with differences $d > 0$ that tend to increase with n. Some sample values follow; the values of U in the table are truncated at one decimal place.

n	p_n	d	$U(y(n-d), \frac{1}{2})$
5000	48611	91	48616.6
10000	104729	188	104732.9
20000	224737	358	224744.7
30000	350377	528	350389.1
40000	479909	681	479910.7
50000	611953	847	611955.9
60000	746773	969	746773.9
70000	882377	1163	882385.4
80000	1020379	1295	1020380.7
90000	1159523	1443	1159525.7
100000	1299709	1598	1299722.5

References

[1] H. E. Richert, *Über Zerfällungen in ungleiche Primzahlen*, Math. Zeitschr. 52 (1950), pp. 342–343.

[2] J. Barkley Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), pp. 64–94.

[3] W. Sierpiński, *Elementary Theory of Numbers*, Warszawa 1964.

UNIVERSITY OF VICTORIA
Victoria, B. C., Canada V8W 2Y2

Received on 30. 11. 1979
and in revised form on 8. 4. 1980 (1186)

ACTA ARITHMETICA
XLI (1982)

Gaps between primes, and the pair correlation of zeros of the zeta-function

by

D. R. HEATH-BROWN (Oxford)

1. Introduction. In studying the finer properties of the distribution of the zeros of the Riemann Zeta-function, Montgomery [7] examined the pair correlation function

$$F_T(X) = \sum_{0<\gamma_1,\gamma_2\leqslant T} W(\gamma_1-\gamma_2)e\big(X(\gamma_1-\gamma_2)\big).$$

Here $W(u) = 4/(4+u^2)$, $e(u) = \exp(2\pi iu)$, and γ runs over imaginary parts $\mathrm{Im}(\varrho)$ of the non-trivial zeros ϱ of $\zeta(s)$ (counted according to multiplicity). Montgomery based his investigation on the assumption of the Riemann Hypothesis, and we shall follow him; for convenience we use the abbreviation RH. It is clear that $F_T(X) = F_T(-X)$, that $F_T(X) \ll T(\log T)^2$, and that $F_T(X) \geqslant 0$ (this follows from Lemma 3 below). Montgomery showed in addition that, on RH,

$$(1)\qquad F_T(X) = TX + \frac{T}{2\pi}x^{-2}(\log T)^2 + O(T) + O(xX) + O(Tx^{-3/2}\log T),$$

for $x = e^{2\pi X}$, $x \geqslant 1$. Actually he stated a slightly less precise result, but it is clear that his analysis leads to the above refinement. When $0 < \delta \leqslant \beta \leqslant 1-\delta$, where, as later, $X = (\beta\log T)/2\pi$, (1) reduces to $F_T(X) = TX + O(T)$, uniformly in β. Moreover, Montgomery conjectured, in general, that

$$(2)\qquad F_T(X) \sim \frac{T}{2\pi}(\log T)\,\mathrm{Min}(1, |\beta|)$$

uniformly in $0 < \delta \leqslant |\beta| \leqslant A$. From (1) he deduced, on RH, several important consequences for the distribution of the γ's, and he showed that the conjecture (2) would lead to more powerful conclusions — for example, that 'almost all' zeros would be simple.

Results connecting the distribution of the primes p_n and the zeros of the Zeta-function have long been known. In particular von Koch [5] showed, on RH, that

$$\Psi(x) = x + O\left(x^{1/2}(\log x)^2\right),$$

Cramér [1], also on RH, that

$$p_{n+1} - p_n \ll p_n^{1/2}\log p_n, \tag{3}$$

and Selberg [9], subject again to RH, that

$$\sum_{p_n \leqslant x} (p_{n+1} - p_n)^2 \ll x(\log x)^3,$$

and that 'almost all' intervals $[x, x+f(x)(\log x)^2]$ would contain a prime, if $f(x) \to \infty$.

In addition, it has generally been expected, in a rather imprecise sense, that information on the distribution of the γ's would improve the above results on primes. It turns out that the conjecture (2) contains just such information. The first result in this connection was due to Gallagher and Mueller [3], who showed that (2), together with RH, implies

$$\Psi(x) = x + o\left(x^{1/2}(\log x)^2\right). \tag{4}$$

Mueller [8] later showed that the weaker hypothesis

$$F_T(X) \ll T\log T,$$

uniformly for $0 < \delta \leqslant |\beta| \leqslant A$, together with RH, also suffices for (4), and further, that these two hypotheses yield

$$p_{n+1} - p_n \ll p_n^{1/2}(\log p_n)^{3/4+\varepsilon}$$

for any $\varepsilon > 0$.

We shall extend these results as follows.

THEOREM 1. *Assume* RH, *and that* $F_T(X) = o\left(T(\log T)^2\right)$ *as* $T \to \infty$, *uniformly for* $1 \leqslant |\beta| \leqslant A$, *for every constant* $A \geqslant 1$. *Then* (4) *follows.*

THEOREM 2. *Assume* RH. *Then*

$$\sum_{\substack{x \leqslant p_n \leqslant 3x/2 \\ d_n \geqslant \Delta}} d_n \ll x\Delta^{-1}\log x + \operatorname*{Max}_{K \geqslant 1}\left\{\left(K(\log 2K)^{-2}\right)^{-2}\operatorname*{Max}_{\tau \leqslant Kx/\Delta} F_\tau\left(\frac{\log x}{2\pi}\right)\right\},$$

where, as later, $d_n = p_{n+1} - p_n$.

COROLLARY 1. *Assume* RH *and that*

$$F_T\left(\frac{\log x}{2\pi}\right) \ll T\log T, \tag{5}$$

uniformly in x *and* T *for* $x\Delta^{-1}(\log x)^{-1} \leqslant T \leqslant x\Delta^{-1}(\log x)^{4/3}$. *Then*

$$\sum_{p_n \leqslant x;\, d_n \geqslant \Delta} d_n \ll x\Delta^{-1}\log x.$$

COROLLARY 2. *Assume* RH *and that* (5) *holds, with* p_n *in place of* x, *uniformly in* p_n *and* T, *for* $p_n^{1/2}(\log p_n)^{-2} \leqslant T \leqslant p_n^{1/2}\log p_n$. *Then*

$$d_n \ll (p_n\log p_n)^{1/2}.$$

COROLLARY 3. *Assume* RH *and that* (5) *holds, uniformly in* x *and* T, *for* $x^{1/2}(\log x)^{-2} \leqslant T \leqslant x$. *Then*

$$\sum_{p_n \leqslant x} d_n^2 \ll x(\log x)^2.$$

COROLLARY 4. *Assume* RH *and that* (5) *holds, uniformly in* x *and* T, *for* $x(\log x)^{-3} \leqslant T \leqslant x$. *Then 'almost all' intervals* $[x, x+f(x)\log x]$ *contain a prime, providing that* $f(x) \to \infty$. *That is to say, the Lebesgue measure of the set of* $x \leqslant x_1$, *such that the interval contains no prime, is* $o(x_1)$ *as* $x_1 \to \infty$.

We have made no attempt to get the smallest possible T-intervals in these results. Corollary 1 is an immediate consequence of Theorem 2 and the bound $F_T(X) \ll T(\log T)^2$. Corollaries 2, 3, and 4 follow trivially from Corollary 1, since (5) holds when $x \leqslant T \leqslant x^2$, by (1). However one could easily give a shorter direct proof of Corollary 2. We expect the conclusion of Corollary 4 to be best possible, but we are unable to disprove even that 'almost all' intervals $[x, x+\log x]$ contain a prime.(1) Regarding Corollary 3, we may conjecture that, in fact

$$\sum_{p_n \leqslant x} d_n^2 \sim 2x\log x.$$

Finally we have

THEOREM 3. *There exist numerical constants* c_1, c_2 *as follows. Assume* RH, *and that*

$$\left|F_T\left(\frac{\log x}{2\pi}\right) - \frac{T}{2\pi}\log x\right| \leqslant T\delta(x)\log T$$

for some $\delta(x)$ *in the range* $(\log x)^{-1/3} \leqslant \delta(x) \leqslant 1$ *and for* $x(\log x)^{-2} \leqslant T \leqslant x\log x$. *Then*

$$\operatorname*{Min}_{x/c_1 \leqslant p_n \leqslant xc_1} \frac{d_n}{\log p_n} \leqslant c_2\delta(x).$$

(1) When $0 < \lambda < 1$ at least a proportion $1-\lambda$, asymptotically, of intervals $[x, x+\lambda\log x]$ contain no prime. The author is grateful to Professor P. X. Gallagher for showing him a very simple proof of this.

Thus, if (2) holds uniformly in T and β, for β in some neighbourhood of 1 then, on RH,

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0.$$

2. Preparations. In this section we lay the groundwork for the proofs of our theorems.

LEMMA 1. *Assume* RH. *Then, if* x, $T \geqslant 2$, *we have*

$$\Psi(x) = x - \sum_{|\gamma|\leqslant T} x^{1/2+i\gamma}(\tfrac{1}{2}+i\gamma)^{-1} + E(x) + F(x),$$

where

$$E(x) = E(x, T) \ll x^{-2} + xT^{-1}(\log xT)^2 \quad \textit{and} \quad F(x) = F(x, T) \ll \Lambda(n)$$

for $n-\frac{1}{2} < x \leqslant n+\frac{1}{2}$.

This follows, for example, from Davenport [2], Chapter 17.

LEMMA 2. *Define* $\vartheta(x) = \sum \log p$, *where the sum is for primes* $p \leqslant x$, *and*

$$\omega(t) = \omega(t, \delta) = \frac{(1+\delta)^{1/2+it}-1}{\frac{1}{2}+it}.$$

Assume RH. *Then, if* $T \geqslant x_0^2 \geqslant 2$ *and* $x_0^{-2} \leqslant \delta \leqslant 1$, *we have*

$$I = \int_{x_0}^{2x_0} \Big|\{\vartheta(x+x\delta)-\vartheta(x)-x\delta\}^2 - \Big\{\sum_{|\gamma|\leqslant T} x^{1/2+i\gamma}\omega(\gamma)\Big\}^2\Big| dx \ll B(x_0, \delta),$$

where

$$B(x_0, \delta) = x_0^{5/2}\delta^2 + x_0^{9/4}\delta^{3/2}(\log x_0)^{1/2} + x_0^2\delta(\log x_0)^{1/2} + x_0^{3/2}\delta^{1/2}(\log x_0) + x_0(\log x_0).$$

For the proof of Lemma 2 we write

$$G(x) = G(x, \delta) = \vartheta(x+x\delta)-\vartheta(x)-x\delta,$$
$$G_1(x) = \Psi(x+x\delta)-\Psi(x)-x\delta-G(x).$$

Then, by Lemma 1 and the Cauchy–Schwarz inequality, we find

$$I = \int_{x_0}^{2x_0} |G(x)^2 - \{G(x)+H(x)\}^2|\,dx \leqslant 2\int_{x_0}^{2x_0} |G(x)H(x)|\,dx + \int_{x_0}^{2x_0} H(x)^2\,dx$$
$$\ll \Big\{\int_{x_0}^{2x_0} G(x)^2\,dx\Big\}^{1/2}\Big\{\int_{x_0}^{2x_0} H(x)^2\,dx\Big\} + \int_{x_0}^{2x_0} H(x)^2\,dx = I_1^{1/2}I_2^{1/2} + I_2,$$

say; here we have written

$$H(x) = G_1(x) - E(x+x\delta) + E(x) - F(x+x\delta) + F(x)$$

for brevity. We proceed to bound I_1 and I_2. We have

$$I_1 \ll x_0^3\delta^2 + \int_{x_0}^{2x_0} \{\vartheta(x+x\delta)-\vartheta(x)\}^2\,dx \ll x_0^3\delta^2 + (\log x_0)^2\int_{x_0}^{2x_0}\Big\{\sum_{x<p\leqslant x+x\delta} 1\Big\}^2 dx$$
$$\ll x_0^3\delta^2 + (\log x_0)^2 \sum_{\substack{x_0\leqslant p,q\leqslant 4x_0\\ |p-q|\leqslant 4\delta x_0}} \delta x_0 \ll x_0^3\delta^2 + x_0^2\delta\log x_0,$$

where p and q run over primes. Here we have used the uniform sieve estimate

$$\sum_{p,q\leqslant x;\, p-q=k} 1 \ll x\frac{\sigma(k)}{k}(\log x)^{-2} \qquad (1\leqslant k\leqslant x), \tag{6}$$

where $\sigma(k)$ is the sum of the divisors of k; this follows, for example, from Halberstam and Richert ([4], Theorem 3.11).

Turning now to I_2, we have

$$G_1(x)^2 = \Big\{\vartheta\big((x+x\delta)^{1/2}\big)-\vartheta(x^{1/2}) + \sum_{3\leqslant n\leqslant \log x_0}\Big(\vartheta\big((x+x\delta)^{1/n}\big)-\vartheta(x^{1/n})\Big)\Big\}^2$$
$$\ll \{\vartheta\big((x+x\delta)^{1/2}\big)-\vartheta(x^{1/2})\}^2 + (\log x_0)\sum_{3\leqslant n\leqslant \log x_0}\{\vartheta\big((x+x\delta)^{1/n}\big)-\vartheta(x^{1/n})\}^2,$$

whence

$$\int_{x_0}^{2x_0} G_1(x)^2\,dx \ll (\log x_0)^2 \sum_{\substack{x_0\leqslant p^2,q^2\leqslant 4x_0\\ |p^2-q^2|\leqslant 4\delta x_0}} \delta x_0 + (\log x_0)^3 \sum_{3\leqslant n\leqslant \log x_0}\ \sum_{\substack{x_0\leqslant p^n,q^n\leqslant 4x_0\\ |p^n-q^n|\leqslant 4\delta x_0}} \delta x_0$$
$$\ll x_0^2\delta^2 + x_0^{3/2}\delta\log x_0$$

on applying (6). Moreover, as $T \geqslant x_0^2$, we have

$$\int_{x_0}^{2x_0} E(x)^2\,dx,\ \int_{x_0}^{2x_0} E(x+x\delta)^2\,dx \ll x_0.$$

Finally, we find

$$\int_{x_0}^{2x_0} F(x)^2\,dx \ll \sum_{x_0-1\leqslant n\leqslant 2x_0+1} \Lambda(n)^2 \ll x_0\log x_0$$

and similarly

$$\int_{x_0}^{2x_0} F(x+x\delta)^2\,dx \ll x_0\log x_0.$$

It follows that

$$I_2 \ll x_0^2\delta^2 + x_0^{3/2}\delta\log x_0 + x_0\log x_0;$$

this, in conjunction with our bound for I_1, proves Lemma 2.

Our next two lemmas show how to bound the sum

$$\Sigma(T, v) = \Sigma(T, v, X) = \sum_{0<\gamma\leqslant T} e\big(\gamma(v+X)\big)$$

in terms of $F_T(X)$. We shall make no use of RH, or even of the fact that γ is the imaginary part of a zero; the results hold good if γ runs over an arbitrary countable set.

LEMMA 3. *Define* $k(v) = 2\pi\exp(-4\pi|v|)$. *Then*

$$\int_{-\infty}^{\infty} k(v)|\Sigma(T, v)|^2 dv = F_T(X).$$

To prove this we have only to note that

$$\int_{-\infty}^{\infty} k(v)e(vx)dv = W(x).$$

LEMMA 4. *Let* $T \geqslant 1$. *Then*

$$\Sigma(T, 0) \ll T^{1/2}\{\operatorname*{Max}_{t\leqslant T} F_t(X)\}^{1/2}.$$

By Montgomery ([6], Lemma 1.1) we have

$$f(0) \ll \int_{-1}^{1} |f'(v)|dv + \int_{-1}^{1} |f(v)|dv$$

if f has a continuous derivative on $[-1, 1]$. Thus

$$\begin{aligned}|\Sigma(T, 0)|^2 &\ll \int_{-1}^{1} |\Sigma(T, v)|\cdot\left|\frac{\partial}{\partial v}\Sigma(T, v)\right|dv + \int_{-1}^{1} |\Sigma(T, v)|^2 dv\\ &\ll \Big\{\int_{-\infty}^{\infty} k(v)|\Sigma(T, v)|^2 dv\Big\}^{1/2}\Big\{\int_{-\infty}^{\infty} k(v)\Big|\sum_{0<\gamma\leqslant T}\gamma e\big(\gamma(v+X)\big)\Big|^2 dv\Big\}^{1/2} +\\ &\qquad + \int_{-\infty}^{\infty} k(v)|\Sigma(T, v)|^2 dv.\end{aligned}$$

However, on summing by parts,

$$\sum_{0<\gamma\leqslant T}\gamma e\big(\gamma(v+X)\big) = T\Sigma(T, v) - \int_0^T \Sigma(t, v)dt,$$

whence

$$\begin{aligned}\int_{-\infty}^{\infty} k(v)\Big|\sum_{0<\gamma\leqslant T}\gamma e\big(\gamma(v+X)\big)\Big|^2 dv &\ll T^2\int_{-\infty}^{\infty} k(v)|\Sigma(T, v)|^2 dv + T\int_0^T\int_{-\infty}^{\infty} k(v)|\Sigma(t, v)|^2 dv\,dt\\ &\ll T^2\operatorname*{Max}_{t\leqslant T} F_t(X).\end{aligned}$$

Lemma 4 follows.

3. Proof of Theorem 1. Taking $T = x$ in Lemma 1, we have, by partial summation

$$\Psi(x) - x \ll x^{1/2}\Big\{1 + T^{-1}|\Sigma(T, 0, X)| + \int_2^T t^{-2}|\Sigma(t, 0, X)|dt\Big\},$$

where $x = e^{2\pi X}$. When $x \geqslant \tau \geqslant x^{2/A}$ we have, by the hypothesis of Theorem 1,

$$\begin{aligned}\operatorname*{Max}_{t\leqslant\tau} F_t(X) \leqslant \operatorname*{Max}_{t\leqslant\tau^{1/2}} F_t(X) + \operatorname*{Max}_{\tau^{1/2}\leqslant t\leqslant\tau} F_t(X) &= O\big(\tau^{1/2}(\log\tau)^2\big) + o\big(\tau(\log\tau)^2\big)\\ &= o\big(\tau(\log\tau)^2\big)\end{aligned}$$

uniformly in τ. Hence, by Lemma 4,

$$1 + T^{-1}|\Sigma(T, 0, X)| + \int_{x^{2/A}}^T t^{-2}|\Sigma(t, 0, X)|dt = o\big((\log x)^2\big).$$

On the other hand

$$\int_2^{x^{2/A}} t^{-2}|\Sigma(t, 0, X)|dt \ll \int_2^{x^{2/A}} t^{-1}(\log t)dt \ll A^{-2}(\log x)^2.$$

Thus

$$|\Psi(x) - x| \leqslant C\big(\varepsilon(A, x) + A^{-2}\big)x^{1/2}(\log x)^2,$$

where C is an absolute constant and, for any fixed A, $\varepsilon(A, x)\to 0$ as $x\to\infty$. Now suppose $\varepsilon > 0$ is given. Choose $A = (2C\varepsilon^{-1})^{1/2}$, and take x_0 such that

$$|\varepsilon(A, x)| \leqslant \varepsilon/(2C) \qquad (x \geqslant x_0).$$

Then $|\Psi(x) - x| \leqslant \varepsilon x^{1/2}(\log x)^2$ for $x \geqslant x_0$, and Theorem 1 follows.

4. Proof of Theorem 2. We shall write x_0 in place of x in Theorem 2. The result is trivial if $\Delta \leqslant \log x_0$ or, by (3), if $\Delta \gg x_0^{1/2}\log x_0$; thus, taking $\delta = \Delta/(4x_0)$, we may assume that $x_0^{-1} \leqslant \delta \ll x_0^{-1/2}\log x_0$.

We begin by examining the integral

$$J = \int_{x_0}^{2x_0} \{\vartheta(x+x\delta) - \vartheta(x) - x\delta\}^2 dx.$$

If $x_0 \leqslant p_n \leqslant \frac{3}{2}x_0$ and $d_n \geqslant \Delta$, we have $p_n \leqslant x \leqslant x+x\delta < p_n + d_n$, and so $\vartheta(x+x\delta) = \vartheta(x)$, for $p_n \leqslant x \leqslant p_n + \frac{1}{2}d_n$. Thus

$$J \gg \sum_{\substack{x_0 \leqslant p_n \leqslant 3x_0/2 \\ d_n \geqslant \Delta}} \int_{p_n}^{p_n+p_n/2} (x\delta)^2 dx \gg x_0^2 \delta^2 \sum_{\substack{x_0 \leqslant p_n \leqslant 3x_0/2 \\ d_n \geqslant \Delta}} d_n.$$

On the other hand, by Lemma 2 with $T = x_0^2$,

$$J \ll B(x_0, \delta) + x_0 \int_{x_0}^{2x_0} \Big| \sum_{|\gamma| \leqslant T} x^{i\gamma} \omega(\gamma) \Big|^2 dx,$$

and for the present range of δ,

$$B(x_0, \delta) \ll x_0^2 \delta \log x_0.$$

Hence it remains to show that

$$(7) \qquad \int_{x_0}^{2x_0} \Big| \sum_{|\gamma| \leqslant T} x^{i\gamma} \omega(\gamma) \Big|^2 dx \ll x_0 \delta^2 \operatorname*{Max}_{K \geqslant 1} \{(K(\log 2K)^{-2})^{-2} \operatorname*{Max}_{\tau \leqslant K/\delta} F_\tau(X)\},$$

where $X = (\log x_0)/2\pi$.

We treat the terms with $|\gamma| \leqslant \delta^{-1}$ and $|\gamma| > \delta^{-1}$ separately. In the first case we have

$$\Big| \sum_{|\gamma| \leqslant 1/\delta} x^{i\gamma} \omega(\gamma) \Big| \ll \Big| \sum_{0 < \gamma \leqslant 1/\delta} x^{i\gamma} \omega(\gamma) \Big|$$

and, by partial summation,

$$\sum_{0 < \gamma \leqslant 1/\delta} x^{i\gamma} \omega(\gamma) = \Sigma(1/\delta, v, X)\, \omega(1/\delta) - \int_0^{1/\delta} \Sigma(t, v, X)\, \omega'(t)\, dt,$$

where $x_0 \leqslant x \leqslant 2x_0$, $x = e^{2\pi(v+X)}$. However we have the trivial bounds

$$(8) \qquad \omega(t) \ll \operatorname{Min}(\delta, |t|^{-1}), \qquad \omega'(t) \ll \delta \operatorname{Min}(\delta, |t|^{-1}),$$

whence

$$\sum_{0 < \gamma \leqslant 1/\delta} x^{i\gamma} \omega(\gamma) \ll \delta |\Sigma(1/\delta, v, X)| + \delta^2 \int_0^{1/\delta} |\Sigma(t, v, X)|\, dt.$$

Then, by the Cauchy–Schwarz inequality and Lemma 3, we find

$$\int_{x_0}^{2x_0} \Big| \sum_{|\gamma| \leqslant 1/\delta} x^{i\gamma} \omega(\gamma) \Big|^2 dx \ll \int_{x_0}^{2x_0} \Big\{ \delta^2 |\Sigma(1/\delta, v, X)|^2 + \delta^3 \int_0^{1/\delta} |\Sigma(t, v, X)|^2 dt \Big\} dx$$

$$\ll x_0 \int_0^{(\log 2)/2\pi} \Big\{ \delta^2 |\Sigma(1/\delta, v, X)|^2 + \delta^3 \int_0^{1/\delta} |\Sigma(t, v, X)|^2 dt \Big\} dv$$

$$\ll x_0 \int_{-\infty}^{\infty} k(v) \Big\{ \delta^2 |\Sigma(1/\delta, v, X)|^2 + \delta^3 \int_0^{1/\delta} |\Sigma(t, v, X)|^2 dt \Big\} dv$$

$$(9) \qquad = x_0 \delta^2 F_{1/\delta}(X) + x_0 \delta^3 \int_0^{1/\delta} F_t(X)\, dt \ll x_0 \delta^2 \operatorname*{Max}_{\tau \leqslant 1/\delta} F_\tau(X),$$

as required.

We turn now to the case $\delta^{-1} < |\gamma| \leqslant T$. Here we use the decomposition

$$\sum_\gamma x^{i\gamma} \omega(\gamma) = (1+\delta)^{1/2} \sum_\gamma (x+x\delta)^{i\gamma} (\tfrac{1}{2}+i\gamma)^{-1} - \sum_\gamma x^{i\gamma} (\tfrac{1}{2}+i\gamma)^{-1},$$

whence

$$\int_{x_0}^{2x_0} \Big| \sum_{1/\delta < |\gamma| \leqslant T} x^{i\gamma} \omega(\gamma) \Big|^2 dx \ll \int_{x_0}^{3x_0} \Big| \sum_{1/\delta < |\gamma| \leqslant T} x^{i\gamma} (\tfrac{1}{2}+i\gamma)^{-1} \Big|^2 dx.$$

Partial summation now yields

$$\sum_{1/\delta < \gamma \leqslant T} x^{i\gamma} (\tfrac{1}{2}+i\gamma)^{-1} = \Sigma(T, v, X)(\tfrac{1}{2}+iT)^{-1} - \Sigma(1/\delta, v, X)(\tfrac{1}{2}+i\delta^{-1})^{-1} + \\ + i \int_{1/\delta}^{T} \Sigma(t, v, X)(\tfrac{1}{2}+it)^{-2} dt,$$

and therefore

$$\sum_{1/\delta < |\gamma| \leqslant T} x^{i\gamma} (\tfrac{1}{2}+i\gamma)^{-1} \ll T^{-1} |\Sigma(T, v)| + \delta |\Sigma(1/\delta, v)| + \int_{1/\delta}^{T} t^{-2} |\Sigma(t, v)|\, dt.$$

After applying the Cauchy–Schwarz inequality we find

$$\Big\{ \int_{1/\delta}^{T} t^{-2} |\Sigma(t, v)|\, dt \Big\}^2 \leqslant \Big\{ \int_{1/\delta}^{T} t^{-1} (\log(1+t\delta))^{-2} dt \Big\} \Big\{ \int_{1/\delta}^{T} t^{-3} (\log(1+t\delta))^2 |\Sigma(t, v)|^2 dt \Big\}$$

$$\ll \int_{1/\delta}^{T} t^{-3} (\log(1+t\delta))^2 |\Sigma(t, v)|^2 dt,$$

whence, by Lemma 3,

$$\int_{x_0}^{2x_0}\Big|\sum_{1/\delta<\gamma\leqslant T} x^{i\gamma}\omega(\gamma)\Big|^2 dx$$

$$\ll \int_{x_0}^{3x_0}\Big\{T^{-2}|\Sigma(T,v)|^2+\delta^2|\Sigma(1/\delta,v)|^2+\int_{1/\delta}^{T} t^{-3}\big(\log(1+t\delta)\big)^2|\Sigma(t,v)|^2dt\Big\}dx$$

$$\ll x_0\int_{-\infty}^{\infty} k(v)\Big\{T^{-2}|\Sigma(T,v)|^2+\delta^2|\Sigma(1/\delta,v)|^2+ + \int_{1/\delta}^{T} t^{-3}\big(\log(1+t\delta)\big)^2|\Sigma(t,v)|^2dt\Big\}dv$$

$$= x_0T^{-2}F_T(X)+x_0\delta^2F_{1/\delta}(X)+x_0\int_{1/\delta}^{T} t^{-3}\big(\log(1+t\delta)\big)^2F_t(X)\,dt$$

$$\ll x_0\delta^2\operatorname*{Max}_{K\geqslant 1}\big\{\big(K(\log 2K)^{-2}\big)^{-2}F_{K/\delta}(X)\big\}.$$

Together with (9), this yields the required estimate (7), and the proof of Theorem 2 is complete.

5. Preparations for the proof of Theorem 3. Theorem 3 is naturally more delicate than Theorems 1 and 2. In this section we begin by evaluating, in terms of $F_t(X)$, the integral

$$I = \int_{-\infty}^{\infty} k(v)|S(v)|^2dv,$$

where

$$S(v) = \sum_{0<\gamma\leqslant T} e\big(\gamma(v+X)\big)\omega(\gamma).$$

On expanding we find

$$I = \sum_{0<\gamma_i\leqslant T} W(\gamma_1-\gamma_2)e\big((\gamma_1-\gamma_2)X\big)\omega(\gamma_1)\overline{\omega(\gamma_2)}.$$

From (8) we have $\omega(\gamma_1) = \omega(\gamma_2)+O(\delta^2|\gamma_2-\gamma_1|)$ whence, for $\gamma_1\leqslant\gamma_2$,

$$\omega(\gamma_1)\overline{\omega(\gamma_2)} = |\omega(\gamma_2)|^2+O(\delta^2|\gamma_2-\gamma_1|/\gamma_2).$$

The error term here contributes

$$\ll \delta^2\sum_{0<\gamma_2\leqslant T}\gamma_2^{-1}\sum_{0<\gamma_1\leqslant\gamma_2}\big(1+(\gamma_2-\gamma_1)\big)^{-1} \ll \delta^2(\log T)^4$$

to I, whence

$$I = \sum_{0<\gamma_i\leqslant T} W(\gamma_1-\gamma_2)e\big((\gamma_1-\gamma_2)X\big)|\omega(\operatorname{Max}\gamma_i)|^2+O\big(\delta^2(\log T)^4\big).$$

Partial summation now yields

$$I = |\omega(T)|^2F_T(X)-\int_0^T F_t(X)\Big\{\frac{d}{dt}|\omega(t)|^2\Big\}dt+O\big(\delta^2(\log T)^4\big).$$

We now consider

$$\int_{-\infty}^{\infty} k(v)|S(v)+\overline{S(v)}|^2dv;$$

this is clearly

$$2I+2\operatorname{Re}\int_{-\infty}^{\infty} k(v)S(v)^2dv.$$

However the last integral is

$$\sum_{0<\gamma_i\leqslant T}\omega(\gamma_1)\omega(\gamma_2)W(\gamma_1+\gamma_2)e\big((\gamma_1+\gamma_2)X\big) \ll \delta^2\sum_{\gamma_i}W(\gamma_1+\gamma_2) \ll \delta^2(\log T)^3,$$

whence

$$\int_{-\infty}^{\infty} k(v)\Big|\sum_{|\gamma|\leqslant T} e\big(\gamma(v+X)\big)\omega(\gamma)\Big|^2dv$$

$$= 2|\omega(T)|^2F_T(X)-2\int_0^T F_t(X)\Big\{\frac{d}{dt}|\omega(t)|^2\Big\}dt+O\big(\delta^2(\log T)^4\big).$$

We now suppose $x\geqslant 2$ and $x^{-1}\leqslant\delta\leqslant x^{-1}\log x$, (however X and x are not yet assumed to be related by $x = e^{2\pi X}$), and we take $T = x^3$. The expression above is then $\ll\delta(\log x)^3$, by (8), and hence we have

$$\int_0^1\Big|\sum_{|\gamma|\leqslant T} e\big(\gamma(v+X)\big)\omega(\gamma)\Big|^2dv \ll \delta(\log x)^3$$

uniformly for all real X. If we now choose $L = \log\log x+O(1)$, we shall therefore find

$$\Big(\int_{-\infty}^{-L}+\int_L^{\infty}\Big)k(v)\Big|\sum_{|\gamma|\leqslant T} e\big(\gamma(v+X)\big)\omega(\gamma)\Big|^2dv \ll e^{-4\pi L}\delta(\log x)^3 \ll \delta(\log x)^{-9}.$$

Consequently

$$\int_{-L}^{L} k(v)\Big|\sum_{|\gamma|\leqslant T} e\big(\gamma(v+X)\big)\omega(\gamma)\Big|^2dv$$

$$= 2|\omega(T)|^2F_T(X)-2\int_0^T F_t(X)\Big\{\frac{d}{dt}|\omega(t)|^2\Big\}dt+O\big(\delta(\log x)^{-9}\big).$$

Our choice of T yields

$$|\omega(T)|^2F_T(X) \ll T^{-1}(\log T)^2 \ll \delta(\log x)^{-9}.$$

Moreover, if

$$A(t) = A(t,\delta) = \{2t^{-1}\sin(\tfrac{1}{2}\delta t)\}^2,$$

then

$$\frac{d}{dt}|\omega(t)|^2 = A'(t)+O(\delta^2t^{-3})+O(\delta^3t^{-1}).$$

Thus, finally, we find

LEMMA 5.

$$\int_{-L}^{L} k(v)\Big|\sum_{|\gamma|\le T} e\big(\gamma(v+X)\big)\omega(\gamma)\Big|^2 dv = -2\int_0^T F_t(X)A'(t)\,dt+O\big(\delta(\log x)^{-9}\big).$$

6. Completion of the proof of Theorem 3. We begin by combinin[g] Lemmas 2 and 5. We write $x=\exp(2\pi X)$, $y=\exp\big(2\pi(v+X)\big)$ $x_0=\exp\big(2\pi(L_0+X)\big)$ and $l=(\log 2)/2\pi$. Then

$$\int_{L_0}^{L_0+l} k(v)\Big|\sum_{|\gamma|\le T} e\big(\gamma(v+X)\big)\omega(\gamma)\Big|^2 dv = (2\pi)^{-1}\int_{x_0}^{2x_0} k(v)y^{-2}\Big\{\sum_{|\gamma|\le T} y^{\frac12+i\gamma}\omega(\gamma)\Big\}^2 d$$

$$= (2\pi)^{-1}\int_{x_0}^{2x_0} k(v)y^{-2}G(y)^2dy+O\big(x_0^{-2}k(L_0)B(x_0,\delta)\big).$$

If we now let L_0 run over integral multiples of l in the range $Nl = -$ $\le L_0\le Ml = L-l$, and sum, then the error terms will be of the forn

$$\sum_{L_0} x_0^{-2}k(L_0)x_0^a\delta^b(\log x_0)^c \ll x^{a-2}\delta^b(\log x)^c\sum_{N\le n\le M}\exp\big\{2\pi l\big((a-2)n-2|n|\big)\big\}$$

$$\ll x^{a-2}\delta^b(\log x)^c,$$

since in each case (that is for $a = 5/2,\ 9/4,\ 2,\ 3/2$ or 1) we have $|a-2|<2$ It follows that

$$\int_{-L}^{L} k(v)\Big|\sum_{|\gamma|\le T} e\big(\gamma(v+X)\big)\omega(\gamma)\Big|^2 dv = \int_{-L}^{L} k(v)y^{-1}G(y)^2dv+O\big(x^{-2}B(x,\delta)\big),$$

whence Lemma 5 yields, for our range of δ,

$$\int_{x/D}^{xD}\mathrm{Min}(x^{-2},x^2y^{-4})G(y)^2dy = -2\int_0^T F_t(X)A'(t)\,dt+O(x^{-1/2}\delta^{1/2}\log x),$$

in which $D=\exp(2\pi L)$.

We now introduce an average with respect to δ, in order to restric the t-integral to the range mentioned in Theorem 3. We multiply b

$K(\delta)=1-|2-\delta/\Delta|$, and integrate for $\Delta\le\delta\le 3\Delta$, where $x^{-1}\le\Delta\le\frac{1}{3}x^{-1}\log x$. $K(\delta)$ is so chosen that

$$\int_{\Delta}^{3\Delta}K(\delta)\frac{\partial}{\partial t}A(t,\delta)\,d\delta = \frac{\partial}{\partial t}\big\{2\Delta t^{-2}\big(1-4t^{-2}\Delta^{-2}(\cos 2t\Delta)(\sin\tfrac12 t\Delta)^2\big)\big\} = C(t,\Delta),$$

say. Here $C(t,\Delta)\ll\mathrm{Min}(t^{-3}\Delta,\ t\Delta^5)$, and so

$$\int_0^T F_t(X)C(t,\Delta)\,dt = \int_{T_1}^{T_2}F_t(X)C(t,\Delta)\,dt+O\big(T_1^3\Delta^5(\log x)^2\big)+O\big(T_2^{-1}\Delta(\log x)^2\big),$$

where $0<T_1\le T_2$. We choose $T_1=x(\log x)^{-2}$, $T_2=x\log x$. Now, if, in accordance with the hypothesis of Theorem 3,

$$|F_t(X)-tX|\le t\delta(x)\log t$$

for $T_1\le t\le T_2$, then

$$\int_0^T F_t(X)C(t,\Delta)\,dt$$

$$=\int_{\Delta}^{3\Delta}K(\delta)\int_{T_1}^{T_2}tX\frac{\partial}{\partial t}A(t,\delta)\,dt\,d\delta+O\big(\Delta^2\delta(x)\log x\big)+O(x^{-1/2}\Delta^{3/2}\log x).$$

Moreover

$$\int_{T_1}^{T_2}tX\frac{\partial}{\partial t}\{(2t^{-1}\sin\tfrac12\delta t)^2\}\,dt = 4\delta X\int_{\delta T_1}^{\delta T_2}v\frac{d}{dv}(v^{-1}\sin\tfrac12 v)^2dv$$

$$=4\delta X\Big([v^{-1}(\sin\tfrac12 v)^2]_{\delta T_1}^{\delta T_2}-\int_{\delta T_1}^{\delta T_2}\Big\{\frac{\sin\frac12 v}{v}\Big\}^2dv\Big)$$

$$=4\delta X\Big(-\int_0^\infty\Big\{\frac{\sin\frac12 v}{v}\Big\}^2dv+O(\delta T_1)+O(\delta^{-1}T_2^{-1})\Big)$$

and

$$\int_0^\infty\Big\{\frac{\sin\frac12 v}{v}\Big\}^2dv = \frac{\pi}{4},$$

whence

$$-2\int_0^T F_t(X)C(t,\Delta)\,dt$$

$$=(\log x)\int_{\Delta}^{3\Delta}k(\delta)\,\delta\,d\delta+O(x^{-1/2}\Delta^{3/2}\log x)+O\big(\Delta^2\delta(x)\log x\big).$$

Thus

$$\Big|\int_{\Delta}^{3\Delta} k(\delta)\Big\{\int_{x/D}^{xD} \mathrm{Min}(x^{-2}, x^2y^{-4})G(y,\delta)^2dy - \delta\log x\Big\}d\delta\Big| \ll x^{-1/2}\Delta^{3/2}\log x + \Delta^2\delta(x)\log x.$$

The integrand of the δ-integration is a continuous function of δ, whence there must exist some δ, $\Delta \leqslant \delta \leqslant 3\Delta$, such that

$$(10)\qquad \int_{x/D}^{xD} \mathrm{Min}(x^{-2}, x^2y^{-4})\{\vartheta(y+y\delta)-\vartheta(y)-y\delta\}^2dy = \delta\log x + O(x^{-1/2}\Delta^{1/2}\log x) + O\big(\Delta\delta(x)\log x\big).$$

We proceed to expand and evaluate the above integral in the form

$$\int_{x/D}^{xD} y^2\delta^2\mathrm{Min}(x^{-2}, x^2y^{-4})dy - 2\sum_p (\log p)\int y\delta\mathrm{Min}(x^{-2}, x^2y^{-4})dy + \sum_{p,q}(\log p)(\log q)\int \mathrm{Min}(x^{-2}, x^2y^{-4})dy = E_1 - 2E_2 + E_3,$$

where $x/D \leqslant y \leqslant xD$ and $p/(1+\delta) \leqslant y \leqslant p$ in E_2, $\mathrm{Max}(p,q)/(1+\delta) \leqslant y \leqslant \mathrm{Min}(p,q)$ in E_3. We readily find $E_1 = \frac{4}{3}x\delta^2 + O(x\Delta^2/D)$. When $2x/D \leqslant p \leqslant \frac{1}{2}xD$, the integral in E_2 is

$$\int_{p/(1+\delta)}^{p} y\delta\mathrm{Min}(x^{-2}, x^2y^{-4})dy = \big(1+O(\Delta)\big)\int_{p/(1+\delta)}^{p} p\delta\mathrm{Min}(x^{-2}, x^2p^{-4})dy = p^2\delta^2\mathrm{Min}(x^{-2}, x^2p^{-4})\big(1+O(\Delta)\big),$$

and otherwise is majorized by this. Hence

$$E_2 = \big(1+O(\Delta)\big)\sum_{2x/D\leqslant p\leqslant\frac{1}{2}xD} p^2\delta^2\mathrm{Min}(x^{-2}, x^2p^{-4})\log p + O\Big(\sum_{p\leqslant 2x/D} p^2\Delta^2x^{-2}\log p\Big) + O\Big(\sum_{p\geqslant\frac{1}{2}xD} p^{-2}\Delta^2x^2\log p\Big) = \tfrac{4}{3}x\delta^2 + O(x\Delta^2/D).$$

A similar calculation shows that the contribution to E_3 from terms with $p = q$ is

$$\big(1+O(\Delta)\big)\sum_{2x/D\leqslant p\leqslant\frac{1}{2}xD} p\delta\mathrm{Min}(x^{-2}, x^2p^{-4})(\log p)^2 + O\Big(\sum_{p\leqslant 2x/D} p\Delta x^{-2}(\log p)^2\Big) + O\Big(\sum_{p\geqslant\frac{1}{2}xD} p^{-3}\Delta x^2(\log p)^2\Big) = \delta\log x + O(\Delta).$$

For the remaining terms of E_3 we suppose $p < q$. The integral, in this case, vanishes unless $q \leqslant p(1+\delta)$, when it is trivially $\ll p\Delta\mathrm{Min}(x^{-2}, x^2p^{-4})$. Thus terms in which $p < x/K$ or $p > xK$, where $1 \leqslant K \leqslant D$, contribute

$$\ll \Big(\sum_{N=2^n\leqslant x/K} + \sum_{2N=2^n\geqslant xK}\Big)\sum_{N\leqslant p\leqslant 2N}\ \sum_{p<q\leqslant p+2\delta N} N\Delta(\log N)^2\mathrm{Min}(x^{-2}, x^2N^{-4}).$$

On applying (6) we may estimate this as $\ll x\Delta^2K^{-1}$. These results, in conjunction with (10), show that the terms with $p < q \leqslant p(1+\delta)$, $x/K \leqslant p \leqslant xK$, must contribute

$$\tfrac{4}{3}x\delta^2 + O(x^{-1/2}\Delta^{1/2}\log x) + O\big(\Delta\delta(x)\log x\big) + O(x\Delta^2K^{-1}).$$

This quantity will be positive if $x\Delta \geqslant K(\log x)^{2/3}$, $x\Delta \geqslant K\delta(x)\log x$, and K is a sufficiently large absolute constant. Consequently there must exist a p and q with $x/K \leqslant p \leqslant xK$ and

$$p < q \leqslant p + 3K^2\mathrm{Max}\big((\log x)^{2/3}, \delta(x)\log x\big)$$

and Theorem 3 follows.

References

[1] H. Cramér, *Some theorems concerning prime numbers*, Arkiv för Mat. Astronom. och Fysik 15 (5) (1920), pp. 1–32.

[2] H. Davenport, *Multiplicative number theory*, Markham, Chicago 1967.

[3] P. X. Gallagher and J. H. Mueller, *Primes and zeros in short intervals*, J. Reine Angew. Math. 303/304 (1978), pp. 205–220.

[4] H. Halberstam and H.-E. Richert, *Sieve methods*, Academic Press, London 1974.

[5] H. von Koch, *Sur la distribution des nombres premiers*, Acta Math. 24 (1901), pp. 159–182.

[6] H. L. Montgomery, *Topics in multiplicative number theory*, Springer, Berlin 1971.

[7] — *The pair correlation of zeros of the Zeta-function*; in: *Analytic Number Theory* (Proc. Sympos. Pure Math., Vol. XXIV St. Louis Univ., St. Louis Mo., 1972), pp. 181–193. (Amer. Math. Soc., Providence, R. I., 1973).

[8] J. H. Mueller, *Exponential sums over the zeros of the Riemann zeta function (I)*, to appear.

[9] A. Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. 47 (6) (1943), pp. 87–105.

Received on 4. 12. 1979 (1187)