Some asymptotic formulas on generalized divisor functions, III

by

P. Erdős and A. Sárközy (Budapest)

1. Throughout this paper, we use the following notation:
   \( a_1, a_2, \ldots, a_s, X_1, X_2, \ldots \) denote positive absolute constants. We denote the number of elements of the finite set \( S \) by \( |S| \). We write \( e^x = \exp(x) \). We denote the least prime factor of \( n \) by \( p(n) \), while the greatest prime factor of \( n \) is denoted by \( P(n) \). We write \( p^x \mid n \) if \( p^x \mid n \) but \( p^{x+1} \nmid n \). \( \omega(n) \) denotes the number of all the prime factors of \( n \) so that \( \omega(n) = \sum_{p \mid n} 1 \) and we write
   \[ \omega(n, x, y) = \sum_{p \mid n} 1. \]

   The divisor function is denoted by \( d(n) \):
   \[ d(n) = \sum_{d \mid n} 1. \]

   Let \( A \) be a finite or infinite sequence of positive integers \( a_1 < a_2 < \ldots \)
   Then we write
   \[ N_A(x) = \sum_{\substack{a \in A \leq x}} 1, \quad f_A(x) = \sum_{\substack{a \in A \leq x}} 1, \quad d_A(n) = \sum_{\substack{a \mid n \text{ and } a \in A}} 1 \]
   (in other words, \( d_A(n) \) denotes the number of divisors amongst the \( a_i \)'s) and
   \[ D_A(x) = \max_{1 \leq a \leq x} d_A(a). \]

   The aim of this series is to investigate the function \( D_A(x) \). (See [1] and [2]; see also Hall [4].) Clearly,
   \[ \sum_{1 \leq a \leq x} d_A(n) = \omega(x) + O(x). \]
   Thus if \( f_A(x) \) is large then we have \( D_A(x) f_A(x) \gg 1 \). In Part II of this
series (see [2]), we showed that \( f_d(x) \to +\infty \) implies that
\[
\lim_{x \to +\infty} \sup D_d(x) = +\infty,
\]
in fact, we have
\[
\lim_{x \to +\infty} D_d(x) \exp\left(\frac{c_1}{\log f_d(x)}\right) = +\infty.
\]
We proved this by showing that if \( f_d(x) \) is large (in fact, it is sufficient to assume that \( N_d(x) \) is large) then there exists an integer \( y \) such that \( x < y \leq \exp((\log x)^2) \) and \( D_d(y) \) is large. In this paper, our aim is to prove that if we have more information about \( f_d(x) \) then \( D_d(x) / f_d(x) \) must be large for the same \( x \). In fact we prove that

**Theorem 1.** For all \( x > 0 \) and \( x > X_e(x) \),
\[
f_d(x) > (\log \log x)^{\alpha x}
\]
implies that
\[
D_d(x) > \alpha f_d(x).
\]
(Note that by Theorem 1 in [1], the lower bound \((\log \log x)^{\alpha x}\) in (1) cannot be replaced by \( \log \log x \).)

Sections 2 and 3 are devoted to the proof of this theorem while in Sections 4 and 5 we discuss some other related results.

2. In order to prove Theorem 1, we need some lemmas.

**Lemma 1.** There exists an absolute constant \( c_2 \) such that for all \( u \geq 0 \) and \( y \geq 1 \) we have
\[
\sum_{\nu \leq u} \frac{1}{\nu} < c_2.
\]

Proof. Lemma 1 can be proved easily by using Brun's sieve. In fact, (3) is trivial for \( u \leq 1 \) (since in this case, the left-hand side is equal to 0), while for \( u > 1 \), (3) is a consequence of [7], p. 53, Theorem 4.10.

**Lemma 2.** Let us write
\[
Q(x) = \nu - (1 + \nu) \log(1 + \nu).
\]
Then for \( 1 \leq y \leq z \leq x \), \( 0 \leq \alpha \leq 1 \) we have
\[
\sum_{\alpha \leq \nu \leq \alpha z} 1 < c_4 \exp\left(\frac{Q(\nu) \log \log u}{\log \log y}\right).
\]
Proof. Let \( 1 \leq y \leq x \), \( 0 \leq \alpha \leq 1 \), and let \( F \) be an arbitrary nonempty set of prime numbers not exceeding \( y \). Put \( F(x) = \sum_{\nu \in F} 1 / \nu \). K. K. Norton proved (see [5], Theorem (5.9); see also Halász [3]) that
\[
\sum_{\nu \in F(x)} 1 < c_4 \exp\left(\frac{Q(\nu) E(\nu)}{\log \log y}\right).
\]
By using this theorem with \( E = \{ p : y < p \leq x \} \) (note that \( E \neq \emptyset \) by \( 2y < x \)), and with respect to the well-known formula
\[
\sum_{\nu \leq x} 1 / \nu = \log x + c_2 + o(1),
\]
we obtain (5).

**Lemma 3.** For \( 1 \leq y \leq z \leq x \), \( 0 \leq \alpha \leq \beta \leq 1 \) we have
\[
\sum_{\alpha \leq \nu \leq \beta \nu} 1 < c_5 (\beta \alpha)^{-1} \exp\left(\frac{Q(\nu) \log \log x}{\log \log y}\right)
\]
(where \( Q(x) \) is defined by (4)).

Proof. Let \( 0 < \alpha \leq \beta < 1 \), and let \( F \) be an arbitrary nonempty set of prime numbers not exceeding \( y \). Put \( E(\nu) = \sum_{p \in F} 1 / \nu \). K. K. Norton proved (see [5], Theorem (5.12); see also Halász [3]) that
\[
\sum_{p \leq y} \frac{1}{p} < c_5 (\beta \alpha)^{-1} \exp\left(\frac{Q(\nu) E(\nu)}{\log \log x}\right).
\]
By using this theorem with \( E = \{ p : y < p \leq x \} \) (again, \( E \neq \emptyset \) by \( 2y < x \)), and with respect to (6), we obtain (7).

3. In this section, we complete the proof of Theorem 1. Define the positive integer \( R \) by
\[
\alpha^1 - \frac{1}{2 \log f_d(x)} < e^{-Q(x)} \leq \alpha^1 - \frac{1}{2 \log f_d(x)}
\]
\[
\text{i.e.,}
\]
\[
2^{R-1} < \frac{3 \log x}{f_d(x)} \leq 2^R, \quad R - 1 < \frac{1}{\log \log f_d(x)} \leq R.
\]
Then for large \( x \), we have
\[
R < \frac{1}{\log \log f_d(x)} + 1 < 2 \log \log x.
\]
For \( i = 0, 1, \ldots, R \), let
\[
\alpha_i = \alpha^{i - \frac{1}{2N}},
\]
and for \( i = 1, 2, \ldots, R \), put
\[
A_i = A \cap [\alpha_{i-1}, \alpha_i].
\]
Then by (1) and (8), for large \( x \) we have

\[
\sum_{i=1}^{R} \left( \sum_{a \leq x} \frac{1}{a} \right) = \sum_{a \leq x} \frac{1}{a} = \sum_{a \leq x} \frac{1}{a} - \sum_{x < a \leq x + R} \frac{1}{a} \\
> f_a(x) - \frac{1}{N} \sum_{x < a \leq x + R} \frac{1}{a} > f_a(x) - \frac{1}{N} \sum_{a > x} \frac{1}{a} = f_a(x) - (1 + o(1)) \log \varepsilon f_a(x) > f_a(x) - \frac{f_a(x)}{2} = \frac{f_a(x)}{2}.
\]

Obviously, there exists an integer \( j \) such that \( 1 \leq j \leq R \) and

\[
\sum_{a \neq d} \frac{1}{a} > \frac{R}{2} \sum_{i=1}^{j} \left( \sum_{a \neq d} \frac{1}{a} \right)
\]

hence with respect to (9) and (10),

\[
f_a(x) = \sum_{a \neq d} \frac{1}{a} > \frac{R}{2} f_a(x) > \frac{f_a(x)}{4 \log \varepsilon f_a(x)}.
\]

Let us fix an integer \( j \) \((1 \leq j \leq R)\) satisfying (11), and write \( A_j \) in the form

\[
A_j = A'_j \cup A''_j
\]

where \( A'_j \) consists of the integers \( a \) such that \( a \in A_j \) and there exists an integer \( d \) satisfying

\[
(\log x)^3 < d \leq \omega^{1/2} + \omega f_a(x)
\]

and \( d \mid a \), while \( A''_j \) consists of the integers \( a \) such that \( a \in A_j \) and \( d \nmid a \) for all \( d \) satisfying (13). (For \( \omega^{1/2} + \omega f_a(x) \leq (\log x)^3 \), we have \( A_j = \emptyset \).)

We have to distinguish two cases.

Case 1. Assume first that

\[
f_a'(x) = \sum_{a \neq d} \frac{1}{a} > \frac{1}{2} f_a(x) > \frac{f_a(x)}{8 \log \varepsilon f_a(x)}.
\]

Then by (11), we have

\[
f_a(x) = \sum_{a \neq d} \frac{1}{a} > \frac{1}{2} f_a(x) > \frac{f_a(x)}{8 \log \varepsilon f_a(x)}.
\]

For \( a \in A'_j \), write \( a \) in the form

\[a = d^*(a) b(a),\]

where \( d^*(a) \) denotes the least integer \( d \) such that \( d \) satisfies (13) and \( d \mid a \). Then for \( a \in A'_j \), we have \( b(a) \leq x \) and \( (\log x)^3 < d^*(a) \) so that

\[
\sum_{a \leq x} \frac{1}{a} = \sum_{a \leq x} \frac{1}{d^*(a) b(a)} = \sum_{b \leq x} \frac{1}{b} \sum_{a \leq x} \frac{1}{d^*(a) b(a)}.
\]

\[
< \sum_{b \leq x} \frac{1}{b} \sum_{a \leq x} \left( \frac{1}{d^*(a) b(a)} \right) = \frac{1}{(\log x)^3} \sum_{b \leq x} \frac{1}{b} \sum_{a \leq x} \frac{1}{d^*(a) b(a)}.
\]

\[
< \frac{1}{(\log x)^3} \left( \max_{1 < b \leq x} \frac{1}{b} \right) \sum_{b \leq x} \frac{1}{b} \leq \frac{2}{(\log x)^2} \left( \max_{1 < b \leq x} \sum_{a \leq x} \frac{1}{a} \right).
\]

If \( x \) is large enough (in terms of \( \omega \)) then (14) and (15) yield that

\[
\max_{1 < b \leq x} \sum_{a \leq x} \frac{1}{a} > \frac{(\log x)^2}{2} \sum_{a \leq x} \frac{1}{a} > \frac{(\log x)^2}{16 \log \varepsilon f_a(x)} f_a(x) > \omega f_a(x) + 1
\]

so that there exists an integer \( b_0 \) for which

\[
1 < b_0 \leq \omega^{1/2}
\]

and

\[
\sum_{a \leq x} \frac{1}{a} > \omega f_a(x) + 1.
\]

Put \( s = [\omega f_a(x)] + 1 \). Then by (17), there exist distinct integers \( a_1, a_2, \ldots, a_s \) such that \( a_i \) can be written in the form

\[a_i = b_i d_i^*(a_i) = b_i d_i^*(a_i),\]

where

\[
(\log x)^3 < d_i^*(a_i) \leq \omega^{1/2} + \omega f_a(x).
\]

Let

\[u = b_1 d_1 \ldots d_s \leq \omega^{1/2} \left( \omega^{1/2} + \omega f_a(x) \right) = \omega^{1/2} \omega^{1/2} \omega f_a(x) = \omega,
\]

Then by (16) and (18), we have

\[
\omega^{1/2} \omega^{1/2} \omega f_a(x) \leq \omega^{1/2} \omega^{1/2} \omega f_a(x) \omega^x = \omega.
\]

\[
\omega^{1/2} \omega^{1/2} \omega f_a(x) \omega^x = \omega.
\]
and obviously, \( a_i = b_i d_i / u \) and \( a_i = b_i d_i \in \mathcal{A} \) so that
\[
\frac{1}{2} f_{\mathcal{A}}(x) = \sum_{a \in \mathcal{A}} \frac{1}{a} \geq \frac{1}{2} f_{\mathcal{A}}(x) = \frac{1}{2} \sum_{a \in \mathcal{A}} \frac{1}{a}.
\]

(19) and (20) yield (2) and this completes the proof of Theorem 1 in this case.

Case 2. Assume now that
\[
f_{\mathcal{A}}(x) = \sum_{a \in \mathcal{A}} \frac{1}{a} \geq \sum_{a \in \mathcal{A}} \frac{1}{a} - \sum_{a \in \mathcal{A}} \frac{1}{a} = f_{\mathcal{A}}(x) - \frac{1}{2} f_{\mathcal{A}}(x) = \frac{1}{2} f_{\mathcal{A}}(x) > f_{\mathcal{A}}(x) = \frac{1}{2 \log \log x}.
\]

For \( u \geq 1 \), let
\[
g(u) = \left( \frac{3 \log f_{\mathcal{A}}(x)}{u} \right)^u.
\]

Then for \( 1 \leq u < \frac{3}{e} \log f_{\mathcal{A}}(x) \), the function \( g(u) \) is increasing since
\[
g'(u) = g(u) \log \left( \frac{3 \log f_{\mathcal{A}}(x)}{eu} \right) > 0
\]
and for large \( x \), we have
\[
g(1) = 3 \log f_{\mathcal{A}}(x) = \frac{f_{\mathcal{A}}(x)}{(\log \log x)^{\frac{1}{3}}}.
\]

Thus for \( u \geq 1 \), let
\[
g(t) = \left( \frac{3 \log f_{\mathcal{A}}(x)}{t} \right)^t = \frac{f_{\mathcal{A}}(x)}{(\log \log x)^{\frac{1}{3}}}.
\]

We need a lower bound for this number \( t \). By (1), we have
\[
g\left( \frac{1}{t} \log f_{\mathcal{A}}(x) \right) = \left( e^{\frac{1}{t} \log f_{\mathcal{A}}(x)} = \left( f_{\mathcal{A}}(x) \right)^{\frac{1}{t} \log e}
\]
\[
< \frac{f_{\mathcal{A}}(x)^{\frac{1}{t} \log e}}{f_{\mathcal{A}}(x)^{\frac{1}{t} \log e}} < \frac{f_{\mathcal{A}}(x)}{(\log \log x)^{\frac{1}{3}}}.
\]

(23), (24) and (25) imply that
\[
\frac{1}{t} \log f_{\mathcal{A}}(x) < t
\]
(since \( g(u) \) is increasing for \( 1 < u < \frac{3}{e} \log f_{\mathcal{A}}(x) \)).

Let us write
\[
z_j = \max\left\{ \omega(a, z_j, w^{1/2}) \mid (\log \omega)^{\frac{1}{3}} \right\}.
\]

Let \( \mathcal{A}^j \) denote the set of the integers \( a \) such that \( a \in \mathcal{A}^j \) and
\[
\omega(a, z_j, w^{1/2}) > t.
\]

Now we are going to give an upper estimate for
\[
\sum_{a \in \mathcal{A}^j} \frac{1}{a} = \sum_{a \in \mathcal{A}^j} \frac{1}{a}.
\]

If \( a \in \mathcal{A}^j \) and \( \omega(a, z_j, w^{1/2}) < t \) then by the definition of \( \mathcal{A}^j \), we have
\[
\omega^{-1} = \omega^{-1} = x_j \leq a < z_j = \omega^{1/2}
\]
and \( a \) can be written in the form
\[
a = \omega^{1/2} \ldots p_{m}^{1/2} \omega
\]
where \( \omega^{1/2} \) and \( \omega < \omega_{1}, \ldots < \omega_{m} \leq \omega^{1/2} \) and
\[
w^{1/2} \omega^{1/2}.
\]

Thus by Lemma 1, we have
\[
\sum_{a \in \mathcal{A}^j} \frac{1}{a} \leq \sum_{a \in \mathcal{A}^j} \frac{1}{a} \sum_{a \in \mathcal{A}^j} \sum_{a \in \mathcal{A}^j} \frac{1}{a} \times
\]
\[
\times \left( \sum_{a, \omega(a) > \omega^{1/2}} \frac{1}{a} \right)_{x_j < \omega_{1}^{1/2} \ldots p_{m}^{1/2} < w^{1/2}}.
\]
\[
< a \left( \sum_{p|a, a > p^3} \frac{1}{m} \right) \left( 1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left( \sum_{r \leq x, \gcd(r, a) = 1} \frac{1}{p^m} \right) \right)
\]
\[
= e \left( \prod_{p|a} \sum_{i=0}^{+\infty} \frac{1}{p^i} \right) \left( 1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left( \sum_{r \leq x, \gcd(r, a) = 1} \frac{1}{p^m} \right) \right)
\]
\[
= e \left( \prod_{p|a} \frac{1}{1 - 1/p} \right) \left( 1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left( \sum_{r \leq x, \gcd(r, a) = 1} \frac{1}{p^m} \right) \right).
\]

It is well-known that
\[
\prod_{p|a} \frac{1}{1 - 1/p} < e \log y
\]
and
\[
\sum_{p|a} \frac{1}{p^m} = \log y + o(1).
\]

Thus with respect to (1), (23), (24) and (26), and by using the Stirling-formula, we obtain from (27) that for \( x > X_0(a) \),

\[
(28) \quad \sum_{a \leq d, a \nmid t} \frac{1}{a} \sum_{a \leq d, a \nmid t} \left( \log \left( \log \alpha \right) \right)^m \left( 1 + \sum_{m=1}^{[t]} \frac{1}{m!} \right) \left( \log \alpha \right)^{m} - \log \alpha + o(1))
\]
\[
= e \log x \left( 1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left( \log \frac{\log \alpha^{1/m} + \log \alpha}{\log \alpha} \right) \right)
\]
\[
\leq e \log x \left( 1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left( \log \frac{\log \alpha^{1/m} + \log \alpha}{\log \alpha^{1/m} + \log \alpha} \right) \right)
\]
\[
= e \log x \left( 1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left( \log \frac{\log \alpha^{1/m} + \log \alpha}{\log \alpha^{1/m} + \log \alpha} \right) \right)
\]
\[
< e \log x \left( \log \frac{\log \alpha^{1/m} + \log \alpha}{\log \alpha^{1/m} + \log \alpha} \right)
\]
\[
< e \log x \left( \log \frac{\log \alpha^{1/m} + \log \alpha}{\log \alpha^{1/m} + \log \alpha} \right) \leq \frac{1}{18} \log x \cdot \frac{f_d(x)}{\log x}
\]
\[
< \frac{1}{18} \log x \cdot \frac{f_d(x)}{\log x} = \frac{1}{18} \log x \cdot \frac{f_d(x)}{\log x}.
\]

(22) and (28) yield that

\[
(29) \quad f_d(x) = \sum_{a \leq d} \frac{1}{a} = \sum_{a \leq d} \frac{1}{a} - \frac{1}{a}
\]
\[
\sum_{a \leq d} \frac{1}{a} = \frac{f_d(x)}{8 \log \log x} + \frac{f_d(x)}{16 \log \log x} = \frac{f_d(x)}{16 \log \log x}.
\]

Let \( S \) denote the set of the integers \( n \) such that \( n \leq a \) and \( n \) can be written in the form

\[
(30) \quad n = \sum_{a \leq d} \frac{1}{a} \quad \text{where} \quad a \in A^*_d \quad \text{and} \quad \omega(n, x, x^{1/2}) > \frac{18}{19} \log f_d(x).
\]

For fixed \( n \in S \), let \( \varphi(n) \) denote the number of representations of \( n \) in the form (30).

Then we have

\[
(31) \quad \sum_{a \leq d} \frac{1}{a} = \sum_{a \leq d} \frac{1}{a} = \left( \sum_{a \leq d} \frac{1}{a} \right) - \frac{1}{a} - \frac{1}{a} = \left( \sum_{a \leq d} \frac{1}{a} \right) - \frac{1}{a} - \frac{1}{a}.
\]

In order to estimate the last sum, we use Lemma 2 with \( z_j \), \( w^{1/2} \), \( x/a \) and \( 1/400 \) in place of \( y, x, v \) and \( a \), respectively. Then \( 1 \leq y \) and \( 2y < x \) hold trivially by the definition of \( z_j \) (and by \( f \leq R \)), and also \( z < a \) holds by

\[
z = \omega^{1/2} = \frac{x}{a^{1/2}} = \frac{x}{a^2} < \frac{x}{a} = v
\]
(since we have \( a \in A^*_d \) and thus \( a < x_j \)). Thus Lemma 2 can be applied, and we obtain with respect to (1) and the definition of \( z_j \) that for large \( x \)

\[
(32) \quad \sum_{a \leq d} \frac{1}{a} = \sum_{a \leq d} \frac{1}{a} = \left( \sum_{a \leq d} \frac{1}{a} \right) - \frac{1}{a} - \frac{1}{a} = \left( \sum_{a \leq d} \frac{1}{a} \right) - \frac{1}{a} - \frac{1}{a}.
\]

(32) and (28) yield that

\[
(29) \quad f_d(x) = \sum_{a \leq d} \frac{1}{a} = \sum_{a \leq d} \frac{1}{a} - \frac{1}{a}.
\]

Let \( S \) denote the set of the integers \( n \) such that \( n \leq a \) and \( n \) can be written in the form

\[
(30) \quad n = a \quad \text{where} \quad a \in A^*_d \quad \text{and} \quad \omega(n, x, x^{1/2}) > \frac{18}{19} \log f_d(x).
\]

For fixed \( n \in S \), let \( \varphi(n) \) denote the number of representations of \( n \) in the form (30).

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\]

In order to estimate the last sum, we use Lemma 2 with \( z_j \), \( w^{1/2} \), \( x/a \) and \( 1/400 \) in place of \( y, x, v \) and \( a \), respectively. Then \( 1 \leq y \) and \( 2y < x \) hold trivially by the definition of \( z_j \) (and by \( f \leq R \)), and also \( z < a \) holds by

\[
z = \omega^{1/2} = \frac{x}{a^{1/2}} = \frac{x}{a^2} < \frac{x}{a} = v
\]
(since we have \( a \in A^*_d \) and thus \( a < x_j \)). Thus Lemma 2 can be applied, and we obtain with respect to (1) and the definition of \( z_j \) that for large \( x \)

\[
(32) \quad \sum_{a \leq d} \frac{1}{a} = \sum_{a \leq d} \frac{1}{a} = \left( \sum_{a \leq d} \frac{1}{a} \right) - \frac{1}{a} - \frac{1}{a} = \left( \sum_{a \leq d} \frac{1}{a} \right) - \frac{1}{a} - \frac{1}{a}.
\]
Furthermore, by (8) we have

$$
\frac{399}{400} \sum_{j < \rho \approx \log x^{1/3}} \frac{1}{p} > \frac{399}{400} \left( \log \frac{\log x^{1/3}}{\log \epsilon} - c_{15} \right).
$$

By (8), we have

$$
\frac{f_d(x)}{x} \leq \log x^{1/3} \leq \log x^{1/3}.
$$

We obtain from (1) and (34) that

$$
\log \eta = \max \{ \log \eta x, \log \eta x^{1/3} \cos \eta \}
= \max \left\{ \log \eta x, \log \eta x^{1/3} \cos \eta \right\}
= \max \left\{ \log \eta x, \frac{3 \log \log x}{2 \log \eta} \log \eta x^{1/3} \right\},
$$

$$
\leq \max \left\{ \log \eta x, \frac{3 \log \log x}{2 \log \eta} \log \eta x^{1/3} \right\}
\leq \max \left\{ \log \eta x, \frac{3 \log \log x}{2 \log \eta} \log \eta x^{1/3} \right\},
$$

$$
= \frac{18 \log \eta x^{1/3}}{(18 \log x)^{1/20}}.
$$

(33) and (35) yield for large $x$ that

$$
\frac{399}{400} \sum_{j < \rho \approx \log x^{1/3}} \frac{1}{p} > \frac{399}{400} \left( \log \frac{\log x^{1/3}}{\log \epsilon} - c_{15} \right)
\geq \frac{399}{400} \left( \log \frac{f_d(x)^{1/20}}{18} - c_{15} \right) > \frac{18}{19} \log f_d(x)
$$

hence

$$
\sum_{\epsilon \Delta = 1} 1 \leq \sum_{\epsilon \Delta = 1} \sum_{\epsilon \Delta = 1} 1.
$$

(32) and (37) yield that

$$
\sum_{\epsilon \Delta = 1} \frac{1}{3 \alpha} < \frac{x}{\log f_d(x)}.
$$

Thus we obtain from (29) and (31) that

$$
\sum_{n \leq x} \varphi(n) \leq \sum_{\epsilon \Delta = 1} \left( \frac{x}{a} - \frac{1}{3} \frac{x}{a} \right) < \sum_{\epsilon \Delta = 1} \frac{x}{2a}
= \frac{x}{2} f_d(x) > \frac{\sqrt{d} f_d(x)}{32 \log x}.
$$

Now we are going to give an upper estimate for $\sum_{n \leq x} \varphi(n)$. Obviously,

$$
\sum_{n \leq x} \varphi(n) \leq \sum_{n \leq x} \varphi(n) \leq \sum_{n \leq x} D_\alpha(x) = |S| D_\alpha(x).
$$

Thus in order to obtain an upper bound for $\sum_{n \leq x} \varphi(n)$, we have to estimate $|S|$.

If $n \in S$ then by (26), (30) and the definition of the set $A_d^\alpha$, we have

$$
\omega(n, x^{1/3}) = \omega(n, x^{1/3} + \omega(n, x^{1/3}) = \omega(n, x^{1/3} + \omega(n, x^{1/3}))
> \frac{18}{19} \log f_d(x) + \frac{18}{19} \log f_d(x) = \frac{55}{38} \log f_d(x)
$$

hence

$$
|S| \leq \sum_{\epsilon \Delta = 1} \sum_{\epsilon \Delta = 1} \frac{1}{3 \alpha}.
$$

In order to estimate this sum, we use Lemma 3 with $\epsilon, x^{1/3}$, $\frac{17}{37}$ and 9/10 in place of $\epsilon, x, n, a$ and $\beta$, respectively. (1 < $y, x, n, a)$ and $\beta$ hold trivially with respect to the definition of $\epsilon, x.$) We obtain with respect to (8) and the definition of $\epsilon, x.$)

$$
\sum_{n \leq x} \sum_{\epsilon \Delta = 1} \frac{1}{3 \alpha} \leq \sum_{\epsilon \Delta = 1} \frac{1}{3 \alpha} \exp \left( \frac{17}{37} \log \frac{\log x^{1/3}}{\log \epsilon} \right)
$$

$$
< \frac{91}{1000} \log \frac{\log x^{1/3}}{\log \epsilon} \log \frac{\log x^{1/3}}{\log \epsilon}
$$

$$
< \frac{91}{1000} \log \frac{\log x^{1/3}}{\log \epsilon} \log \frac{\log x^{1/3}}{\log \epsilon}
$$

$$
< \alpha \exp \left( - \frac{91}{1000} \log \frac{\log x^{1/3}}{\log \epsilon} \log \frac{\log x^{1/3}}{\log \epsilon} \right)
$$

$$
< \alpha \exp \left( - \frac{91}{1000} \log \frac{\log x^{1/3}}{\log \epsilon} \log \frac{\log x^{1/3}}{\log \epsilon} \right)
$$
Furthermore, with respect to (6) and the definition of \( z \), we have
\[
\frac{54}{37} \sum_{y < p \leq x} \frac{1}{p} - \frac{54}{37} \sum_{y < p \leq x} \frac{1}{p^{1 + \varepsilon}} \leq \frac{54}{37} \left( \sum_{p < x} \frac{1}{p} - \sum_{p < x} \frac{1}{p^{1 + \varepsilon}} \right) = \frac{54}{37} (\log \log x)^{1/10} - \log \log x^{1/10} + o(1) + o(1) = \frac{54}{37} (\log \omega(\log \log x) + o(1)) < \frac{54}{37} \log f_4(x) + o(1) < \frac{54}{37} \log f_4(x) + o(1)
\]
and thus
\[
\sum_{y < p \leq x} \frac{1}{p} \leq \sum_{y < p \leq x} \frac{1}{p^{1 + \varepsilon}} + o(1).
\]

(40), (41) and (42) yield that
\[
|S| \leq f_4(x)^{-1/10}.
\]

Finally, we obtain from (38), (39) and (43) that
\[
\frac{D_4(x)}{\log \log x < \sum_{\eta(x) \leq \xi(\log \log x)^{1/10}} \sum_{\eta(x) \leq \xi(\log \log x)^{1/10}} D_4(x) - \log \log x^{1/10} D_4(x)
\]
hence with respect to (1),
\[
D_4(x) > f_4(x) \frac{D_4(x)^{1/10}}{\log \log x} > f_4(x) \frac{D_4(x)^{1/10}}{\log \log x} > f_4(x) \frac{D_4(x)^{1/10}}{\log \log x} = f_4(x) - \frac{1}{32} (f_4(x)^{1/10}) > \omega f_4(x)
\]
for \( x > X_4(\omega) \). Thus (2) holds also in Case 2 and this completes the proof of Theorem 1.

4. By using the same method, we can show that Theorem 1 is true also with \((\log \log x)^{1/10}\) in place of \((\log \log x)^{1/10}\) on the right-hand side of (1). In fact, in order to prove this, the only non-trivial modifications are that \( t \) must be defined as \( t = \eta \log f_4(x) \) where \( \eta = \eta(x) \) is sufficiently small in terms of \( \varepsilon \), and in (30), the condition \( \omega(\log \log x)^{1/10}/19 \) must be replaced by \( \omega(\log \log x)^{1/10}/19 \) for \( K(\varepsilon) \) is sufficiently large in terms of \( \varepsilon \). Furthermore, then Lemmas 2 and 3 must be replaced by lower and upper estimates for
\[
\sum_{y < p \leq x} \frac{1}{p} - \frac{1}{p}
\]
where \( L \) is arbitrary large but fixed. Such estimates could be deduced by the methods used by K. K. Norton in [1]. (Norton’s estimates cannot be used in the original form since the error terms in his lower and upper estimates depend implicitly on the set \( \mathcal{K} \) of the prime numbers whose multiples we investigate. Thus in our case, these results would yield lower and upper bounds depending implicitly on \( \{p : y < p \leq x\} \), i.e., on \( y \) and \( x \), instead of the explicit estimates needed by us.)

On the other hand, we guess that also the exponent \( 2 + \varepsilon \) could be improved, and, perhaps, Theorem 1 is true also with \((\log \log x)^{1/10}\) or even \( \omega(\log \log x)^{1/10}\) on the right-hand side of (1). This is the reason of that we preferred to work out the slightly weaker estimate given in Theorem 1 whose proof is much simpler.

5. One may expect that if we know that \( f_4(y) \) is large for all \( y \), then Theorem 1 can be sharpened in the sense that the lower bound given for \( f_4(x) \) in (1) (for fixed \( x \)) can be replaced by a much smaller lower bound for \( f_4(x) \) (for all \( y \)). In fact, we show in this section that

**Theorem 2.** For all \( \omega > 0 \), there exists a real number \( X_4(\omega) \) such that if \( x > X_4(\omega) \) and writing \( y = \exp \left( \frac{\log x}{(\log \log x)^{1/10}} \right) \), we have
\[
f_4(y) > 22 \log \log y,
\]
then
\[
D_4(x) > \omega f_4(x).
\]

Furthermore, we show that Theorem 2 is best possible except the value of the constant factor on the right of (44):

**Theorem 3.** There exist positive constants \( c_1, c_2, X_4 \) and an infinite sequence \( A \) such that
\[
f_4(x) > c_1 \log \log x \quad \text{for all } x > X_4
\]
and
\[
\lim \inf_{x \rightarrow \infty} \frac{D_4(x)}{f_4(x)} < c_2.
\]

In order to prove Theorem 2, we need the following lemma:

**Lemma 4.** If \( x > 1 \), \( t \geq 1 \) and \( A \) is an arbitrary sequence of positive integers such that
\[
D_4(x) < t
\]
then we have

$$N_{A}(a^{\log x}) \leq t.$$  

Proof of Lemma 4. Assume indirectly that

$$N_{A}(a^{\log x}) > t,$$

i.e.,

$$N_{A}(a^{\log x}) \geq [t]+1.$$  

Then there exist integers $a_1, a_2, \ldots, a_{[t]+1}$ such that $a_1 \in A$, $a_2 \in A$, ..., $a_{[t]+1} \in A$ and

$$a_1 < a_2 < \ldots < a_{[t]+1} \leq a^{\log x}.$$  

Put $u = a_1 a_2 \ldots a_{[t]+1}$. Then $a_i | u$ for $1 \leq i \leq [t]+1$ and thus

$$D_A(u) > [t]+1 > t.$$  

On the other hand, by (49) we have

$$u = a_1 a_2 \ldots a_{[t]+1} \leq (a^{\log x})^{[t]+1} \leq (a^{\log x})^{x+1} = x.$$  

(50) and (51) imply that

$$D_A(x) > t,$$

in contradiction with (48) which completes the proof of Lemma 4.

Proof of Theorem 2. We have to distinguish two cases.

Case 1. Let

$$f_A(x) > (\log \log x)^{\delta_0}.$$  

Then for $x > X_{\delta}(\omega)$, (45) holds by Theorem 1.

Case 2. Let

(52)  

$$f_A(x) \leq (\log \log x)^{\delta_0}.$$  

Assume indirectly that

(53)  

$$D_A(x) \leq \omega f_A(x).$$

Then by using Lemma 4 with $t = \omega f_A(x)$, we obtain that

(54)  

$$N_{A}(a^{\log x}) = N_{A}(a^{\log (\log \log x)^{\delta_0} + 1}) \leq t = \omega f_A(x).$$

Put $M = N_{A}(a^{\log (\log \log x)^{\delta_0} + 1})$ and let $a_1 < a_2 < \ldots < a_M$ denote the $a_i$'s not exceeding $a^{\log (\log \log x)^{\delta_0} + 1}$. Then by (52) and (54), we have

$$f_A\left(\frac{x}{a_i}\right) \leq \log \frac{x}{a_i} \leq \log x + \omega \log \log x + \epsilon_{\delta_0} < 21 \log \log \log x.$$  

On the other hand, by (51) we have

$$a^{\log (\log \log x)^{\delta_0} + 1} = \exp\left(\frac{\log x}{\omega f_A(x)}\right) = \exp\left(\frac{\log x}{\omega (\log \log x)^{\delta_0} + 1}\right) \Rightarrow \exp\left(\frac{\log x}{(\log \log x)^\alpha}\right) = y.$$  

Thus (44) yields that

$$f_A(x) = \omega f_A(x) > 21 \log \log \log x$$  

in contradiction with (55) which completes the proof of Theorem 2.

Proof of Theorem 3. In the proof of Theorem 1 in [1], for $x > X_\beta$ we constructed a sequence $B(x)$ such that

(56)  

$$f_{B_{\beta}}(x) > \omega_1 \log \log x,$$

and

(57)  

$$D_{B_{\beta}}(x) < 2 \log \log x.$$  

Let us define the infinite sequence $a_1 < a_2 < \ldots$ by the following recursion: let

$$a_1 = X_\beta$$  

and

$$a_k = \exp(\exp(\exp(a_{k-1}))).$$

For $x > 1$, let

$$E(x) = \{n: \exists x < n \leq x\}.$$  

Finally, let

$$A = \bigcup_{k=1}^{\infty} B(x_k) \cup E(\log \log x_k).$$

We are going to show that this sequence $A$ satisfies both (46) and (47).

First we prove (46). Assume that $x > X_\beta$. Then there exists a uniquely determined positive integer $k (\geq 2)$ such that $a_{k-1} < x \leq a_k$. Then either

$$x_{k-1} < x \leq \exp(a_{k-1}) = \log \log a_k$$

or

$$\exp(a_{k-1}) = \log \log a_k < x \leq a_k$$

holds. If (58) holds, then by (56) we have

$$f_A(x) > f_A(x_{k-1}) > f_{B_{\beta}(x_{k-1})} > \omega_1 \log \log a_{k-1} > \omega_1 \log \log \log x,$$

and
while if (50) holds then
\[ f_A(x) \geq E(\log \log x_k) = \sum_{0 < \log \log x_k < \log \log x_k} \frac{1}{n} > \frac{1}{3} \log \log x_k \geq \frac{1}{3} \log \log \log x. \]

Thus in fact, (46) holds in both cases.

In order to prove that also (47) holds, it is sufficient to show that for \( k = 1, 2, \ldots \), we have
\[ \frac{D_A(x_k)}{f_A(x_k)} < \epsilon_{21}. \]

If \( u < x_k \) then by (57) we have
\[ d_A(u) = \sum_{a \leq u} = \sum_{a \leq \frac{v}{\log x_k}} 1 + \sum_{a \leq x_k} \frac{1}{\log x_k} \leq \sum_{a \leq x_k} 1 + \sum_{a \leq x_k} 1 = D_{BC}(u) \]
\[ \leq \log x_k + D_{BC}(u) < 3 \log \log x_k \]

hence
\[ D_A(x_k) < 3 \log \log x_k. \]

Furthermore, by (56), we have
\[ f_A(x_k) = \sum_{a \leq x_k} \frac{1}{a} = \sum_{a \in \mathcal{D}_2} \frac{1}{a} = f_{BC}(x_k) > \epsilon_{21} \log \log x_k. \]

(61) and (62) yield (60) and the proof of Theorem 3 is completed.

References