[5] R. Dedekind, *Über Zusammenhang zwischen der Theorie der Ideale und der Theorie der höhere Kongruenzen*, Abh. König. Ges. der Wissen. zu Göttingen, 23 (1878), pp. 1–23.
[6] D. S. Dummit and H. Kisilevsky, *Indices in cyclic cubic fields*, in: *Number Theory and Algebra*, Academic Press, New York 1977, pp. 29–42.
[7] H. T. Engstrom, *On the common index divisor of an algebraic field*, Trans. Amer. Math. Soc. 32 (1930), pp. 223–237.
[8] M. N. Gras, *Sur les corps cubiques cycliques dont l'anneau des entiers est monogène*, C. R. Acad. Sci., Paris, 278 (1974), pp. 59–62.
[9] M. Hall, *Indices in cubic fields*, Bull. Amer. Math. Soc. 43 (1937), pp. 104–108.
[10] K. Hensel, *Arithmetische Untersuchungen über die gemeinsamer ausserwesentlicher Diskriminantenteiler einer Gattung*, Journal für Mathematik 113 (1894), pp. 128–160.
[11] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, PWN, Warszawa 1974.
[12] O. Ore, *Über der Zusammenhang zwischen den definierenden Gleichungen und der Idealtheorie in algebraischen Körper*, Math. Ann. 96 (1926), pp. 313–352.
[13] P. A. B. Pleasants, *The number of generators of the integers of a number field*, Mathematika 21 (1974), pp. 160–167.
[14] A. A. Sukallo, *On determination of the index of a field of algebraic numbers*, Rostov. Gos. Univ. Uč. Zap., Fiz.-Mat. Fak., 32 (1955), pp. 37–42.
[15] L. Tornheim, *Minimal basis and inessential discriminant divisors for a cubic field*, Pacific J. Math. 5 (1955), pp. 621–631.
[16] E. Zylinsky, *Zur Theorie der ausserwesentlicher Diskriminantenteiler algebraischer Körper*, Math. Ann. 73 (1913), pp. 273–274.

Received on 24.11.1980 (1233)

ACTA ARITHMETICA
XLII (1982)

Some new estimates for $G(k)$ in Waring's problem

by

K. Thanigasalam (Monaca, Penn.)

1. Introduction. In a recent paper [3], some new estimates were obtained for $G(k)$ when $k \geq 9$. In this paper they will be improved a little further. For large k the method does not give significant results.

Theorem. $G(9) \leqslant 88$, $G(10) \leqslant 104$, $G(11) \leqslant 119$, $G(12) \leqslant 134$, $G(13) \leqslant 150$, $G(14) \leqslant 165$, $G(15) \leqslant 181$, $G(16) \leqslant 197$, $G(17) \leqslant 213$, $G(18) \leqslant 229$, $G(19) \leqslant 245$, $G(20) \leqslant 262$.

When $k = 8$ the argument gives $G(8) \leqslant 73$ which is the same as that obtained by Davenport's method.

As in [3] we take

$$2P = N^{1/k}, \quad P_0 = \sqrt{P}, \quad \tau = P^{k-1+\delta} \tag{1}$$

where N is a large positive integer and δ is a small positive constant. Let

$$\eta = \frac{1}{2k-1}, \quad P_1 = P_0^{1-\eta}, \quad P_2 = P_0^{1+\eta}, \tag{2}$$

let $\mathscr{U}$ denote the set of numbers u of the form

$$u = \sum_{i=1}^{s_2} x_i^k$$

with

$$P_1^{k-\delta} < u < s_2 2^k P_1^{k-\delta}, \tag{3}$$

and let

$$U_1 = \operatorname{card} \mathscr{U}. \tag{4}$$

Suppose further that $\mathscr{P}$ is the set of primes v with

$$\tfrac{1}{2}P_2^{1-\delta/2} \leqslant v \leqslant P_2^{1-\delta/2}. \tag{5}$$

Then, as in [3], we take

(6) $$Q(a) = \sum_{p\in\mathscr{P}} \sum_{u\in\mathscr{U}} e(ap^k u).$$

For $1 \leqslant a \leqslant q$, $(a, q) = 1$, $q \leqslant P_2$ let

(7) $$\mathfrak{m}_{a,q} = \{a\colon |a-a/q| \leqslant q^{-1}\tau^{-1}\}$$

denote a basic interval and write $\mathfrak{m}$ for their union and

(8) $$m = (\tau^{-1}, 1+\tau^{-1})\setminus\mathfrak{m}$$

for the supplementary intervals.

2. Estimation of $Q(a)$ over m. The method is generally as in [3] except that Lemmas 1, 2, 3 are replaced by the following lemma.

LEMMA. *Suppose that* $a = a/q+\beta$ *with* $(a, q) = 1$, $q \leqslant 2(2X)^k$, $|\beta| \leqslant \frac{1}{2}q^{-1}(2X)^{-k}$ *and suppose further that if* $q \leqslant X$, *then* $|\beta| \geqslant q^{-1}X^{1-k}Y^{-1}$. *Then*

$$Q_0(a) = \sum_{X\leqslant p\leqslant 2X} \sum_{y\leqslant Y} b_y e(ap^k y)$$

satisfies

$$Q_0(a) \ll X^{\varepsilon}(XY+X^k)^{1/2}\Big(\sum_{y\leqslant Y}|b_y|^2\Big)^{1/2}.$$

Proof. When $(b, q) = 1$, the congruence $x^k \equiv b \pmod q$ has $\ll q^{\varepsilon}$ solutions modulo q. Thus the primes p in $(X, 2X)$ can be divided into r classes $\mathscr{P}_1, \ldots, \mathscr{P}_r$ with $r \ll q^{\varepsilon}$ such that if $p_1 \in \mathscr{P}_j$, $p_2 \in \mathscr{P}_j$, then $p_1^k \equiv p_2^k \pmod q$ if and only if $p_1 \equiv p_2 \pmod q$. Hence

$$Q_0(a) = \sum_{j=1}^{r} Q_j(a), \quad Q_j(a) = \sum_{y\leqslant Y} b_y \sum_{p\in\mathscr{P}_j} e(ap^k y)$$

and it suffices to obtain the corresponding result for Q_j. By Cauchy's inequality,

(9) $$Q_j(a)^2 \ll \Big(\sum_{y\leqslant Y}|b_y|^2\Big)\sum_{y\leqslant Y}\Big|\sum_{p\in\mathscr{P}_j} e(ap^k y)\Big|^2$$
$$\ll \Big(\sum_{y\leqslant Y}|b_y|^2\Big)\Big(YX+\sum_{\substack{p_1\in\mathscr{P}_j\\ p_1\neq p_2}}\sum_{p_2\in\mathscr{P}_j} \|a(p_1^k-p_2^k)\|^{-1}\Big).$$

Note that for $p_1 \in \mathscr{P}_j$, $p_2 \in \mathscr{P}_j$ one has

(10) $$|\beta|\,|p_1^k-p_2^k| \leqslant \tfrac{1}{2}q.$$

If there are any terms in (9) with $p_1 \neq p_2$, $p_1^k \equiv p_2^k \pmod q$, then $p_1 \equiv p_2 \pmod q$ and $q \leqslant X$. Hence

$$\|a(p_1^k-p_2^k)\| = |\beta(p_1^k-p_2^k)| \geqslant q^{-1}Y^{-1}|p_1-p_2|.$$

Thus the contribution from such terms is

$$\ll \sum_{y\leqslant Y}|b_y|^2 \sum_{p\in\mathscr{P}_j}\sum_{h\leqslant 2X/q}\frac{qY}{hq} \ll \sum_{y\leqslant Y}|b_y|^2 X^{1+\delta}Y.$$

It remains to consider those p_1, p_2 with $p_1 \not\equiv p_2 \pmod q$. By (10),

$$\|a(p_1^k-p_2^k)\| \geqslant \tfrac{1}{2}\|a(p_1^k-p_2^k)/q\|.$$

Since the number of solutions of $p_1^k-p_2^k = h$ is $\ll X^{\varepsilon}$ when $h \neq 0$ it follows that the contribution from these terms is

$$\ll \sum_{y\leqslant Y}|b_y|^2 \sum_{\substack{|h|\leqslant(2X)^k\\ q\nmid h}} X^{\varepsilon}\|ah/q\|^{-1} \ll \sum_{y\leqslant Y}|b_y|^2(X^k+q)(Xq)^{\varepsilon}.$$

Collecting the estimates together gives the lemma at once.

3. Outline of the method. Suppose that

(11) $$U_1 > (P_1^{k-\delta})^{\gamma_1-\varepsilon}.$$

Then it follows at once from the above lemma that

(12) $$Q(a) \ll Q(0)\,N^{-\sigma+10\delta} \quad (a\in m),$$

where

(13) $$\sigma = \frac{1}{4k}\Big(1+\frac{1}{2k-1}\Big)-\frac{1}{4}(1-\gamma_1)\Big(1-\frac{1}{2k-1}\Big).$$

Then, as in [3] (cf. (58)), provided that

(14) $$\gamma_2-\frac{1}{4}(1-\gamma_1)\Big(1-\frac{1}{2k-1}\Big)+\frac{1}{4k}\Big(1+\frac{1}{2k-1}\Big) > 1$$

for suitable s_1, s_2 with

(15) $$U_{s_j}(k) \geqslant N^{\gamma_j-\varepsilon},$$

k	s_1	γ_1	s_2	γ_2	$2s_2+s_1$
10	32	0.97052	36	0.980953	104
11	35	0.96831	42	0.98395	119
12	40	0.972075	47	0.984914	134
13	46	0.976871	52	0.985704	150
14	49	0.975501	58	0.987439	165
15	55	0.979078	63	0.98796	181
16	59	0.979249	69	0.989125	197
17	65	0.981754	74	0.989432	213
18	71	0.983719	79	0.989698	229
19	73	0.981737	86	0.990962	245
20	80	0.984209	91	0.991053	262

one obtains

$$G(k) \leqslant 2s_2+s_1. \tag{16}$$

For $k \geqslant 10$ the method of [3] enables $s_1, s_2, \gamma_1, \gamma_2$ to be found, as in the table above, that satisfy (14) and (15). This gives the theorem for $k \geqslant 10$.

4. The case $k = 9$. For $k = 9$, it seems necessary to use Theorem 4 of [1] also in the estimate of $U_s(9)$.

Starting as in [3] (using (42) of [3]), we have

$$U_7(9) \gg N^{a^{(7)}(9)-\varepsilon} \quad \text{where} \quad a^{(7)}(9) > 0.591135. \tag{17}$$

Now consider Theorem 4 of [1] with $k = 9$. With the notations in the theorem, we can consider the following cases:

(a) $l = 5,\ 0 < \delta < \dfrac{1}{2^{l-1}} = \dfrac{1}{16}$

$$\delta + \frac{9\lambda a}{2^5} = \frac{(\delta+6)}{2^5} + \frac{(1-\lambda)}{2^5};$$

(b) $l = 6,\ 0 < \delta < \dfrac{1}{2^{l-1}} = \dfrac{1}{32}$

$$\delta + \frac{9\lambda a}{2^6} = \frac{(\delta+7)}{2^6} + \frac{(1-\lambda)}{2^6}.$$

In both cases, it is easily seen that the number of solutions s of

$$x^9+p^9u_i = y^9+p^9u_j$$

is $\ll PU\Pi \cdot P^{\varepsilon}$.

Now it can be deduced in a standard way (as in [1]) that if $U_s(9) \gg N^{a-\varepsilon}$, then $U_{s+1}(9) \gg N^{\beta-\varepsilon}$ where

$$\beta \geqslant \frac{1+9\lambda a}{9}. \tag{18}$$

In case (a), we can take

$$\delta = \frac{55-72a}{280+9a}, \quad \lambda = \frac{8+\delta}{9} \quad \text{for} \quad \frac{600}{1161} < a < \frac{55}{72} \tag{19}$$

(to ensure $0 < \delta < 1/16$).

In case (b), we take

$$\delta = \frac{64-72a}{568+9a}, \quad \lambda = \frac{8+\delta}{9} \quad \text{for} \quad \frac{1480}{2313} < a < \frac{8}{9} \tag{20}$$

(to ensure $0 < \delta < 1/32$).

(19) and (20) respectively give the estimates (using (18))

$$\beta \geqslant \frac{1}{9}\left\{\frac{280+2304a}{280+9a}\right\}, \tag{21}$$

$$\beta \geqslant \frac{1}{9}\left\{\frac{568+4617a}{568+9a}\right\}. \tag{22}$$

Starting with (17), we use (21) twice and then (22) thrice to get

$$a^{(12)}(9) > 0.78095. \tag{23}$$

Now we use (54) of [3] namely $\beta \geqslant \dfrac{127+1025a}{9(127+a)}$ repeatedly to get

$$a^{(26)}(9) > 0.96180, \tag{24}$$

$$a^{(31)}(9) > 0.97959. \tag{25}$$

It should be noted that Theorem 2 of [1] is used 19 times to estimate $a^{(31)}(9)$ from $a^{(12)}(9)$, and this provides sufficient number of terms to deal with the basic intervals. The use of Theorem 4 of [1] (up to $a^{(12)}(9)$) gives rise to exponential sums of the form

$$\sum_p \sum_x e(ap^kx^k), \quad \sum_p \sum_q \sum_x e(ap^kq^kx^k), \quad \ldots, \quad (p, q, \ldots \text{ being primes})$$

for which the standard approximating functions do not apply. From (24) and (25), we see that condition (14) is satisfied with $s_1 = 26$ and $s_2 = 31$. Hence

$$G(9) \leqslant 2(31)+26 = 88.$$

5. The case $k = 8$. It is possible by the method of this paper to reestablish the known bound $G(8) \leqslant 73$ (obtained by Davenport's method). Following the computations in [2] (end of chapter IX), we have by using Theorem 2 of [1],

$$U_9(8) \gg N^{a^{(9)}(8)-\varepsilon} \quad \text{where} \quad a^{(9)}(8) > 0.73318.$$

Now using $\beta \geqslant \dfrac{1}{8}\left\{\dfrac{63+449a}{63+a}\right\}$, we get

$$a^{(21)}(8) > 0.95434 \quad \text{and} \quad a^{(26)}(8) > 0.97827.$$

(Use of Theorem 4 of [1] does not give significant improvements.) Taking $s_1 = 21$, $s_2 = 26$, we see that condition (14) is satisfied, so that

$$G(8) \leqslant 2(26)+21 = 73.$$

Acknowledgment. The author is greatly indebted to the referee for making many important changes. He in fact, rewrote § 2 of the paper, avoiding complicated arguments given by the author.

References

[1] H. Davenport, *On sums of positive integral k-th powers*, Amer. J. Math. 64 (1942), pp. 189–198.

[2] L. K. Hua, *Additive Primzahltheorie*, Leipzig 1959.

[3] K. Thanigasalam, *On Waring's problem*, Acta Arith. 38 (1979), pp. 141–155.

DEPARTMENT OF MATHEMATICS
PENNSYLVANIA STATE UNIVERSITY
Beaver Campus
Monaca, PA 15061

Received on 26.11.1980
and in revised form on 9.2.1981 (1234)

ACTA ARITHMETICA
XLII (1982)

Semigroup-valued multiplicative functions

by

IMRE Z. RUZSA (Budapest)

Concepts and notations. The following concepts and notations will be used throughout the paper without any further reference.

$\boldsymbol{N}$: the set of natural numbers.

arithmetical function: any function whose domain is $\boldsymbol{N}$.

G-valued function: any function whose values lie in G.

multiplicative function: an arithmetical function f, on whose range an operation, written multiplicatively, is defined, and which satisfies $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$.

For a G-valued multiplicative function, where G is a semigroup, we shall use the term *G-multiplicative function*.

An arithmetical function is *completely multiplicative* if it satisfies the above functional equation for all, not necessarily coprime pairs of integers.

An arithmetical function f is *strongly multiplicative* if it is multiplicative and satisfies $f(p^k) = f(p)$ for every prime p and natural number k.

p always stands for a prime number.

$\sum_p$, $\prod_p$ denote sum, resp. product over primes.

The *asymptotic density* of a set A of natural numbers is defined by

$$d(A) = \lim_{x\to\infty} x^{-1} |A \cap [1, x]|$$

if this limit exists.

The *logarithmic density* is defined by

$$\mathrm{dl}(A) = \lim_{x\to\infty} (\log x)^{-1} \sum_{a\leqslant x, a\in A} a^{-1}$$

if it exists. Taking the limit superior in the above formula we obtain the *upper logarithmic density*, denoted by $\mathrm{dl\,sup}\,A$.

$G < G_1$ among groups denotes that G is a subgroup of G_1.

1. Introduction. In a sense this paper is a continuation to my paper *General multiplicative functions*, [3], to which I shall refer as GMF, but the reader is not supposed to have read it, every concept and result needed will be restated.