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Oscillatory properties of $M(x) = \sum_{n\leqslant x}\mu(n)$, I

by

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1. Mertens conjectured for the partial sums of the Möbius function the inequality

(1.1) $$|M(x)| \leqslant \sqrt{x} \quad (x \geqslant 1).$$

Though very improbable, it has not been disproved so far. The best result, due to H. J. J. te Riele [9], gives

(1.2) $$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} > 0.86.$$

In the present work we shall discuss the more general problem of the lower estimation of

(1.3) $$S_M(Y) = \max_{x\leqslant Y} |M(x)|$$

and

(1.4) $$D_M(Y) = \frac{1}{Y}\int_1^Y |M(x)|\,dx.$$

Although one can easily prove by standard methods $M(x) = \Omega(\sqrt{x})$ and $D_M(Y) = \Omega(\sqrt{Y})$, no satisfactory results are known for $S_M(Y)$ and $D_M(Y)$, even supposing very deep conjectures such as the Riemann hypothesis or the simplicity of all zeros additionally. The second problem would be to get "effective" estimates.

The third problem is to give lower bounds for the modified functions $S_M^*(Y)$, $D_M^*(Y)$ where the maximum and the integral, resp. are taken on some restricted interval $[A(Y), Y]$.

Such problems were considered more difficult for the function $M(x)$ than for the remainder term of the prime number formula $\Delta(x) = \psi(x) - x$ because in the explicit formula of $M(x)$ (in the case where all zeros are assumed to be simple) the terms have the form $x^\varrho/(\varrho\zeta'(\varrho))$, which are far more inconvenient than the corresponding terms x^{ϱ}/ϱ in the case of $\Delta(x)$.

2. S. Knapowski devoted a series of papers ([5], [6], [7], [8]) published in Acta Arithmetica to these questions. He attacked the problem with Turán's method, but owing to the difficulties mentioned above his results were not comparable with those obtained by Turán's method in the investigation of $\Delta(x)$ (see. e.g. Turán [10]), although the proofs were more complicated. His strongest result gives, supposing the Riemann hypothesis,

$$(2.1)\qquad \int_{A_1(Y)}^{Y} \frac{|M(x)|}{x}\,dx > \sqrt{Y}\exp\left(-12\frac{\log Y}{\log_2 Y}\log_3 Y\right)$$

where $\log_\nu Y$ denotes the ν times iterated logarithmic function, $Y > c_1$ (explicit constant) and

$$(2.2)\qquad A_1(Y) = Y\exp\left(-100\frac{\log Y}{\log_2 Y}\log_3 Y\right).$$

This implies naturally

$$(2.3)\qquad \max_{A_1(Y)\leqslant x\leqslant Y}|M(x)| \geqslant \frac{1}{Y}\int_{A_1(Y)}^{Y}|M(x)|\,dx > \sqrt{Y}\exp\left(-112\frac{\log Y}{\log_2 Y}\log_3 Y\right)$$

(supposing the Riemann hypothesis).

The first effective unconditional bounds were proved by I. Kátai ([1], [2], [3], [4]) with the use of a different method.

He was able to prove that for $Y > c_2$ (in what follows c_ν will always denote explicitly calculable positive absolute constants)

$$(2.4)\qquad S_M(Y) \geqslant \max_{Y^{\varkappa}\leqslant x\leqslant Y}|M(x)| \geqslant Y^{\varkappa/2}$$

and

$$(2.5)\qquad \int_{1}^{Y}\frac{|M(x)|}{x}\,dx > c_3 Y^{\varkappa/2},$$

with $\varkappa = (2-\sqrt{3})^2 = 0.07\ldots$ in [1]. Using the numerical result of Rosser and Schoenfeld (that the first two million zeros are on the line $\sigma = 1/2$) and other ideas, he proved (2.4)–(2.5), and somewhat later [4] with $\varkappa = 0.36$ which gives 0.18 in the exponent. But unlike (2.1)–(2.3) the inequality (2.5) does not imply even $D_M(Y)\to\infty$.

He also reached stronger but ineffective results. Supposing the existence of a zeta-zero $\varrho_0 = \beta_0 + i\gamma_0$, he proved [3] for $Y > Y_0(\varrho_0, \varepsilon)$ (ineffective constant) that

$$(2.6)\qquad \int_{Y^{1-\varepsilon}}^{Y}\frac{|M(x)|}{x}\,dx > Y^{\beta_0-\varepsilon},$$

which already implies that

$$(2.7)\qquad \max_{Y^{1-\varepsilon}\leqslant x\leqslant Y}|M(x)| \geqslant \frac{1}{Y}\int_{Y^{1-\varepsilon}}^{Y}|M(x)|\,dx > Y^{\beta_0-\varepsilon}.$$

This naturally gives the unconditional but ineffective inequalities

$$(2.8)\qquad S_M(Y) \geqslant D_M(Y) \geqslant Y^{1/2-\varepsilon}\quad (Y > Y_2(\varepsilon)).$$

Supposing the Riemann hypothesis, he proved [2] for $Y > c_3$ that

$$(2.9)\qquad S_M(Y) \geqslant \sqrt{Y}\exp\left(-c_4(\log_2 Y)^2\right).$$

(2.7) and (2.9) together clearly imply that the estimation (2.9) also holds unconditionally but only for $Y > Y_3(\varepsilon)$, an ineffective constant.

3. In the present work we shall show that by using a new and simpler method the above-mentioned problems can be treated with very satisfactory effective lower estimations and good localizations. Further, we do not need any unproved hypothesis or extended computational results. (The only computational result we use is that the first zeta-zero ϱ_1 has the form $1/2 + i\gamma_1$ with $\gamma_1 = 14.13\ldots$, but even this plays a role only if we want to give the value of some explicitly calculable constants.)

We shall prove the following

THEOREM. *If $\varrho_0 = \beta_0 + i\gamma_0$ is a zeta-zero, then for $Y > e^{|\gamma_0|+4}$*

$$(3.1)\qquad D_M(Y) \geqslant \frac{1}{Y}\int_{Y/(100\log Y)}^{Y}|M(x)|\,dx > \frac{c_5 Y^{\beta_0}}{|\varrho_0|^3}$$

(*where some computation shows that c_5 can be chosen as* 1/6).

We want to note that instead of $c_5|\varrho_0|^{-3}$ one can write $c|\varrho_0|^{-5/2}\times\log|\varrho_0|$ too.

(3.1) trivially implies the

COROLLARY I. *For $Y > e^{|\gamma_0|+4}$ one has*

$$(3.2)\qquad S_M(Y) \geqslant \max_{Y/(100\log Y)\leqslant x\leqslant Y}|M(x)| \geqslant c_5 Y^{\beta_0}/|\varrho_0|^3.$$

If we use our theorem and the corollary with $\varrho_0 = \varrho_1 = 1/2 + i\gamma$ we immediately get

COROLLARY II. *For* $Y > 2$

$$\max_{Y/(100\log Y)\leqslant x\leqslant Y} |M(x)| \geqslant \frac{1}{Y}\int\limits_{Y/(100\log Y)}^{Y} |M(x)|\,dx \geqslant c_6\sqrt{Y} \tag{3.3}$$

(*with the value* $c_6 = 1/17000$).

This shows that not only individual values of $|M(x)|$ are $\geqslant c\sqrt{x}$ but also their average and every constant can be calculated effectively. Unfortunately the value of the constant c_6 is not large enough to disprove the Mertens conjecture. With finer calculation the value of c_6 can be somewhat increased but not very much. At the same time we remark that Corollary I is the first effective proof of the invalidity of the Mertens conjecture if the Riemann hypothesis is supposed to be false. By this we mean that no earlier methods could prove even any effective inequality of the type

$$\max_{1\leqslant x\leqslant Y}\frac{|M(x)|}{\sqrt{x}} > 1 \tag{3.4}$$

for $Y = Y(\varrho_0)$ an effective constant depending on $\varrho_0 = \beta_0+i\gamma_0$, where $\beta_0 > 1/2$ is supposed, not to speak of the sharper inequality

$$S_M(Y) > Y^{\beta_0-\varepsilon} \quad (Y > Y(\varrho_0, \varepsilon)) \tag{3.5}$$

or

$$D_M(Y) > Y^{\beta_0-\varepsilon} \quad (Y > Y(\varrho_0, \varepsilon)). \tag{3.6}$$

The analogue of (3.5) for the case $\Delta(x) = \psi(x)-x$ was a famous problem of Littlewood, solved by Turán [10] in 1950. But owing to the intractable form of the residues $(\varrho\zeta'(\varrho))^{-1}$ S. Knapowski was not able to solve it for the case of $M(x)$ by using Turán's method. It is interesting to note that our method works more easily for the Möbius function than for the remainder term of the prime number formula.

Concerning the sign changes of $M(x)$, the present author showed that $M(x)$ changes sign in every interval of the form $[Y\exp(-c'\log_2^{3/2}Y), Y]$ for $Y > c''$, where c' and c'' are explicitly calculable positive constants. This theorem will be proved in part II.

4. In order to prove our theorem let the entire function $g(s)$ be defined by

$$g(s) \stackrel{\text{def}}{=} \frac{(s+i\gamma_0-2)\zeta(s+i\gamma_0-1)}{(s-1-\beta_0)\prod_{\nu=1}^{3}(s+i\gamma_0-1+2\nu)}; \tag{4.1}$$

further, let

$$\lambda \stackrel{\text{def}}{=} \log Y - 2, \tag{4.2}$$

$$r_\lambda(H) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int\limits_{(3)} e^{s^2/\lambda+Hs}g(s)\,ds. \tag{4.3}$$

Using the well-known formula ($\sigma > 2$)

$$\int\limits_1^\infty \frac{M(x)}{x^s}\,ds = \frac{1}{(s-1)\zeta(s-1)} \tag{4.4}$$

with $s+i\gamma_0$ instead of s and interchanging the integrals, we get our basic formula

$$\begin{aligned} U \stackrel{\text{def}}{=} &\int\limits_1^\infty \frac{M(x)}{x^{i\gamma_0}} r_\lambda(\lambda-\log x)\,dx \\ &= \frac{1}{2\pi i}\int\limits_{(3)} e^{s^2/\lambda+\lambda s}g(s)\int\limits_1^\infty \frac{M(x)}{x^{s+i\gamma_0}}\,dx\,ds \\ &= \frac{1}{2\pi i}\int\limits_{(3)} e^{s^2/\lambda+\lambda s}\frac{(s+i\gamma_0-2)}{(s-1-\beta_0)(s+i\gamma_0-1)\prod_{\nu=1}^{3}(s+i\gamma_0-1+2\nu)}\,ds \\ &= e^{(1+\beta_0)^2/\lambda+(1+\beta_0)\lambda}\Big(1-\frac{1}{\varrho_0}\Big)\prod_{\nu=1}^{3}(\varrho_0+2\nu)^{-1}+O(Y). \end{aligned} \tag{4.5}$$

This immediately gives

$$|U| \geqslant c_7\frac{Y^{1+\beta_0}}{|\varrho_0|^3} \tag{4.6}$$

(with $c_7 > 0.9e^{-4}$ by using $|\gamma_0| > 14$).

5. Now if $H \leqslant -2$ then, integrating along the line $\sigma = \lambda$ instead of $\sigma = 3$ in (4.3), we get

$$|r_\lambda(H)| \leqslant \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{\lambda-t^2/\lambda-|H|\lambda}\frac{2}{\lambda-2}\,dt \leqslant e^{-\lambda(|H|-1)}, \tag{5.1}$$

and so

$$\Bigg|\int\limits_{e^{\lambda+2}}^{\infty}\frac{M(x)}{x^{i\gamma_0}}r_\lambda(\lambda-\log x)\,dx\Bigg| \leqslant \int\limits_{e^{\lambda+2}}^{\infty} xe^{-\lambda(\log x-\lambda-1)}dx = \frac{e^{\lambda+4}}{\lambda-2} < \frac{Y}{2}. \tag{5.2}$$

On the other hand, if $H \geqslant \log\lambda+2+\log(3/2)$ then, changing the way of integration in (4.3) for $\sigma = -\lambda$, we have by $\lambda - 2 \geqslant |\gamma_0|$

$$(5.3) \qquad |r_\lambda(H)| \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda - t^2/\lambda - H\lambda} \frac{2\left(\frac{\lambda + |\gamma_0 + t| + 2}{2}\right)^{\lambda + 5/2}}{\lambda\left(\frac{\lambda - 5 + |\gamma_0 + t|}{2}\right)^4} dt$$

$$\leqslant \frac{e^{-\lambda(H-1)}}{\pi}\left((\tfrac{3}{2}\lambda)^\lambda + \frac{1}{\lambda}\int_\lambda^\infty e^{-t^2/\lambda}(\tfrac{3}{2}t)^\lambda\, dt\right)$$

$$= \frac{e^{-\lambda(H-1)}(\tfrac{3}{2}\lambda)^\lambda}{\pi}\left(1 + \int_1^\infty (e^{-y^2}y)^\lambda dy\right)$$

$$< e^{-\lambda(H - \log\lambda - 1 - \log(3/2))} \leqslant e^{-\lambda},$$

and so

$$(5.4) \qquad \left| \int_1^{e^{\lambda - \log\lambda - 2 - \log(3/2)}} \frac{M(x)}{x^{i\gamma_0}} r_\lambda(\lambda - \log x)\, dx \right| < e^\lambda < \frac{Y}{2}.$$

Finally, transforming the way of integration for $\sigma = 0$, we get for an arbitrary H:

$$(5.5) \quad |r_\lambda(H)| \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{4 + (\gamma_0 + t)^2} \cdot c_8\left(\sqrt{1 + (\gamma_0 + t)^2}\right)^{3/2}}{\sqrt{(1 + \beta_0)^2 + t^2} \prod_{\nu=1}^{3} \sqrt{(2\nu - 1)^2 + (\gamma_0 + t)^2}} dt < c_9$$

(where one can choose $c_9 = 0.09$ with some computation).

Now (4.6), (5.2), (5.4), and (5.5) imply that

$$(5.6) \qquad c_9 \int_{Y/(100\log Y)}^{Y} |M(x)|dx \geqslant c_7 \frac{Y^{1+\beta_0}}{|\varrho_0|^3} - Y \geqslant c_{10} \frac{Y^{1+\beta_0}}{|\varrho_0|^3},$$

which proves the theorem.

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