Now in view of the work of Schinzel and Tijdeman [10] and Baker [1] on the equation (15), the assertions follows immediately.

(iii) It is easy to see that the equation (14) has only finitely many solutions in integers \( x > 1, y > 1, n > 1, m \) with \( x \neq y, y/\sqrt{m}, m-n \geq 2 \) and \( m(m-n) \geq 6 \) if and only if the conjecture of Pillai [7] that (1) has only finitely many solutions in integers \( m > 1, n > 1, x > 1, y > 1 \) with \( mn \geq 6 \) is correct. This conjecture of Pillai is still open. If \( b = c = d = 1 \), Tijdeman [13] proved that (14) has only finitely many solutions in integers \( x > 1, y > 1, n > 1, m \) with \( x \neq y \) and \( m-n \geq 2 \).

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An application of a formula of Western to the evaluation of certain Jacobsthal sums

by

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1. Introduction and summary. Let \( k \geq 2 \) be a positive integer and let \( p \) be a prime such that \( p \equiv 1 \) (mod 2k). The Jacobsthal sum \( \Phi_k(D) \) is defined by

\[
\Phi_k(D) = \sum_{a=1}^{p-1} \frac{\omega(a^k + D)}{p}
\]

where \( D \) is an integer not divisible by \( p \) and \( \left( \frac{D}{p} \right) \) is the Legendre symbol.

When \( k = 2 \), Jacobsthal ([5]), pp. 240-241 evaluated \( \Phi_2(D) \) when \( D \) is a quadratic residue (mod \( p \)) but left a sign ambiguity in its evaluation when \( D \) is a quadratic non-residue (mod \( p \)). Recently, the authors [3] have shown how to remove this ambiguity by using the law of quartic reciprocity in a form given by Gosset [2]. When \( k = 3 \), von Schurtschas ([9], p. 288) evaluated \( \Phi_3(D) \) when \( D \) is a cubic residue (mod \( p \)) but left an ambiguity in its evaluation when \( D \) is a cubic non-residue (mod \( p \)), and the authors [3] have shown how to remove this ambiguity by using a form of the law of cubic reciprocity given by Emmanuel Lehmer [6].

When \( k = 4 \), Whiteman [12], [13] has shown that

\[
\Phi_4(D) = \begin{cases} 
-4(-1)^{\frac{p-1}{2}}, & \text{if } D \text{ is an octic residue (mod } p), \\
4(-1)^{\frac{p-1}{2}}, & \text{if } D \text{ is a quartic but not an octic residue (mod } p), \\
\pm 4d, & \text{if } D \text{ is a quartic non-residue (mod } p) \end{cases}
\]

where \( p = e^3 + 2d^2 \equiv 1 \) (mod 8), \( e \equiv 1 \) (mod 4).

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Since $2$ is a quadratic residue of a prime $p \equiv 1 \pmod{8}$, $\Phi(p)$ is known from (1.2). In Section 4 of this paper we show how to remove the sign ambiguity in the evaluation of $\Phi(q)$, where $q$ is an odd prime which is a quadratic non-residue (mod $p$), by means of a form of the law of cubic reciprocity given by Western [11] (see Section 2). For example, we prove the following:

**Theorem 2 (b).** Let $p = a^2 + b^2 = c^2 + 2d^2$, $a \equiv c \equiv 1 \pmod{4}$ be a prime $\equiv 1 \pmod{8}$ such that $3$ is a quadratic non-residue (mod $p$). Then

$$\Phi(q) = -2(-1)^{p-13}d,$$

where $b$ and $d$ are chosen to satisfy one of $a \equiv b \equiv d \pmod{5}$ or $a \equiv b \equiv -2d \pmod{5}$.

In order to evaluate $\Phi(q)$ by this method, it is necessary to determine $q^{p-13}/(mod p)$ when $q$ is a quadratic non-residue (mod $p$), i.e. when $q^{p-13}$ is a primitive eighth root of unity (mod $p$). In Section 3 we explicitly evaluate $q^{p-13}/(mod p)$ for $q = 3, 5, 7, 11, 13, 17$ and 19 when

$$\left( \frac{q}{p} \right) = -1, \ p \equiv 1 \pmod{8},$$

in terms of the representations $p = a^2 + b^2 = c^2 + 2d^2$, by giving necessary and sufficient criteria in terms of $a$, $b$, $c$ and $d$, for $q$ to satisfy

$$q^{p-13} = ((a-b)/ac)^j \pmod{p},$$

for $j = 1, 3, 5, 7$. We state our results only for $j = 1$ as the analogous results for $j = 3, 5$ and 7 may be obtained from these by self-evident transformations. Illustrative of the results in this section is the following.

**Theorem 1 (d).** Let $p = a^2 + b^2 = c^2 + 2d^2 \equiv 1 \pmod{8}$ be a prime with $a$ and $c$ chosen so that $a \equiv c \equiv 1 \pmod{4}$, and let $k = 1, 5$, or $3$ according as $c = 0, \pm 2d$, or $\pm 4d \pmod{11}$. Then

$$(-1)^{p-13}/(mod p) = \begin{cases} a \equiv b \equiv -3d \pmod{11}, \\ a \equiv 3b \equiv -2kd \pmod{11}, \\ a \equiv 4b \equiv 3kd \pmod{11}. \end{cases}$$

The results in Section 3 complement those of von Lienen [8] who gave necessary and sufficient criteria for each prime $q \leq 41$ to be an octic residue (mod $p$), given that $q$ is a quartic residue (mod $p$). This leaves the problem of evaluating $q^{p-13}/(mod p)$ when $q$ is a quartic but not a quartic residue (mod $p$), in other words, when $q^{p-13} \equiv \pm b/a \pmod{p}$. In Section 5 we give necessary and sufficient criteria for each prime $q \leq 19$ to satisfy $q^{p-13} \equiv \pm b/a \pmod{p}$.

Illustrative of these results is the following:

**Theorem 3 (d).** Let $p = a^2 + b^2 = c^2 + 2d^2 \equiv 1 \pmod{8}$ be a prime $\equiv 1 \pmod{8}$ with $a$ and $c$ chosen so that $a \equiv c \equiv 1 \pmod{4}$, and let $k = 1, 5$, or $-3$ according as $d = 0$, $c = \pm d$, or $c = \pm 5d \pmod{11}$. Then

$$11^{p-13}/(mod p) = \begin{cases} b \equiv -3c \pmod{11}, \\ b \equiv -5c \pmod{11}. \end{cases}$$

The fact that the values of $k$ in Theorem 3 (d) coincide with those in Theorem 1 (d) in magnitude, sign, and order is fascinating—it is not a coincidence. In Section 6 we include a proof of this phenomenon, apart from a possible ambiguity in the sign of the $k$'s, occurs for all primes $q > 3$ which are $\equiv \pm 3 \pmod{8}$.

2. Western's formulae. Let $p \equiv 1 \pmod{8}$ be a prime. Set $\zeta = e^{i\pi/8} = (1+i)/\sqrt{2}$ and let $R$ denote the ring of integers of the quartic field $Q(\zeta)$. The elements of $R$ are of the form $a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3$, where $a_0, a_1, a_2, a_3$ are rational integers; moreover, $R$ is a unique factorization domain. In $R$, $p$ factors as a product of four primes

$$p = \pi_1 \pi_2 \pi_3 \pi_4,$$

where

$$\pi_k = \pi_k^j \ (j = 1, 3, 5, 7), \ \pi_k^j = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3.$$  

Replacing $\pi_k$ by a suitable associate, we can suppose that

$$\pi_k = \pi_k^1 \equiv 1 \pmod{2}$$

(see, for example, [11], p. 69), so that

$$a_0 \equiv 1 \pmod{2}, \ a_1 \equiv a_2 \equiv a_3 \equiv 0 \pmod{3}.$$  

Next we note that

$$\pi_k \pi_3 = a + bi, \ \pi_k \pi_4 = c + di \sqrt{2},$$

where

$$p = a^2 + b^2 = c^2 + 2d^2, \ a = a_0^2 - a_1^2 + 2a_2a_3 \equiv 1 \pmod{4},$$

$$c = a_0^2 - a_1^2 + a_2^2 - a_3^2 \equiv 1 \pmod{4}.$$  

For $q$ an odd prime and $j = 0, 1, \ldots, 7$, it follows from Western's formulae ([11], p. 248) and (2.5) that for $q = 1, 3, 5, 7$ and $7 \pmod{8}$ respectively, we have

$$q^{p-13}/(mod p) \equiv ((a-b)/ac)^j \pmod{p},$$

if $p^{a-b}(a-b)^a(b+c\sqrt{2})b^{a-13}/(mod q)$, 

$$((-a-b)/ac)^j \pmod{p},$$

if $p^{a-b}(a+b)^a(b+c\sqrt{2})b^{a-13}/(mod q)$,
(2.11) \( q^{p-1/2} \equiv [(a-b)d/ac](\mod p) \)
if \( p^{p-2/3}(a-b)q^{p-1/4}(e+di\sqrt{2})q^{p+1/2} \equiv \xi^2(\mod q) \),

(2.12) \((-1)^{p-1/2} \equiv [(a-b)d/ac](\mod p) \)
if \( p^{p-1/2}(a-b)q^{p-1/4}(e+di\sqrt{2})q^{p+1/2} \equiv \xi^2(\mod q) \).

3. Evaluation of \( q^{p-1/2}(\mod p) \) when \( \left( \frac{q}{p} \right) = -1 \). Let \( p = a^2 + b^2 = c^2 + 2d^2 \equiv 1(\mod 8) \) be a prime with \( a \) and \( c \) chosen so that \( a \equiv c \equiv 1(\mod 4) \). Let \( q \) be an odd prime such that \( \left( \frac{q}{p} \right) = -1 \). In this section we give necessary and sufficient conditions for \( q \) to satisfy
\[ (-1)^{p-1/2} q^{p-1/2} \equiv (a-b)d/ac(\mod p) \]
for each prime \( q \leq 19 \). When \( q = \pm 1(\mod 8) \) these conditions involve congruences of the form \( a \equiv lb(\mod q) \) and \( c \equiv lc(\mod q) \) (see Theorem 1 (c), (f)), and when \( q \equiv 3(\mod 8) \) they involve congruences of the form \( a \equiv lb \equiv ac(\mod q) \) (see Theorem 1 (a), (b), (d), (e), (g)).

We just give the details of the proof of Theorem 1 for part (g), as, apart from the differences mentioned above, the other parts are proved similarly.

**Theorem 1.** Let \( p = a^2 + b^2 = c^2 + 2d^2 \equiv 1(\mod 8) \) be a prime with \( a \) and \( c \) chosen so that \( a \equiv c \equiv 1(\mod 4) \). Then we have

(a) \((-1)^{p-1/2} \equiv (a-b)d/ac(\mod p) \Rightarrow a \equiv -b \equiv d(\mod 3) \),

(b) \( 5^{p-1/2} \equiv (a-b)d/ac(\mod p) \Rightarrow \{ \begin{array}{l} a \equiv b \equiv d(\mod 5) , \\
a \equiv b \equiv -2d(\mod 5) , \\
\end{array} \)

(c) \((-7)^{p-1/2} \equiv (a-b)d/ac(\mod p) \Rightarrow \{ \begin{array}{l} a \equiv 2b(\mod 7) \text{ and } c \equiv kc(\mod 7) , \\
a \equiv 3b(\mod 7) \text{ and } c \equiv -kd(\mod 7) , \\
\end{array} \)
where \( k \equiv 1, 5, \text{ or } -3 \) according as \( c \equiv 0, \pm 2d, \text{ or } \pm 4d(\mod 11) \),

(d) \( 13^{p-1/2} \equiv (a-b)d/ac(\mod p) \Rightarrow \{ \begin{array}{l} a \equiv -b \equiv -kd(\mod 13) , \\
a \equiv -2b \equiv 6kd(\mod 13) , \\
\end{array} \)

where \( k \equiv 1, 4, 5, \text{ or } -3 \) according as \( c \equiv 0, \pm 2d, \pm 3d, \text{ or } \pm 4d(\mod 13) \),

(e) \( 17^{p-1/2} \equiv (a-b)d/ac(\mod p) \Rightarrow \{ \begin{array}{l} a \equiv -2b(\mod 17) \text{ and } c \equiv -kd(\mod 17) , \\
a \equiv 3b(\mod 17) \text{ and } c \equiv kd(\mod 17) , \\
\end{array} \)

where \( k \equiv 1, 4, 5, \text{ or } -3 \) according as \( c \equiv 0, \pm 2d, \pm 3d, \text{ or } \pm 4d(\mod 13) \),

(f) \( 19^{p-1/2} \equiv (a-b)d/ac(\mod p) \Rightarrow \{ \begin{array}{l} a \equiv -2b(\mod 17) \text{ and } c \equiv -kd(\mod 17) , \\
a \equiv 3b(\mod 17) \text{ and } c \equiv kd(\mod 17) , \\
\end{array} \)

where \( k \equiv 1, 4, 5, \text{ or } -3 \) according as \( c \equiv 0, \pm 2d, \pm 3d, \text{ or } \pm 4d(\mod 13) \),

where \( k \equiv 1, 5, \text{ or } -3 \) according as \( c \equiv 0, \pm 2d, \pm 4d(\mod 11) \),

\[ where \( k = 1, -2, 3, \text{ or } 5 \) according as \( c \equiv \pm d, \pm 2d, \pm 3d, \text{ or } \pm 5d(\mod 17) \),

\[ a \equiv -b \equiv -kd(\mod 19) , \\
a \equiv 3b \equiv -6kd(\mod 19) , \\
a \equiv -6b \equiv 2kd(\mod 19) , \\
a \equiv 7b \equiv 9kd(\mod 19) , \\
a \equiv -8b \equiv -4kd(\mod 19) , \]

where \( k = 1, 5, 3, -2, \text{ or } -4 \) according as \( c \equiv 0, \pm d, \pm 4d, \pm 5d, \text{ or } \pm 7d(\mod 19) \).

**Proof of Theorem 1 (g).** For brevity, all congruences will be assumed to be modulo 10 in the following proof unless otherwise stated.

It is straightforward to check that for each congruence on the right-hand side of (g), we have
\[ p^2(a+bi)^2(c-di\sqrt{2})^2 = 10(1+i)\sqrt{2} \equiv \xi \]
and so, by (2.10), with \( q = 19 \), we have
\[ (-19)^{p-1/2} \equiv (a-b)d/ac(\mod p) \].

For example, for \( a = -b = -d, c = 0 \), we have, as \( b^2 = 1 \),
\[ p^2(a+bi)^2(c-di\sqrt{2})^2 = 4(-1+i)^2(-4i\sqrt{2})^2 = -4(4-4i)(10i\sqrt{2}) \equiv 10(1+i)\sqrt{2} \equiv \xi \].

Conversely, suppose that
\[ (-19)^{p-1/2} \equiv (a-b)d/ac(\mod p) \].

Then \((-19)^{p-1/2} \equiv -1(\mod p) \), and by the law of quadratic reciprocity, we have
\[ \left( \frac{p}{19} \right) = -1 \].

It is clear from (3.2) and \( p = a^2 + b^2 = c^2 + 2d^2 \) that \( b \neq 0 \) and \( d \neq 0 \). Setting \( c = \mu d \) in (3.2) we obtain
\[ \left( \frac{\mu^2 + 2}{19} \right) = -1 \],
so that \( \mu = 0, \pm 1, \pm 4, \pm 5, \pm 7 \).

Next, from (3.1), we have \((-19)^{p-1/2} \equiv b/a(\mod p) \). Setting \( a = \mu d \), and appealing to the law of quartic reciprocity (see for example [2]), we obtain
\[ \left( \frac{a-bi}{a+bi} \right) = \left( \frac{\lambda-1}{\lambda+1} \right) = \xi \],
so that \( \lambda = -1, 3, -6, 7, -8 \).
Next by the law of octic reciprocity, see (2.10), we have

\begin{equation}
\rho^4(a+bi)^3(c-di\sqrt{2})^3 = \zeta.
\end{equation}

(3.5)

Using the congruences \( a \equiv kb, \ c \equiv \mu d, \) and setting \( d \equiv \beta b, \) so that \( c \equiv \beta d, \) in (3.5), we obtain

\begin{equation}
(\lambda^2+1)^3(\lambda+i)^3(\mu-i\sqrt{2})^3 \left( \frac{\beta}{19} \right) \equiv -9(1+i)\sqrt{2},
\end{equation}

as \( b^{14} = 1, \) \( \beta^6 = \left( \frac{\beta}{19} \right). \) Next, as

\begin{equation}
(\lambda+i)^6 = (\lambda^2+9\lambda^2+5\lambda)+(5\lambda^2+9\lambda^2+1)i
\end{equation}

and

\begin{equation}
(\mu-i\sqrt{2})^6 = (\mu^5+4\mu^5+10\mu^5+12\mu^5+11\mu)(-9\mu^6+3\mu^6+10\mu^6+16\mu^6+16)i\sqrt{2},
\end{equation}

we obtain

\begin{equation}
(\lambda^2+1)^3(\lambda+i)^3 = \begin{cases} -3+3i, & \text{if } \lambda = -1, 3, -6, 7; \\ 3-3i, & \text{if } \lambda = -8; \end{cases}
\end{equation}

and

\begin{equation}
(\mu-i\sqrt{2})^6 = \begin{cases} 3i\sqrt{2}, & \text{if } \mu = 0, \pm1, \pm5; \\ -3i\sqrt{2}, & \text{if } \mu = \pm4, \pm7. \end{cases}
\end{equation}

Since

\((3+3i)(\pm3i\sqrt{2}) = \mp9(1+i)\sqrt{2}, \) (3-3i)(\pm3i\sqrt{2}) = \pm9(1+i)\sqrt{2},

we must have from (3.6), (3.9), and (3.10), that

\begin{equation}
\left( \frac{\beta}{19} \right) = +1 \text{ if } \lambda = -1, 3, -6, 7 \text{ and } \mu = 0, \pm1, \pm5, \text{ or }
\lambda = -8 \text{ and } \mu = \pm4, \pm7,
\end{equation}

and

\begin{equation}
\left( \frac{\beta}{19} \right) = -1 \text{ if } \lambda = -1, 3, -6, 7 \text{ and } \mu = \pm4, \pm7 \text{ or }
\lambda = -8 \text{ and } \mu = 0, \pm1, \pm5.
\end{equation}

From \( a^2+b^3 = c^2+2d^2 \) we obtain

\begin{equation}
\beta^2 = \frac{\lambda^2+1}{\mu^6+2}.
\end{equation}

(3.13)

Appealing to (3.11), (3.12), and (3.13) we have the following table of values of \( \beta: \)

\begin{equation}
\begin{array}{ccccccc}
\mu & 0 & 1 & 9 & 16 & 5 & 2 \\
\pm1 & 11 & 4 & 5 & 17 & 3 \\
\pm4 & 13 & 3 & 8 & 7 & 17 \\
\pm8 & 9 & 5 & 11 & 7 & 18 \\
\pm17 & 14 & 12 & 15 & 10 & 9 & 4
\end{array}
\end{equation}

(3.14)

The \( r \)th row in this array is determined by multiplying the entries of the first row by the first entry in the \( r \)th row, \( r = 1, \ldots, 5 \) (and similarly for the \( r \)th column).

Finally set

\begin{equation}
\frac{\lambda}{\beta k} = -1, -6, 2, 9, \text{ or } -4
\end{equation}

according as \( \mu = 0, \pm1, \pm4, \pm5, \) or \( \pm7 \) respectively. Then from (3.14) and (3.15) we have for all 20 values of \( \beta, \)

\begin{equation}
\frac{\lambda}{\beta k} = -1, -6, 2, 9, \text{ or } -4
\end{equation}

according as \( \lambda = -1, 3, -6, 7, \) or \( -8. \)

This completes the proof of Theorem 1 (a).

Remark. Set \( a \equiv kb \equiv \mu d \) if \( q \equiv \pm3 \pmod{8} \) and set \( a \equiv kb, \ c \equiv \mu d, \) if \( q \equiv \pm1 \pmod{8}. \) If \( (a-b)d \mid \sigma \) on the left-hand side of Theorem 1 (a)–(g), is replaced by \( (a-b)d \mid \sigma \), \( j = 5, 0, \) or \( 7 \), then the congruences on the right-hand side are satisfied if and only if \( \lambda \) is replaced by \( -\lambda \) if \( j = 3, \mu \) is replaced by \( -\mu \) if \( j = 5, \lambda \) is replaced by \( -\lambda \) and \( \mu \) is replaced by \(-\mu \) if \( j = 7.\)

Example 1 \((q = 5, \text{ see Theorem 1 (b))}.

(i) Let \( p = 1297 = 1^2+36^2 = (-35)^2+2-6^2 \) so that \( a \equiv b \equiv d \pmod{5}. \) Then we have

\begin{equation}
\frac{(a-b)d}{ac} = \frac{(-35)(6)}{-35} \equiv 6 \pmod{1297},
\end{equation}

and it is easily checked that for \( p = 1297 \) we have

\begin{equation}
5^{p-1}/a \equiv 5^{12} \pmod{p}.
\end{equation}

(ii) Let \( p = 137 = (-11)^2+4^2 = (-3)^2+2(8)^2 \) so that \( a \equiv b \equiv -2d \pmod{5}. \) Then we have

\begin{equation}
\frac{(a-b)d}{ac} = \frac{(-15)(8)}{(-11)(-3)} \equiv -120 \pmod{1297},
\end{equation}

and it is easily checked that for \( p = 137 \) we have

\begin{equation}
5^{p-1}/a \equiv 5^{12} \pmod{p}.
\end{equation}
(iii) Let \( p = 17 = 1^2 + 4^2 = (-3)^2 + 2(3)^2 \) so that the right-hand side of (b) is not satisfied, rather, \( a \equiv -b \equiv -2d \pmod{5} \). Setting \( a = b \) and noting that only the sign of \( \lambda \) differentiates \( a \equiv -b \equiv -2d \) from the congruence \( a = b \equiv -2d \) in (b), we must have

\[
5^{(p-1)/4} \equiv \frac{(a+b)d}{ac} \pmod{p},
\]

and it is easily checked that this is, indeed, the case.

4. Evaluation of \( \Phi_q(g) \) when \( \left( \frac{q}{p} \right) = -1 \). We now use the results of Section 3 to show how to evaluate \( \Phi_q(g) \) for a prime \( q \) which is a quadratic non-residue \( \pmod{p} \). Explicit results are obtained for \( q \leq 13 \).

From the work of Whiteman ([13], p. 90) or Lehmer ([7], p. 65), we have

\[
\Phi_q(D) \equiv -\left(D^{p-1/2}(p-1)/2 \right) + D^{p-1/2}(p-1)/2 \pmod{p},
\]

as \( \left( \frac{p-1}{2} \right) = \left( \frac{p-1}{2} \right) \) we obtain

\[
\Phi_q(D) \equiv -\left( \frac{p-1}{2} \right) \left(D^{p-1/2}(p-1)/8 \right) \pmod{p}.
\]

Next as

\[
\left( \frac{p-1}{2} \right) = 2c(-1)^{(p-1)/4} \pmod{p},
\]

see, for example, Jacobi [4], p. 168, Stern [10], or Whiteman [13], p. 97, we obtain

\[
\Phi_q(D) \equiv -2c(-1)^{(p-1)/4} \pmod{p}.
\]

The congruence (4.4) can be used to evaluate \( \Phi_q(D) \) when \( |\Phi_q(D)| \) and \( D^{p-1/2}(p-1/8) \pmod{p} \) are known. Appealing to Theorem 1, we use (4.4) to evaluate \( \Phi_q(g) \) for \( q = 3, 5, 7, 11, 13, 17, 19, \left( \frac{q}{p} \right) = -1 \). We just give the details for \( q = 5 \) as the details are similar for the other values of \( q \).

For a prime \( p = 1 \pmod{8} \) with \( \left( \frac{5}{p} \right) = -1 \) we can choose the signs of \( b \) and \( d \) so that

\[
a \equiv b \equiv d \pmod{5},
\]

and it follows from Theorem 1 (b), that

\[
5^{(p-1)/4} \equiv \frac{(a-b)d}{ac} \pmod{p}.
\]
Since (see for example [1] and [7])

\[(4.14) \quad 2^{p-a/n} = \begin{cases} 
(-1)^{p-1/n}b \pmod{p} & \text{if } b \equiv 0 \pmod{16}, \\
(-1)^{p-1/n}b/a \pmod{p} & \text{if } b \equiv 4 \pmod{16}, \\
(-1)^{p+1/n}b \pmod{p} & \text{if } b \equiv 8 \pmod{16}, \\
(-1)^{p+1/n}b/a \pmod{p} & \text{if } b \equiv 12 \pmod{16},
\end{cases}
\]

we obtain from (4.13) and (4.14)

\[(4.15) \quad (-6)^{p-1/n} = \begin{cases} 
(-1)^{p-1/n}(a-b)d/e \pmod{p} & \text{if } b \equiv 0 \pmod{16}, \\
(-1)^{p-1/n}(a+b)d/e \pmod{p} & \text{if } b \equiv 4 \pmod{16}, \\
(-1)^{p+1/n}(a-b)d/e \pmod{p} & \text{if } b \equiv 8 \pmod{16}, \\
(-1)^{p+1/n}(a+b)d/e \pmod{p} & \text{if } b \equiv 12 \pmod{16}.
\end{cases}
\]

It follows at once from (4.15) that

\[(4.16) \quad (-6)^{p-1/n} + (-6)^{2p-1/n} = \begin{cases} 
(-1)^{p-1/n}2d/e \pmod{p} & \text{if } b \equiv 0 \pmod{4}, \\
(-1)^{p+1/n}2d/e \pmod{p} & \text{if } b \equiv 8 \pmod{16},
\end{cases}
\]

so that, by (4.4), we have \(\Phi_4(-6) = -4d \pmod{p}\) if \(b \equiv 0 \pmod{4}\) and \(\Phi_4(-6) = 4d \pmod{p}\) if \(b \equiv 8 \pmod{16}\). Thus, by (1.2), we have

\[(4.17) \quad \Phi_4(-6) = \begin{cases} 
4d & \text{if } b \equiv 0 \pmod{4}, \\
-4d & \text{if } b \equiv 8 \pmod{16}.
\end{cases}
\]

**Example 3.** Let \(p = 17 = 1^2 + (-4)^2 = (-3)^2 + 2(-2)^2\) so that \(a \equiv c \equiv 1 \pmod{4}\), \(a \equiv -b \equiv 3 \pmod{3}\), and \(b \equiv 12 \pmod{16}\). From (4.17) we have \(\Phi_4(-6) = 4d = -8\), and indeed,

\[
\Phi_4(-6) = \sum_{n=1}^{16} \frac{a(a^2 - 6)}{17}
= 2 \left( \frac{3}{17} + \frac{4}{17} + \frac{14}{17} + \frac{5}{17} + \frac{2}{17} + \frac{11}{17} \right)
= -8.
\]

**5. Evaluation of** \(2^{p-1/n} \pmod{p}\) **when** \(\left\lfloor \frac{q}{p} \right\rfloor = +1\). Let \(p = a^2 + b^2 = a^2 + 2d = 1 \pmod{8}\) be a prime with \(a\) and \(c\) chosen so that \(a = c = 1 \pmod{4}\). Let \(q\) be an odd prime such that \(\left\lfloor \frac{q}{p} \right\rfloor = +1\). In this section we give necessary and sufficient conditions for \(q = b/a \pmod{p}\) for each prime \(q \equiv 0 \pmod{4}\). When \(q \equiv \pm 1 \pmod{8}\), these conditions involve congruences of the form \(a \equiv 2b \pmod{q}\) (see Theorem 3 (c), (f)), and when \(q \equiv \pm 3 \pmod{8}\) they involve congruences of the form \(b \equiv \pm c \pmod{q}\) (see Theorem 3 (a), (b), (d), (e), (g)). Apart from this difference, the proofs of (a)-(g) are similar, and may easily be written down by analogy with the proof of (g) which is given below.

**Theorem 3.** Let \(p = a^2 + b^2 = c^2 + 2d^2 = 1 \pmod{8}\) be a prime with \(a\) and \(c\) chosen so that \(a = c \equiv 1 \pmod{4}\). Then we have

(a) \((-3)^{p-1/n} \equiv b/a \pmod{p} \iff b \equiv c \pmod{3}\),

(b) \(3^{p-1/n} \equiv b/a \pmod{p} \iff \begin{cases} 
 b \equiv -c \pmod{5}, \\
 b \equiv 2c \pmod{5},
\end{cases}\)

(c) \((-7)^{p-1/n} \equiv b/a \pmod{p} \iff \begin{cases} 
 a \equiv b \pmod{7} \text{ and } cd \equiv 0 \pmod{7}, \\
 a \equiv -b \pmod{7} \text{ and } cd \not\equiv 0 \pmod{7},
\end{cases}\)

(d) \((-11)^{p-1/n} \equiv b/a \pmod{p} \iff \begin{cases} 
 b \equiv -ka \pmod{11}, \\
 b \equiv -5ka \pmod{11},
\end{cases}\)

where \(k = 1, 5 \text{ or } -3\) according as \(d \equiv 0 \pmod{11}\), \(c \equiv \pm d\) or \(\pm 5d \pmod{11}\),

(e) \(13^{p-1/n} \equiv b/a \pmod{p} \iff \begin{cases} 
 b \equiv ka \pmod{13}, \\
 b \equiv 6ka \pmod{13},
\end{cases}\)

where \(k = 1, 4, -5\) or \(-3\) according as \(d \equiv 0\), \(c \equiv \pm d\), \(c \equiv \pm 5d\), or \(\pm 6d \pmod{13}\),

(f) \(17^{p-1/n} \equiv b/a \pmod{p} \iff \begin{cases} 
 a \equiv 3kb \pmod{17}, \\
 a \equiv -7kb \pmod{17},
\end{cases}\)

where \(k = 1\) if \(cd \equiv 0\) or \(c \equiv \pm 6d \pmod{17}\), \(k = -1\) if \(c \equiv \pm 4d\) or \(\pm 8d \pmod{17}\),

(g) \((-19)^{p-1/n} \equiv b/a \pmod{p} \iff \begin{cases} 
 b \equiv ke \pmod{19}, \\
 b \equiv 2ke \pmod{19}, \\
 b \equiv 7ke \pmod{19},
\end{cases}\)

where \(k = 1, 7, 3, -2, -4\) according as \(d \equiv 0\), \(c \equiv \pm 2d\), \(\pm 9d\), \(\pm 8d\), or \(\pm 3d \pmod{19}\) respectively.

**Proof of Theorem 3 (g).** For brevity, all congruences are to be taken modulo 19 unless otherwise stated.

Case (i): \(d \equiv 0\). It is easy to check that for each congruence on the right-hand side of (g), we have

\[p^2(a+bi)^2(c-di\sqrt{2})^2 \equiv i,\]
and so by (2.10), with \( q = 19 \), we deduce that
\[
(-19)^{p^{-1/3}} \equiv b/a \pmod{p}.
\]
For example, for \( a = 0, b = -2 \), we have
\[
p^3(a + bi)^4(c - d\sqrt{2})^3 = i^4(9 + 6i\sqrt{2})^6 = i,
\]
as \( a \equiv 0, c \equiv 0 \) imply \( b^2 = 5b + 2d^2 \), that is, \( d \equiv \pm 6b; b^{12} \equiv 1 \);
and, appealing to (3.8) and noting that \( 6^9 \equiv \frac{6}{19} \equiv +1 \),
\[
(9 + 6i\sqrt{2})^6 = 6^9(\pm i\sqrt{2})^9 \equiv 1.
\]
Conversely, suppose that
\[
(5.1) \quad (-19)^{p^{-1/3}} \equiv b/a \pmod{p}.
\]
Then \((-19)^{p^{-1/3}} \equiv +1 \pmod{p}\), and by the law of quadratic reciprocity, we have
\[
(5.2) \quad \left(\frac{p}{19}\right) = +1.
\]
Setting \( c = \mu d \) in (5.2), we obtain
\[
\left(\frac{\mu^2 + 2}{19}\right) = +1,
\]
so that
\[
\mu \equiv \pm 2, \pm 3, \pm 8, \pm 9.
\]
Also, from (5.1), we have \((-19)^{p^{-1/3}} \equiv -1 \pmod{p}\), and so by the law of quartic reciprocity, see [2], we have
\[
(5.4) \quad \left(\frac{a - bi}{a + bi}\right)^5 = -1.
\]
Clearly \( b \neq 0 \) and we can set \( a = \lambda b \) in (5.4) to get
\[
(5.5) \quad \lambda = 0, \pm 2, \pm 5.
\]
Next, by the law of octic reciprocity, see (2.10), we have
\[
(5.6) \quad p^3(a + bi)^4(c - d\sqrt{2})^3 = i.
\]
Using \( a = \lambda b, c = \mu d \), and setting \( b = \gamma c \) so that \( a = \lambda \mu d \) and \( b = \mu d \)
in (5.6) we obtain, as \( d^{12} \equiv 1 \) and \( (\mu c)^9 \equiv \left(\frac{\mu}{19}\right) \),
\[
(5.7) \quad (\lambda^2 + 1)^2(\lambda + i)^i(\mu - i\sqrt{2})^9\left(\frac{\mu}{19}\right) = i.
\]
From (3.7) and (3.8) we obtain
\[
(5.8) \quad (\lambda^2 + 1)(\lambda + i)^i \equiv \begin{cases} i & \text{if } \lambda \equiv 0, \pm 5, \\ -i & \text{if } \lambda \equiv \pm 2, \pm 9. \end{cases}
\]
and
\[
(5.9) \quad (\mu - i\sqrt{2})^9 \equiv \begin{cases} +1 & \text{if } \mu \equiv -2, -3, -8, -9, \\ -1 & \text{if } \mu \equiv +2, +3, +8, +9. \end{cases}
\]
From (5.7), (5.8), and (5.9) we must have
\[
(5.10) \quad \left(\frac{\mu y}{19}\right) = +1 \quad \text{if} \quad \begin{cases} \lambda \equiv 0, \pm 5 \wedge \mu \equiv -2, +3, -8, -9, \\ \lambda \equiv \pm 2 \wedge \mu \equiv +2, +3, +8, +9. \end{cases}
\]
and
\[
(5.11) \quad \left(\frac{\mu y}{19}\right) = -1 \quad \text{if} \quad \begin{cases} \lambda \equiv 0, \pm 5 \wedge \mu \equiv -2, -3, +8, +9, \\ \lambda \equiv \pm 2 \wedge \mu \equiv -2, -3, +8, -9. \end{cases}
\]
From \( a^2 + b^2 = c^2 + 2d^2 \) we obtain
\[
(5.12) \quad \gamma^a = \frac{-\mu^2 + 2}{\mu^2(\lambda^2 + 1)}.
\]
Next, from (5.10), (5.11), and (5.12), we have the following table of values of \( \gamma \):

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>0</th>
<th>( \pm 2 )</th>
<th>( \pm 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \pm 2 )</td>
<td>7</td>
<td>-5</td>
<td>-8</td>
</tr>
<tr>
<td>( \pm 4 )</td>
<td>-4</td>
<td>-8</td>
<td>-9</td>
</tr>
<tr>
<td>( \pm 8 )</td>
<td>-2</td>
<td>-4</td>
<td>5</td>
</tr>
<tr>
<td>( \pm 9 )</td>
<td>3</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

Finally, setting
\[
(5.13) \quad k = 7, -4, -2, -a + 3 \quad \text{or} \quad +3 \quad \text{according to} \quad c = \pm 2d, \pm 3d, \pm 8d, \quad \text{or} \quad \pm 9d,
\]
we obtain for all 12 values of \( \gamma \),
\[
(5.14) \quad \gamma^a = \frac{1}{2}, \quad \text{or} \quad \gamma = 1 \quad \text{or} \quad 7 \quad \text{according to} \quad \lambda = 0, \pm 2, \text{or} \pm 5.
\]
This proves Theorem 3 (g) in case (i).

Case (ii): \( d = 0 \). The proof is the same as in case (i) except that we clearly cannot set \( c = \mu d \). However, setting \( b = \gamma c \), (5.8) becomes in this case,
\[
(5.15) \quad (\lambda^2 + 1)(\lambda + i)^i\left(\frac{\gamma}{19}\right) = i.
\]
Hence by (5.8) we have

\[(5.17) \quad \left(\frac{\gamma}{19}\right) = +1 \iff \lambda = 0, \pm 5.\]

Next from \(a^2 + b^2 = c^2 + 2ad\) we have

\[(5.18) \quad \gamma = \frac{1}{2^2 + 1}.\]

Putting (5.17) and (5.18) together, we get (just as in (5.15) taking \(b = 1\)),

\[(5.19) \quad \gamma = 1, 2, \text{ or } 7 \text{ according as } \lambda = 0, \pm 2, \text{ or } \pm 5.\]

This completes the proof of Theorem 3 (g) in case (ii).

**Example 4.** \((q = 5): 5^2 + 16^2 = 9^2 + 2(10)^2\) so that \(b = -c (\text{mod } 5)\).

Then we have

\[
\frac{b}{a} = \frac{16}{5} \equiv 228 \pmod{281},
\]

and it is easily checked that for \(p = 281\) we have

\[5^{p-1}/b = 5^{25} \equiv 228 \pmod{281}.\]

(ii) Let \(p = 1289 = (-35)^2 + 8^2 = 33^2 + 2(10)^2\) so that \(b = c (\text{mod } 5)\).

Then we have

\[5^{p-1}/b = 5^{161} \equiv 479 = \frac{8}{35} \equiv -\frac{b}{a} \pmod{p}.\]

(iii) Let \(p = 89 = 5^2 + 8^2 = 9^2 + 2(2)^2\) so that \(b = 2c (\text{mod } 5)\). Then we have

\[5^{p-1}/b = 5^{88} \equiv 55 \equiv \frac{8}{5} \equiv -\frac{b}{a} \pmod{p}.\]

(iv) Let \(p = 241 = (-15)^2 + 4^2 = 13^2 + 2(6)^2\) so that \(b = 2c (\text{mod } 5)\). Then we have

\[5^{p-1}/b = 5^{120} \equiv 177 \equiv \frac{4}{15} \equiv -\frac{b}{a} \pmod{p}.\]

Remark. Although the number of cases to be considered grows quite rapidly as \(q\) increases, there are no technical difficulties in using the methods given in the proofs of Theorems 1, 2, and 3 to extend the results in this paper beyond \(q = 19\).

6. **Values of \(f\) in Theorems 1 and 3 when \(q = \pm 3 (\text{mod } 8)\).** We observe with considerable interest that the values of \(k\) which occur in Theorem 1 (d), (e), (g) are identical with the values of \(k\) which occur in Theorem 3 (d), (e), (g), both in magnitude and sign. The values of \(k\) in Theorem 3 have been written in an order which corresponds to their order in Theorem 1. In this section we investigate the pattern which underlies this ordering for every \(q > 3\) which is \(\equiv \pm 3 (\text{mod } 8)\).

Let \(k_{1,j}, k_{2,0}\) be the value of \(k\) corresponding to \(e \equiv \pm r_j d (\text{mod } q)\) when \(\left(\frac{q}{p}\right) = -1, j = 1, 2, \ldots\), and let \(k_{2,1}\) be the value of \(k\) corresponding to \(e \equiv \pm s_j d (\text{mod } q)\) when \(\left(\frac{q}{p}\right) = +1\), where \(k_{1,j}\) and \(k_{2,1}\) are equal to 1 and correspond to \(e \equiv 0 (\text{mod } q)\), and \(d \equiv 0 (\text{mod } q)\), respectively. For example, when \(q = 10\) we have from Theorem 1 (g) and Theorem 3 (g) that \(k_{1,2} = k_{1,3} = 3, r_2 = 4, s_2 = 2\). Observing that \(r_j^2 - s_j^2 \equiv (-3)(5) = -1 (\text{mod } 10)\) and that the same congruence is satisfied (mod \(q\)) for every choice of \(j\) in Theorem 1 and 3, parts (d), (e), (g) as well as part (b) when reformulated in analogy to (d), (e), and (g), we are led (with the above notation) to a general theorem relating the values of \(k\) in Theorem 1 and 3 when \(q\) is a prime \(\equiv \pm 3 (\text{mod } 8), q > 3\).

**Theorem 4.** Let \(q\) be a prime \(> 3\) which is \(\equiv \pm 3 (\text{mod } 8)\). Then

\[(6.1) \quad (k_{1,j})^2 = (k_{2,1})^2 \equiv (r_j)^2 (s_j)^2 \equiv 4 (\text{mod } q), \quad j = 1, 2, \ldots\]

**Proof.** Let \(q > 3\) so that \(r_j\) and \(s_j\) exist. Applying (3.4) with the exponent 5 replaced by \((q + 1)/4\) if \(q \equiv 3 (\text{mod } 8)\) and by \((q - 1)/4\) if \(q \equiv 5 (\text{mod } 8)\), one can show that one of \(\lambda \equiv 1\) or \(\lambda \equiv -1\) is always a solution. Then, by (3.13) and (3.18), we have \((r_j) \equiv \mu (3.13)),

\[(6.2) \quad (k_{1,j})^2 \equiv \frac{1}{4} (r_j)^2 (s_j)^2 \equiv 4 (\text{mod } q), \quad j = 1, 2, \ldots\]

Similarly, applying (5.4) with the same replacement, it is obvious that this congruence is always satisfied if \(\lambda \equiv 0 (a \equiv \overline{b})\), taking \(\gamma/k_{1,j} \equiv 1 (\text{mod } q), \quad j = 1, 2, \ldots\) (see (5.15)), and using (5.12), we have \((s_j) \equiv \mu (5.12))

\[(6.3) \quad (k_{2,1})^2 \equiv \frac{1}{4} (s_j)^2 \equiv 4 (\text{mod } q), \quad j = 1, 2, \ldots\]

The result follows then from (6.2) and (6.3), as the \(k_{1,j}\) and \(k_{2,1}\) are chosen in the range \(-\frac{1}{2} q < k < \frac{1}{2} q\).

**References**


A note on recurrent mod $p$ sequences

by

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Important arithmetical functions, namely the integral valued linear combinations of polynomials multiplied by exponentials functions, have the striking property of being periodic mod $p$ for all sufficiently large primes $p$.

In this paper we are concerned with the following problem: which other sequences, apart from the above mentioned ones, satisfy some periodicity condition mod $p$ for almost all primes $p$?

Our result is that no other such sequence exists, provided a certain kind of growth condition is satisfied.

We consider sequences satisfying a more general property, i.e. those which are solutions of recurrence equations mod $p$ for large $p$. (Periodicity is actually a special kind of recurrence.)

In the sequel $C_1, C_2, \ldots$ will denote numbers which depend only on the sequence.

We have the following

THEOREM. Let $f : \mathbb{N} \to \mathbb{Z}$. Suppose that

(i) for every prime $p > p_0$, $f$ satisfies a non trivial recurrence equation in $\mathbb{Z}[p\mathbb{Z}]$, of length $r_p < p^k$, for some fixed $k$.

(ii) $|f(n)| \leq n^B$ for some constant $B$.

Then $f$ satisfies a non trivial recurrence equation over $\mathbb{Z}$.

Proof. We recall the following Siegel’s classical lemma (see for example [1]): “Let $M, N$ denote integers, $N > M > 0$, and let $u_j (1 \leq i \leq M, 1 \leq j \leq N)$, denote integers satisfying $|u_j| \leq U$. Then there exists a non trivial integral solution $x_1, x_2, \ldots, x_N$, of the linear system

$$\sum_{j=1}^{N} u_j x_j = 0 \text{ for } i = 1, 2, \ldots, M$$

such that

$$|x_j| \leq (NU)^{M(N-M)}.$$