let $s \in S$ be arbitrary. (The case $S = \emptyset$ is trivial.) By Lemma (3.3) we have

$$s \in S \Rightarrow t_1 s \in S \Rightarrow t_1 t_2 s \in S,$$

thus $t_1 t_2 \in T$ by definition.

(b) Let $s \in S$. By (3.3) we have $t_1 s \in S$ and $t_2 s \in S$, therefore

$$t_2/t_1 = (t_2 s)/(t_1 s) \in T. \ \blacksquare$$

(3.5) Lemma. *There exists an Abelian group G_0 and a G_0-multiplicative f_0 such that*

$$T = f_0^{-1}(e),$$

where e is the unity of G_0.

Proof. Immediate consequence of Lemmas (3.4) and (2.6). T will be the inverse image of the unity because $1 \in T$. ∎

Now we proceed with the theorem. (3.3) implies

$$ST = \bigcup_{s \in S} sT = S. \tag{3.6}$$

Let the elements of S be $s_1, s_2, \ldots$ and let

$$S_k = \bigcup_{i=1}^{k} s_i T.$$

According to (3.6) we have $S_k \subset S$.

(3.7) Lemma. *As $k \to \infty$,*

$$\mathrm{dl}\sup(S \setminus S_k) \to 0.$$

Proof. Let U_k be the sequence of numbers which are divisible by some element of S but not by any of $s_1, \ldots, s_k$. We show

$$S \setminus S_k \subset U_k.$$

Namely if $s \in S$, then s is divisible by an element of S (by itself). If it is not divisible by $s_1, \ldots, s_k$, then it belongs to U_k. If it is divisible by s_j, $j \leqslant k$, then $s/s_j \in T$ and thus

$$s \in s_j T \subset S_k.$$

The proof is completed by an application of Lemma (2.9).

(3.8) Lemma. *S_k possesses asymptotic density.*

Proof. Applying the sieve formula we can express the counting function of S_k as a finite linear combination of counting functions of sets of the form

$$V = r_1 T \cap \ldots \cap r_m T.$$

($r_1, \ldots, r_m$ are taken from among $s_1, \ldots, s_k$.)

Hence it is sufficient to show that these sets have asymptotic densities. According to Lemmas (3.5) and (2.6) we have $T = Q_0 \cap \boldsymbol{N}$, where Q_0 is a multiplicative subgroup of rational numbers. Therefore $rT = rQ_0 \cap r\boldsymbol{N}$, thus

$$V = \bigcap r_i T = \left(\bigcap r_i Q_0\right) \cap \left(\bigcap r_i \boldsymbol{N}\right).$$

$\bigcap r_i Q_0$, being the intersection of cosets of a fixed subgroup, is either empty or itself a coset, say rQ_0. The first case is obvious, consider the second. We have

$$r_i \boldsymbol{N} = u\boldsymbol{N}, \quad u = [r_1, \ldots, r_m],$$

hence

$$V = rQ_0 \cap u\boldsymbol{N} = u(ru^{-1}Q_0 \cap \boldsymbol{N}) = uW.$$

The sequence W is, again by Lemma (2.6), the inverse image of an element under a group-valued multiplicative function, thus it has an asymptotic density by Lemma (2.1); therefore obviously so does V and the proof is completed. ∎

Proof of the theorem. Lemmas (3.7) and (3.8) yield that S has logarithmic density, thus for completely multiplicative functions Theorem 1 is verified. To turn to the general case, let f be simply multiplicative and let f_0 be the completely multiplicative function defined by $f_0(p) = f(p)$ for all primes p. The functions f_0/ϱ_a are also completely multiplicative and consequently logarithmically stable, thus from Lemma (2.14) we can conclude that f is logarithmically stable as well. ∎

4. Stabilizing semigroups. We deal with commutative semigroups G with unity e.

(4.1) Definition. We say that *a divides b*, symbolically $a \mid b$ if $b = ac$ for some c $(a, b, c \in G)$.

(4.2) Definition. We say that a and b are *associates*, symbolically $a \sim b$, if both $a \mid b$ and $b \mid a$.

The relation $\sim$ is clearly a congruence relation.

In GMF the following theorem was stated. *Let G be a semigroup and $G_1 = G/\sim$. If in G_1 each element has only a finite number of divisors* (we call these semigroups *P-semigroups*), *then G is stabilizing.* It is easily seen that the class of P-semigroups is closed under direct multiplication and it contains all the almost groups in the sense defined in Section 1; a subgroup of a P-semigroup, however, need not be one. Even a cancellative semigroup can fail to be a P-semigroup, regard e.g. the additive semigroup of positive real numbers. Of course this problem can be removed formally regarding the class P_1 of those semigroups that can be embedded into a P-semigroup, but we prefer an inner characterization. This will be achieved by Theorem 2.

We recall the relation ϱ_a, which has been defined in (2.13) by

$$x \varrho_a y \Leftrightarrow ax = ay.$$

(4.3) Definition. a is a *quasi-divisor* of b, symbolically $a \oplus b$, if $\varrho_a \subset \varrho_b$. (This is equivalent to that a divides b in a suitable extension of G.)

(4.4) Definition. a and b are *quasiassociates*, symbolically $a \mathrel{\tilde{\sim}} b$, if $\varrho_a = \varrho_b$ (i.e. $a \oplus b$ and $b \oplus a$).

Obviously $\tilde{\sim}$ is a congruence relation.

(4.5) Definition. We say that G is an *R-semigroup* if for every $g \in G$ all the divisors of g lie in a finite number of quasiassociate classes.

Theorem 2. *Every R-semigroup is stabilizing.*

This will be proved in the next section. Now we shall establish some properties of R-semigroups and show that our Theorem 2 implies the theorem stated in GMF and repeated at the beginning of this section, i.e.

(4.6) Lemma. *Every P-semigroup is also an R-semigroup.*

Proof. In a P-semigroup the divisors of a fixed element are contained in finitely many associate classes. Now observe that associates are quasiassociates as well. ∎

(4.7) Statement. *A subsemigroup of an R-semigroup is also an R-semigroup.*

Proof. Let G be an R-semigroup and G' its sub-semigroup. Regard the divisors of a $g \in G'$. They are its divisors in G as well, therefore they lie in finitely many quasiassociate classes in G. Now we have only to note that if x and y are quasiassociates in G, then

$$\forall a, b \in G \ (ax = bx \Leftrightarrow ay = by),$$

thus they are *a fortiori* quasiassociates in G'. ∎

(4.8) Statement. *The direct product of two R-semigroups is also an R-semigroup.*

Proof. Let $G = G_1 \times G_2$. It is easy to see that the corresponding relations are also multiplied, that is

$$(a_1, a_2) \mid (b_1, b_2) \Leftrightarrow a_1 \mid b_1 \text{ and } a_2 \mid b_2,$$

$$(a_1, a_2) \ \widetilde{\circ}\ (b_1, b_2) \Leftrightarrow a_1 \ \widetilde{\circ}\ b_1 \text{ and } a_2 \ \widetilde{\circ}\ b_2,$$

from which the statement follows immediately. ∎

Statements (4.7) and (4.8), together with the observation that an almost group (see Section 1) is an R- or even a P-semigroup, show that Corollary (1.3) is really a corollary of Theorem 2.

5. Proof of Theorem 2. Without any further reference we assume that the occurring semigroups are commutative with an unity and that multiplicative functions assume this unity at 1.

The proof will be based on a series of lemmas.

(5.1) Lemma. *Let G be a semigroup, f G-multiplicative and $g \in G$. We have either*

$$d\big(f^{-1}(g)\big) = 0$$

or

(5.2) $$\sum_{f(p) \nmid g} 1/p < \infty.$$

Proof. Let $p_1, p_2, \ldots$ be those primes for which $f(p) \nmid g$. For a fixed prime p the density of numbers in which the exponent of p is not 1, is $1-p^{-1}+p^{-2}$. Consequently the density of numbers that contain none of $p_1, \ldots, p_k$ with exponent 1, is

$$\prod_{i=1}^{k} (1-p_i^{-1}+p_i^{-2}) \to 0 \quad (k \to \infty)$$

if (5.2) does not hold. But if $f(n) = g$, then n satisfies the above requirement, as otherwise we would have

$$g = f(n) = f(p_i) f(n/p_i),$$

contradicting the definition of the p_i's. ∎

(5.3) Lemma. *Let f be a G-multiplicative function, where G is an R-semigroup. Either f has a deconcentrated distribution (i.e. $d\big(f^{-1}(g)\big) = 0$ for all g), or a completely G-multiplicative function f_0 and a finite number*

of quasiassociate classes $K_1, \ldots, K_m$ can be found so that

(i) *f and f_0 satisfy the requirements of Lemma* (2.5), *and*

(ii) *for every prime p we have*

$$f_0(p) \in \bigcup_{j=1}^{m} K_j$$

and for each $j = 1, \ldots, m$

$$\sum_{f_0(p) \in K_j} 1/p = \infty.$$

Proof. Apply the previous lemma for each $g \in G$. If f does not have a deconcentrated distribution, it yields $f(p) \mid g$ for some g and all primes p with the exception of some having a convergent sum of reciprocals. The divisors of g belong to finitely many quasiassociate classes; let $K_1, \ldots, K_m$ be those of them for which

$$\sum_{f(p) \in K} 1/p = \infty.$$

Let h be any element of K_1 and define f_0 by setting

$$f_0(p) = \begin{cases} f(p), & \text{if } f(p) \in \bigcup K_j, \\ h & \text{otherwise.} \end{cases}$$ ∎

(5.4) Lemma. *If G is a semigroup and a G-multiplicative function f satisfies the condition* (ii) *of Lemma* (5.3), *then it is stable.*

This is the crucial step towards Theorem 2. To the proof we need some preparation.

(5.5) Definition. A semigroup is *associate-free*, if no two elements of it are associates, save they are equal.

(5.6) Lemma. *For any subgroup G the factor-subgroup $G' = G/\widetilde{\circ}$ is associate-free.*

Proof. Let $x', y' \in G'$ and $x' \sim y'$. This means $x' = u'y'$ and $y' = v'x'$. Let x, y, u, v be elements from the classes which form x' etc. We have $x \ \widetilde{\circ}\ uy$ and $y \ \widetilde{\circ}\ vx$. From $x \ \widetilde{\circ}\ uy$ we can conclude $y \ \Phi\ x$, because if $ay = by$, then $auy = buy$, and thus $ax = bx$ by $uy \ \widetilde{\circ}\ x$. Similarly from $y \ \widetilde{\circ}\ vx$ we obtain $x \ \Phi\ y$. These two statements together mean $x \ \widetilde{\circ}\ y$, i.e. $x' = y'$. ∎

(5.7) Lemma. *Let G be an associate-free semigroup, $g_1, \ldots, g_m \in G$ and f a completely G-multiplicative function satisfying*

$$f(p) \in \{g_j \colon 1 \leqslant j \leqslant m\}$$

for all primes p and

(5.8) $$\sum_{f(p) = g_j} 1/p = \infty$$

for each $1 \leqslant j \leqslant m$. Regard the sequence

$$(g_1 g_2 \ldots g_m)^n.$$

If it does not stabilize, then f has a deconcentrated distribution; if it does stabilize at an element a, then f has a distribution concentrated into a, that is, $d(f^{-1}(a)) = 1$.

Proof. Denote

$$g_1 \dots g_m = b.$$

Suppose first that the sequence stabilizes, i.e. $b^n = a$ for all $n \geqslant r$. Then for $n_1 \geqslant r+1, \dots, n_m \geqslant r+1$ we have

$$g_1^{n_1} \dots g_m^{n_m} = a. \tag{5.9}$$

Namely the left-hand side is a multiple of $b^r = a$ and is a divisor of $b^s = a$, where $s = 1+\max n_j$; since G is associate-free, it must be equal to a.

(5.8) implies that almost all numbers (i.e. with the exception of a sequence of density 0) are divisible by more than r primes satisfying $f(p) = g_j$, thus for almost all n, $f(n)$ is of the form (5.9) with $n_j > r$, and consequently $f(n) = a$ for a.a. numbers n.

Next suppose that $f^{-1}(a)$ has a positive upper density for some $a \in G$. We are going to prove that the sequence b^n stabilizes at a.

Since $a \in \operatorname{im} f$, it can be represented in the form

$$a = g_1^{k_1} \dots g_m^{k_m}.$$

Put $r = 1+\max k_j$ and $c = b^r$. By the above considerations $f(n)$ is a multiple of c for a.a. value of n, thus $c \mid a$; on the other hand, we have $a \mid c$ evidently, thus $a = c$. Moreover we have $ag_j \mid c$ for each j be the definition of a and c and also $c = a \mid ag_j$, thus $ag_j = a$. By this equality $ab = a$ follows, which immediately yields $b^n = a$ for $n > r$. ∎

Remark. Similarly we can show that in the first case (i.e. when the sequence stabilizes at a) a.a. numbers n are representable in the form

$$n = n_1 n_2, \quad f(n_1) = f(n_2) = a. \tag{5.10}$$

This fact will also be used later.

Proof of Lemma (5.4). Let $G' = G/\sim$, $f' = f/\sim$. If f' has a deconcentrated distribution, then obviously so does f, hence we may assume the opposite. We apply Lemma (5.7) for f'; it yields that for a.a. n we have $f'(n) = a'$ with some $a' \in G'$. Being the limit of a sequence b^n, a' must be idempotent. Regarding a' as a quasiassociate class of G, this means that it is closed under multiplication, i.e. it is a subsemigroup; to emphasize this fact let us denote it by G_1. Thus G_1 is a semigroup in which any two elements are quasiassociates.

Now define

$$f_1(n) \begin{cases} = f(n), & \text{if } f(n) \in G_1, \\ \text{not defined}, & \text{if } f(n) \notin G_1. \end{cases}$$

f_1 is a partial completely multiplicative function in the sense that if $f_1(m)$ and $f_1(n)$ are defined, then so is $f_1(mn)$ and it is equal to $f_1(m)f_1(n)$. Moreover, f_1 is defined on a.a. numbers.

Call two elements a, b, of G_1 *merging* and write $a \rceil b$ if $ax = bx$ for all $x \in G_1$. Since G_1 consists of a single quasiassociate class, if the above equality holds for any x, it must hold for all, i.e. $a \rceil b$.

$\rceil$ is clearly a congruence relation. Let

$$G_2 = G_1/\rceil, \quad f_2 = f_1/\rceil.$$

We show that G_2 is cancellative. For G_1 it means

$$ax \rceil ay \Rightarrow x \rceil y.$$

But this is evident:

$$ax \rceil ay \Rightarrow a^2x \rceil a^2y \Rightarrow x \rceil y.$$

G_2, being a commutative cancellative semigroup, can be embedded into an Abelian group G_3. We extend f_2 to a completely G_3-multiplicative function f_3. Let n be an arbitrary natural number. As f_2 is defined a. everywhere, we can find a number $m \in \operatorname{dom} f_2$ such that $mn \in \operatorname{dom} f_2$. Now let

$$f_3(n) = f_2(mn)f_2(m)^{-1}.$$

By the partial complete multiplicativity of f_2 it easily follows that this definition is unique, f_3 is completely multiplicative and it is an extension of f_2.

For f_3 we can apply Lemmas (2.1) and (2.3). We obtain that f_3 has a distribution; if it is deconcentrated, we are finished since for any $g \in G_1$ we have

$$f^{-1}(g) = f_1^{-1}(g) \subset f_2^{-1}(g_2) \subset f^{-1}(g_2),$$

where $g_2 = \varphi(g)$ and $\varphi\colon G_1 \to G_2$ is the natural homomorphism, while the n's for which $f(n) \notin G_1$ also form a sequence of density 0.

Hence we may assume that f_3 is concentrated. According to (2.3) we have $d(f^{-1}(g)) > 0$ for every $g \in \operatorname{im} f_3$; on the other hand, $f_3(n) \in G_2$ for a.a. n, thus $\operatorname{im} f_3 \subset G_2$, in particular we have $f_3(1) \in G_2$. Let e be an arbitrary element of $\varphi^{-1}(f_3(1))$. For an arbitrary $g \in \operatorname{im} f_3$ consider the elements ea, $a \in \varphi^{-1}(g)$. The elements of $\varphi^{-1}(g)$ are merging, therefore the elements ea coincide; denote their common value by $\psi(g)$. As $g = f_3(n)$ with a suitable n, we have

$$\varphi(\psi(g)) = \varphi(ea) = f_3(1)f_3(n) = f_3(n) = g. \tag{5.11}$$

Furthermore by $\varphi(ea) = \varphi(a) = g$ we have

$$ea \rceil a \quad \text{if} \quad \varphi(a) \in \operatorname{im} f_3. \tag{5.12}$$

(5.11) shows that ψ is one-to-one; since f_3 is stable, so is the function

$$f_4 = \psi \circ f_3.$$

If $f(n) \in G_1$, then by virtue of (5.12) we have

$$f_4(n) = ef(n) \rceil f(n). \tag{5.13}$$

Now if n can be represented in the form

$$n = n_1 n_2, \quad f(n_1), f(n_2) \in G_1, \tag{5.14}$$

then in view of (5.13) we have $ef(n_1) \,\breve{\sim}\, f(n_1)$, thus

$$f_4(n) = ef(n_1)f(n_2) = f(n_1)f(n_2) = f(n). \tag{5.15}$$

By the remark made after the proof of Lemma (5.7), a.a. numbers admit a representation of type (5.14), consequently $f(n) = f_4(n)$ holds for a.a. natural numbers n. Since f_4 has been proved to be stable, so is f. ∎

Proof of Theorem 2. First we apply Lemma (5.3). If f has a deconcentrated distribution, we are finished. If it does not, we get a function f_0 satisfying requirements (i) and (ii) of the lemma. To obtain that f is stable it is sufficient to show, according to Lemma (2.14), that all the functions f_0/ϱ_a are stable. The stability of f_0 is asserted in Lemma (5.4) and the same lemma will be applied to the functions f_0/ϱ_a; we have only to show that these functions also satisfy the condition (ii) of Lemma (5.3). To this end we show that if $G' = G/\varrho_a$ and $\varphi\colon G \to G'$ is the natural homomorphism, then the image of a quasiassociate class under φ will be contained a quasiassociate class of G', i.e.

$$x \,\breve{\sim}\, y \Rightarrow \varphi(x) \,\breve{\sim}\, \varphi(y).$$

Write $\varphi(x) = x'$, $\varphi(y) = y'$. We have to prove that for arbitrary elements $u' = \varphi(u)$ and $v' = \varphi(v)$ of G' the equivalence

$$u'x' = v'x' \Leftrightarrow u'y' = v'y'$$

holds. But by the definition of ϱ_a we have

$$u'x' = v'x' \Leftrightarrow aux = avx \Leftrightarrow auy = avy \Leftrightarrow u'y' = v'y',$$

and this concludes the proof. ∎

References

[1] A. S. Besicovitch, *On the density of certain sequences of integers*, Math. Ann. 110 (1934), pp. 336–341.

[2] H. Davenport and P. Erdös, *On sequences of positive integers*, Acta Arith. 2 (1937), pp. 147–151.

[3] I. Z. Ruzsa, *General multiplicative functions*, ibid. 32 (1977), pp. 313–347.

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On an average of primes in short intervals

by

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Introduction and statement of the result. The distribution of prime numbers in short intervals is a classical topic in analytic number theory. It mainly consists of establishing asymptotic formulae or estimates for $\pi(x+F(x))-\pi(x)$, where $F(x)$ is a monotonically increasing function.

The best unconditional results concerning asymptotic formulae were obtained by Huxley [5], those concerning estimates from below were obtained by Iwaniec–Jutila [7] and Heath-Brown–Iwaniec [4], while under the Riemann Hypothesis there are results of Cramér [2] and Selberg [10]. Moreover Gallagher [3] obtained interesting results assuming a certain uniform version of the prime r-tuple conjecture of Hardy–Littlewood.

In this paper we define the functions

$$\pi_{2k}(x, h) = \sum_{\substack{p_1,\ldots,p_{2k} \\ p_{\min} \leqslant x \\ 0 \leqslant p_{\max}-p_{\min} \leqslant h}} 1, \quad h = h(x), \tag{1}$$

where $p_{\min} = \min(p_1, \ldots, p_{2k})$ and $p_{\max} = \max(p_1, \ldots, p_{2k})$, and we obtain the following

Theorem. *Assuming the Riemann Hypothesis, for every integer $k \geqslant 1$ we have*

$$\pi_{2k}(x, h) \sim 2kh^{2k-1} x \log^{-2k} x, \tag{2}$$

provided $h/f_{2k}(x) \to \infty$, where

$$f_{2k}(x) = x^{(k-1)/(2k-1)} \log^{2k/(2k-1)} x \log\log^{1/(2k-1)-\varepsilon_k} x, \qquad \varepsilon_k = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

Remarks. In the sequel the Riemann Hypothesis is always assumed, unless the contrary is explicitly stated.

(a) Evidently

$$\pi_{2k}(x, h) = \sum_{\substack{p_1 \neq p_2 \neq \ldots \neq p_{2k} \\ p_{\min} \leqslant x \\ 0 < p_{\max}-p_{\min} \leqslant h}} 1 + \sum_{\substack{p_1,\ldots,p_{2k} \\ p_{\min} \leqslant x \\ 0 \leqslant p_{\max}-p_{\min} \leqslant h}}{}' \; 1,$$