COROLLARY. Let \( F \) be a field with \( m = 4, R = F^4 \), and \( 8 \neq q < \infty \). Then anisotropic ternary forms over \( F \) are determined by their value sets.

Proof. It suffices to show that \( D(\langle 1, a, b \rangle) = D(\langle 1, c, d \rangle) \) implies \( \langle a, b \rangle \cong \langle c, d \rangle \) for anisotropic \( \langle 1, a, b \rangle, \langle 1, c, d \rangle \). By a cardinality argument on the value sets mod \( F^2 \) and by Theorem 8, it is clear that it is impossible for exactly one of \( \langle -a, -b \rangle, \langle -c, -d \rangle \) to be \( Q_2 \).

If \( \langle -a, -b \rangle = \langle -c, -d \rangle = Q_2 \), then by Theorem 8, \( -abB_i \cup -abB_\bar{i} = -cdB_i \cup -cdB_\bar{i} \). Thus \( ab \in B_\bar{i} \cap B_i = F^2 \). But then Corollary 2.9 \([0, \text{P. 00}] \) shows \( \langle -a, -b \rangle \cong \langle -c, -d \rangle \). If \( \langle -a, -b \rangle = \langle -c, -d \rangle = Q_2 \), \( i = 1 \text{ or } 2 \), then Theorem 3 yields \( abB_i = cdB_i \). \( \{i, j\} = \{1, 2\} \). But the quaternion algebra equality also implies \( -a, -b, -c, -d \in B_i \). Thus \( ab \in B_i \cap B_i = F^2 \) and again \( \langle -a, -b \rangle \cong \langle -c, -d \rangle \). Finally assume, \( \langle -a, -b \rangle = Q_2 \), and \( \langle -c, -d \rangle = Q_2 \). Then from Theorem 8, \( abB_\bar{i} = cdB_\bar{i} \). But this cannot happen since \( B_\bar{i} \neq B_\bar{i} \).

References


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PARTITIONS INTO DISTINCT SMALL PRIMES

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Introduction. Throughout, \( p_n \) will denote the \( n \)-th prime. In Theorem 1 we find estimates for \( \sum_{k=1}^{n} p_k \) (see (3)), but because of the application of Theorem 3 later, it will be more convenient to work with the sum \( y(n) = 3 + \sum_{k=4}^{n} p_k \).

THEOREM 1.

(1) \( y(n) < \frac{1}{n} n^2 (\log n + \log \log n) \) for \( n \geq 4 \).

Given \( 0 < \alpha < 1 \), there exists an integer \( N(\alpha) \) such that
(2) \( \frac{1}{n} n^2 (\log n + \log \log n) < y(n) \) for \( n \geq N(\alpha) \), and \( N(0) = 5 \). Moreover, (2) is true if
(3) \( \sum_{k=1}^{n} p_k < \frac{1}{n} n^2 (\log n + \log \log n) \) for \( n \geq 6 \),
and lower bounds for this sum are given by those for \( y(n) \).

COROLLARY. \( \sum_{k=1}^{n} p_k \sim \frac{1}{n} n^2 (\log n + \log \log n) \).

We shall see later that \( N(1) = 5 \) and \( N(0.56) = 140 \) (see Remark 1, following the proof of (2)). The inequalities (1) and (2) are used in proving

THEOREM 2. Let \( \varepsilon > 0 \) and write \( y(n) = y \). Then
\( \sqrt{2y} \sqrt{\log y \sqrt{2y} + (\frac{1}{2} + \varepsilon) \log \log y \sqrt{2y}} < p_n < \sqrt{2y} \sqrt{\log y \sqrt{2y} + (\frac{1}{2} + \varepsilon) \log \log y \sqrt{2y}} \)

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provided \( n \) is sufficiently large. The right-hand inequality holds for \( n \geq \max \{20, n_0(e)\} \), the left for \( n \geq n_1(e) \).

\( n_1(e) \) will be defined when the left inequality is proved.

**Corollary.** \( p_n \sim \sqrt{2y} \sqrt{\log y \log y + \frac{1}{2} \log \log y} \).

**Theorem 3.** Let \( n \geq 4 \). Then every integer in the closed interval \([7, y(n)]\) can be partitioned into distinct primes \( \leq p_n \).

Through an adaptation of the proof of the right-hand inequality in Theorem 2, and the application of Theorem 3, we obtain lastly Theorem 4. The proof could be done independently of that of Theorem 2, but it is simpler to proceed as we have done.

**Theorem 4.** Let \( 0 < e \leq 1, n \geq \max \{20, n_0(e) + 1\} \), and \( y(n - 1) < y \leq y(n) \). Then \( y \) can be partitioned into distinct primes \( p \) satisfying

\[
p < \sqrt{2y} \sqrt{\log y \log y + \frac{1}{4} + e} \log \log y.
\]

We remark on the possibility of dropping the \( e \), and the impossibility of reducing the \( \frac{1}{4} \), after introducing the notation

\[
U(y, e) = \sqrt{2y} \sqrt{\log y \log y + e} \log \log y,
\]

which we shall use in the sequel. We have computed \( y(n) \) and some related functions, including \( U(y(n), e) \) for several values of \( e \), for \( 4 \leq n \leq 100000 \).

Some of the results of these computations are mentioned here, in the introduction, and others appear later, in Remarks 1, 2, and 3; partial print-outs of the computations are available from the author.

**Corollary to Theorem 4.** Given a positive integer \( k \), there is an integer \( n_k = n_k(k) \) such that every integer \( y > n_k \) can be partitioned into \( k \) or more distinct primes.

**Proof.** The number of distinct primes used in representing \( y \) in the theorem exceeds \( y/U(y, \frac{1}{4} + e) \). Hence for given \( k \) we need merely take \( n_k \) such that \( n_k / U(y_k, \frac{1}{4} + e) \geq k \).

The inequality in Theorem 4, without the \( e \), would follow from

\[
p_n < U(y(n - 1), \frac{1}{4});
\]

cf. (33) and the ensuing remarks. Our computations yield (5) for \( 105 \leq n \leq 100000 \), with the difference between the two sides tending to increase; see Remark 3 at the end of the paper. To see that the \( \frac{1}{4} \) in Theorem 4 cannot be reduced, observe that for \( n \geq 6 \), a prime at least as large as \( p_n \) is needed in any partition of \( y(n) \) into distinct primes, because the only primes \( \leq p_n \) not appearing in the sum defining \( y(n) \) are 2 and 3, which are too small to be used in place of \( p_n \). By Theorem 2, \( p_n > U(y(n), \frac{1}{4} - e) \).

First efforts to find an upper estimate for \( p_n \) in terms of \( y(n) \) involved some rough work which led to the conjecture:

\[
p_n < U(y(n), \frac{1}{4});
\]

for all sufficiently large \( n \). The attempt to prove this has led to Theorems 1, 2, and 4. Our computations show that for \( n \leq 100000 \),

\[
p_n < U(y(n), e) \quad \text{with} \quad e = 0.5, 0.6, 0.7
\]

for \( n \geq 49, 11, 6 \) respectively;

with \( e = 3/2 \), we can prove this inequality for all \( n \geq 6 \) (see (28) in Remark 2, following the proof of Theorem 2). On the other hand we find that for \( n \leq 100000 \)

\[
U(y(n), e) < p_n \quad \text{with} \quad e = 0, 0.3
\]

for \( n \geq 31, 37374 \) respectively.

For more on the case \( e = 0 \), see (29) in Remark 2, and also Table 2 following it, where some sample values of \( U(y(n), e) \) for \( e = 0 \) and \( e = 0.5 \) are listed.

Our work with these estimates for \( p_n \) consistently indicated that the lower estimates are the more difficult to fit. More precisely, in Theorem 2 it looks as though \( n_1(e) \) will be very much larger than \( n_0(e) \). For example, \( n_1(\frac{1}{4}) \) seems likely to exceed \( 10^{17} \) (Remark 2), while \( n_0(\frac{1}{4}) = N(\frac{1}{4}) < 3 \times 10^{54} \) (Remark 1). In the numerical work we find that

\[
U(y(n), 3) < p_n
\]

fails for many \( n \leq 37373 \), and that \( p_n - U(y(n), 3) < 1700 \) for \( n \leq 100000 \), whereas the difference between the two sides of (5a) increases from 1010.4... when \( n = 5000 \) to 22873.0... when \( n = 100000 \) (see Table 2). In spite of all this, we know that (6) is true for nearly all \( n \), and we do not know this about (5a).

Regarding Theorem 3, Richert [1] proved in 1950 that every integer \( \geq 7 \) can be partitioned into distinct primes. Neither in his proof nor in any other proof of this or of similar results have we seen an examination of the size of the primes that can be used in the partitions. By proving the somewhat more precise result (Theorem 3) we can see how small these primes can be chosen (Theorem 4). This consequence of such a simple partition theorem (the proof of Theorem 3 is as simple as Richert's proof, except that we keep track of the sums involved) has been overlooked heretofore, probably because the emphasis in additive prime number problems has been on seeing how few primes are needed in a partition rather than on how small they can be.

One can prove results similar to Theorem 3 using similar arguments.

For example, every integer in \([10, 5 + \sum_{i=5}^5 p_i]\) (\( n \geq 7 \)) can be partitioned into distinct odd primes \( \leq p_n \) (cf. Sierpinski [3], p. 144, Theorem 12; Theorem 11 is the above-mentioned result of Richert).
We prove Theorem 3 first; the proofs of the other theorems are of a different nature.

Proof of Theorem 3. First observe that every even integer in the interval $[-14, 14]$ can be expressed in the form
\[ \pm 2 \pm 3 \pm 5 \pm 7 \pm 11 \]
for an appropriate choice of signs. Adding $\pm 13$ to these numbers gives every odd in $[-27, 27]$, and then $\pm 17$ gives every even in $[-44, 44]$, and so on. To see that this indeed continues, define, for $n \geq 5$,
\[ T_n = -2 + 3 - 5 + 7 + 11 + \ldots + p_n, \]
and assume for some such $n$ that every integer of the same parity as $T_n$ in $[-T_n, T_n]$ is expressible in the form
\[ \pm 2 \pm 3 \pm 5 \pm 7 \pm 11 \pm p_n. \]
Then, by Bertrand's postulate, adding $\pm p_{n+1}$ gives every integer of the same parity as $T_{n+1}$ in $[-T_{n+1}, T_{n+1}]$.

The details here are that the intervals
\[ [-T_n - p_{n+1}, T_n - p_{n+1}] \quad \text{and} \quad [-T_n + p_{n+1}, T_n + p_{n+1}] \]
will overlap for each $n \geq 5$ if
\[ T_n \geq p_{n+1} \]
for $n \geq 5$. Now, $T_5 = 14 > p_6 = 13$, and if (8) holds for a certain $n$, then
\[ T_{n+1} = T_n + p_{n+1} \geq 2p_{n+1}, \]
and this exceeds $p_{n+1}$ because of Bertrand's postulate.

We conclude that, given $n \geq 5$, every integer of the same parity as $T_n$ in the interval $[-T_n, T_n]$ can be expressed in the form (7) for an appropriate choice of signs. Now, if we add to these numbers the sum
\[ \sum_{k=1}^{n} p_k, \]
which also has the parity of $T_n$, we obtain all the even integers in
\[ [2(2+5), 2(3+7+11+\ldots+p_n)], \]
each represented as a sum of primes $\leq p_n$, and with each prime in any representation appearing exactly twice. Hence each integer in $[7, 3+\sum_{k=4}^{n} p_k]$ can be partitioned into distinct primes $\leq p_n$. While the argument given is valid only for $n \geq 5$, the final result holds for $n = 4$ also.

Proof of Theorem 1; The lower estimate (2). We shall require the following lemma for both of (1) and (2).

Lemma. For $n \geq 4$,
\[ \frac{n^2}{2} \log \log n \geq \int_4^n x \log \log x \, dx - \frac{n^2}{2} \log \log n. \]

Proof. The right-hand inequality is clear for $n = 4$, and by comparing derivatives one can see that $\frac{1}{2} n^2 \log \log n$ increases faster than the integral does for $n \geq 4$.

The left-hand inequality is obvious for $n = 4$ and 5, and will follow for all larger $n$ once we have shown that
\[ \frac{d}{dn} \frac{n^2}{2} \log \log n \geq \frac{d}{dn} \int_4^n x \log \log x \, dx \]
for $n \geq 5$. This is equivalent to
\[ \frac{1}{2 \log n} < \log \frac{1}{\log n} = \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\log 2}{\log n} \right)^k, \]
which for $5 \leq n \leq 12$ can be proved by direct computation and for $n \geq 13$ is a consequence of the stronger inequality
\[ \frac{\log n}{\log (n/2)} < \log 4. \]
This holds firstly for $n = 13$, and since $\frac{\log n}{\log (n/2)}$ is a decreasing function, it holds for all larger $n$.

To proceed with (2), we have from Rosser and Schoenfeld ([2], Theorem 3)
\[ p_k > k \left( \log (k \log k) - \frac{3}{2} \right) \quad \text{for } k \geq 2. \]
Hence
\[ \sum_{k=4}^{n} p_k > \sum_{k=4}^{n} f(k) \geq \frac{1}{2} (n^2 + n - 12) \]
where $f(x) = x \log (x \log x)$. By the Euler–Maclaurin summation formula,
\[ \sum_{k=4}^{n} f(k) = \int_4^n f(x) \, dx + \frac{1}{2} \left( f(4) + f(n) \right) + \int_4^{n} (x - \lfloor x \rfloor - \frac{1}{2}) f'(x) \, dx \]
\[ = \frac{n^3}{2} \left( \log n - \frac{1}{2} \right) + \frac{n}{2} \log (n \log n) + \int_4^n x \log \log x \, dx \]
\[ - 6 \log 4 + 2 \log 4 + 4 + R; \]
here,

$$R = \int^{n}_{1} (x^2 - [x] - \frac{1}{2}) f'(x) dx \geq 0$$

since \(f'(x) = \log x + \log \log x + 1 + \frac{1}{2} \log x\) is increasing. From (9), (10), (11), and the lemma,

$$\sum_{k=1}^{n} p_k > \frac{n^3}{2} \log n + \frac{n^2}{2} \log \log n - n^2 + \frac{n^2}{2} \log n + \frac{n}{2} \log \log n - \frac{1}{2} n + 5.$$ 

For \(n \geq 5\), the sum of the last four terms here is positive, and

$$g(n) = 3 + \sum_{k=1}^{n} p_k > \frac{n^3}{2} \log n + \frac{n^2}{2} \log \log n - n^2.$$ 

One can obtain (2) for the case \(a = 0\) by observing that

$$\frac{n^3}{2} \log \log n > n^2 \text{ for } n > 2e^2 = 3236.3\ldots,$$

whence by (12)

$$g(n) > \frac{1}{5} n^2 \log n$$

for \(n \geq 3237\); by direct computation we have found that (14) holds for

$$5 \leq n < 3237,$$

and so \(N(0) = 5\).

To prove (2) for \(0 < a < 1\) we make somewhat better use of (12). Note that the inequality

$$\frac{n^3}{2} \log \log n - n^2 > \frac{a n^3}{2} \log \log n$$

is equivalent to

$$a < \frac{\log \log (n/2) - 2}{\log n} = 1 - \frac{2 - \log [1 - \log 2/\log n]}{\log \log n}.$$ 

Therefore, given \(a < 1\), it follows from (12) that (2) holds for all sufficiently large \(n\), and so \(N(a)\) exists. The statement involving \(\varepsilon\) in Theorem 1 also follows.

Remark 1. Some values of \(N(a)\). Let

$$A(n) = \frac{g(n) - \frac{1}{2} n^2 \log n}{\frac{1}{2} n^2 \log \log n},$$

and let \(B(n)\) denote the right side of (15). The proof of (2) depended on the implication

$$a < B(n) \Rightarrow a < A(n),$$

the latter inequality being equivalent to (2). We hence apply our computations of \(g(n)\) and \(A(n)\) to finding \(N(a)\) for \(a = 0\) and for \(a = 1\).

Regarding \(N(1)\), we first find that \(B(22000) > 1\). Since \(B(n)\) increases with \(n\), \(a = 1\) satisfies (16) for all \(n \geq 22000\), and since by our computations

$$A(n) > A(30) = 0.106\ldots \text{ for } 5 \leq n \leq 22000,$$

it will satisfy (17) for \(n \geq 5\). Thus we can take

$$N(1) = 5.$$

The largest value we can find is

$$N(1.56) = 140:\$$

$$B(100000) = 0.1567\ldots, \text{ and } A(n) > .56 \text{ for } 140 \leq n \leq 100000.$$ 

These are optimal values of \(N(a)\), since \(A(4)\) is negative and \(A(139) < .356\).

Speculating further, we find from the computations that \(A(n)\) is increasing for \(20000 \leq n \leq 100000\), with \(A(100000) = .384\ldots\), and so perhaps \(N(.384) < 100000\). To go further yet, we find that \(B(3 \times 10^{24}) = .5008\ldots\), and here about all we can say is that

$$N(.5) < 3 \times 10^{24}.$$ 

However, if the relation between \(A(n)\) and \(B(n)\) should continue as the earlier figures would indicate, \(A(n)\) would be .5 for a much smaller \(n\).

The upper estimates (1) and (3). Here we start from the inequality

$$p_k < k[\log(k \log k) - \frac{1}{2}] \text{ for } k \geq 20$$

(Rosser and Schoenfeld, ibid.). Although this inequality is false for most \(k\) between 4 and 20, it turns out that the upper estimate we find for

$$g(n) = 3 + \sum_{k=1}^{n} k[\log(k \log k) - \frac{1}{2}] = \sum_{k=1}^{n} f(k) - \frac{1}{2} n^2 - \frac{1}{2} n + 6$$

yields the desired one for \(g(n)\).

Regarding an upper estimate for the quantity \(R\) of (10) and (11), we first observe that

$$f(x) \sim f'(x) - \log x = \log \log x + 1 + 1/\log x < \frac{1}{2} \log x$$

for
for \( x \geq 4 \), since \( H'(4) = 2.04\ldots < 2.07\ldots = \frac{3}{4} \log 4 \) and \( \frac{3}{4} \log x \) increases faster than \( H(n) \). Therefore

\[
R < \frac{3}{4} \int_{[x]}^x \{ x - [x] \} \frac{1}{x} \log x \, dx < \frac{3}{4} \int \log x \, dx = \frac{3}{4} (n \log n - n) - 5 (\log 4 - 1).
\]

Hence from (10) and the lemma we find that for \( n \geq 27 \),

\[
x(n) < \frac{1}{4} n^2 (\log n + \log \log n) - \frac{3}{4} n^2 + \frac{1}{4} n \log n + \frac{1}{2} n \log \log n - \frac{3}{4} n - 11 \log 4 + 2 \log 4 + 15
\]

\(< \frac{1}{4} n^2 (\log n + \log \log n)\)

(in fact these inequalities hold for smaller \( n \), but \( n \geq 27 \) suffices for our argument).

One can now see

\[
y(n) < \frac{1}{4} n^2 (\log n + \log \log n)\]

for \( n \geq 4 \)

firstly by direct calculation for \( 4 \leq n \leq 26 \) (see Table 1), and then by showing

\[
y(n) < z(n)\]

for \( n \geq 27 \).

To see the latter, note that

\[
y(27) = 1257 < 1260.8 \ldots = z(27)
\]

and that \( y(n) \) grows more slowly than \( z(n) \) for \( n \geq 27 \) because of (18). This proves (1).

Regarding (3), we see from Table 1 that

\[
\sum_{k=1}^{n} p_k < \frac{1}{4} n^2 (\log n + \log \log n)
\]

for \( 6 \leq n \leq 27 \), and that

\[
\sum_{k=1}^{28} p_k = 1371 < 1375.8 \ldots = z(28),
\]

whence (19) holds for all \( n \geq 6 \). This completes the proof of Theorem 1.

Proof of Theorem 2. We first adopt a convenient notation for some of the lengthy expressions that are involved in the sequel. To illustrate the notation we use the first inequalities that are to be applied in the argument, namely the previously quoted result of Rosser and Schoenfeld,

\[
\frac{3}{4} \{ \log (n \log n) - \frac{3}{4} \} < p_n < \frac{3}{4} \{ \log (n \log n) - \frac{3}{4} \}.
\]

With such an expression we shall use (20LI) to denote the left inequality and (20RI) to denote the right, while (20L), (20C), and (20R) shall denote the three expressions on the left, centre, and right respectively. Thus (20C) = \( p_n \), (20LI) holds for \( n \geq 2 \), and (20RI) holds for \( n \geq 20 \).

From Theorem 1 we have

\[
\frac{1}{4} n^2 (\log n + \log \log n) < y(n) < \frac{1}{4} n^2 (\log n + \log \log n)
\]

where (21LI) holds for \( n \geq N(a) \) and (21RI) for \( n \geq 4 \). We have to prove that, given \( \varepsilon > 0 \),

\[
U(y(n), \frac{1}{2} - \varepsilon) < p_n < U(y(n), \frac{1}{2} + \varepsilon)
\]

for all sufficiently large \( n \).
Since $U(y, c)$ is an increasing function, (21) is equivalent to

\begin{align*}
&n \sqrt{\log n + \alpha \log \log n} \\
&\quad \times \sqrt{\log n + \frac{1}{2} \log (\log n + \alpha \log \log n) + \log \log n + \frac{1}{2} \log (\log n + \alpha \log \log n)} \\
&< U(y(n), c)
\end{align*}

\begin{align*}
&\quad < n \sqrt{\log n} \sqrt{\log n + \frac{1}{2} \log \log n} \\
&\quad \times \sqrt{\log n + \frac{1}{2} \log (\log n + \alpha \log \log n) + \log (\log n + \frac{1}{2} \log \log n)}
\end{align*}

in the sense that (21LI) $\iff$ (23LI) and (21RI) $\iff$ (23RI). Using the notation (35L, c) for the left expression in (23), and (35R, c) for the right, we want to prove

\begin{align*}
(24) \quad U(y(n), c_1) &< (23R, c_1) < (20L) < p_n < (20R) < (23L, c_2) \\
&< U(y(n), c_1)
\end{align*}

with $c_1 = \frac{1}{2} - \varepsilon$ and $c_2 = \frac{1}{3} + \varepsilon$. The two central inequalities are just (20), while the outermost inequalities come from (23), whence from (21). Therefore we have to prove only the two inequalities involving the expressions from (20) and (23).

Regarding the right side of (24), we write $c_2 = c$ and make the following observations (which will serve again in the proof of Theorem 4):

\begin{align*}
(20R) &< (23L, c) \\
&\iff \log n + \frac{1}{2} \log \log n - \frac{1}{2} < \sqrt{\log n + \alpha \log \log n} \\
&\quad \times \sqrt{\log n + \frac{1}{2} \log (\log n + \alpha \log \log n) + \log \log n + \frac{1}{2} \log (\log n + \alpha \log \log n)} \\
&\quad < (c + \frac{1}{2} + a) \log n \log \log n + \frac{1}{2} \log \log n \log \left[1 + \frac{\log \log n}{\log n}\right] \\
&\quad + \frac{a}{2} \log \log n \log \left[1 + \frac{\log \log n}{\log n}\right] + a(c + \frac{1}{2}) (\log \log n)^2 + t(n)
\end{align*}

where $t(n)$ is the sum of the underlined expressions. It is clear that $t(n) = O(\log \log n)$ and $t(n) > 0$ for $n \geq 3$. In order for (25) to hold for all large $n$, it is necessary that $c \geq \frac{1}{2} - a$. In fact, with $c = \frac{1}{2} - a$, (25) holds for all $n \geq 3$; this is the optimal value for $c$, given $a$, since for sharpness in (24) we want $c = c_2$ to be as small as possible.

Thus follows the right side of (24), with $c_2 = \frac{1}{2} - a$, so that

\begin{align*}
(26) \quad p_n < U(y(n), \frac{1}{2} - a) \quad \text{for } n \geq \max \{20, N(a)\}
\end{align*}

That is, for all $n$ such that (20R) and (21LI) both hold. By Theorem 1, given $0 < \varepsilon \leq 1$, we can take $a = 1 - \varepsilon$ here provided $n \geq n_1(\varepsilon) = N(1 - \varepsilon)$ (and $n \geq 20$), and so we have the right-hand inequality in (22) and in Theorem 2.

Regarding the left side of (24), we find in a similar way that

\begin{align*}
(27) \quad (c_1 - \frac{1}{2}) \log \log n \log n + 3 \log n + (c_1 + \frac{1}{2}) (\log \log n)^2 + s(n) < (\log \log n - \frac{1}{2})^2
\end{align*}

where $0 < s(n) = O(\log \log n)$. In order for (27) to hold for all large $n$, it is necessary that $c_1 < \frac{1}{2}$, and indeed, given $\varepsilon > 0$, it is clear that there is a number $n_1(\varepsilon)$ such that (27) will hold with $c_1 = \frac{1}{2} - \varepsilon$ for all $n \geq n_1(\varepsilon)$. Thus we have the left inequality in (22) and in Theorem 2 for $n \geq n_1(\varepsilon)$. (The first and third inequalities on the left side of (24) hold for $n \geq 4$ and $n \geq 2$ respectively.)

Remark 2. (Take $c_1 = \frac{1}{2} - \varepsilon$ in (27).)

(a) The presence of the term $3 \log n$ in (27) requires that $n_1(\varepsilon)$ be very large. For example with $\varepsilon = \frac{1}{2}$, (27) requires

\begin{align*}
(3 - \frac{1}{2} \log \log n)^2 + (\log \log n)^2 + s(n) < (\log \log n - \frac{1}{2})^2,
\end{align*}

and this will be false for $n$ smaller than about $\exp 300$.

(b) The inequality

\begin{align*}
(28) \quad p_n < U(y(n), \frac{1}{2})
\end{align*}

for $n \geq 6$ follows from the above argument: since $N(0) = 5$, (26) yields it for $n \geq 20$, while for $6 \leq n < 20$ it is proved by direct calculation.

The inequality

\begin{align*}
(29) \quad U(y(n), 0) < p_n
\end{align*}

follows from (27), and so holds for $n \geq n_1(\frac{1}{2})$; all that (27) allows us to say about $n_1(\frac{1}{2})$ is that it likely exceeds $\exp 300^{10^{10}}$ — cf. the discussion regarding $\varepsilon = 1$ in (a). Our computations show that (29) holds for $31 \leq n \leq 100 000$; values of the two sides and their difference, for selected $n$, are shown in Table 2.
Proof of Theorem 4. This is similar to the proof of the right-hand inequality in Theorem 2. We shall use the same special notation involving statement numbers as before, and adapt it further to our present purpose by writing for example [23L, c] (n−1) to denote [23L, c] with n−1 in place of n. Instead of the right-side of (24), we shall prove

\[ p_n < (n-1)\left[\log(n-1) + \log\log(n-1) - \frac{1}{4}\right] < [23L, c](n-1) \]

\[ < U(y(n-1), c). \]

The first inequality of (30) holds for n ⩾ 20: For n = 20 it can be seen to hold by direct computation, and for larger n it follows from (20R1) and the inequality

\[ \log n + \log\log n - \frac{1}{2} < \left(1 - \frac{1}{n}\right) \left[\log(n-1) + \log\log(n-1) - \frac{1}{4}\right], \]

which holds for n ⩾ 21 (the left side increases more slowly than the right, and for n = 21, (31L) = 3.657... < (31R) = 3.659...)

<table>
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<tr>
<th>n</th>
<th>U(y(n), 0)*</th>
<th>p_n - U(y(n), 0)*</th>
<th>p_n</th>
<th>U(y(n), 1/4) - p_n</th>
<th>U(y(n), 1)*</th>
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* Truncated at one decimal place

The last inequality in (30) is [23LI] (n−1), which follows from [23LI] (n−1) and so holds for n − 1 ⩾ N(a). Hence, once the central inequality in (30) is proved, with c = 1/4 − α and for n ≥ 20, we shall have

(32)

\[ p_n < U(y(n-1), 1/4 - \alpha) \quad \text{for} \quad n \geq \max\{20, N(a) + 1\} \]

(cf. (26)). Given this result, then for y an integer satisfying y(n−1) < y, where n ⩾ max{20, N(a) + 1}, we shall have

\[ p_n < U(y, 1/4 - \alpha); \]

if further y ≤ y(n), then by Theorem 3 y can be partitioned into distinct primes ≤ p_n. Putting α = 1 − ε, we can then conclude that y can be partitioned into distinct primes p with

(33)

\[ p < U(y, 1 + \varepsilon), \]

and Theorem 4 will have been proved.

In proving the central inequality in (30), we use the same argument as that used in showing

(20R)< (23L, c) ≪ (25)

in the proof of Theorem 2, except with (n−1) in place of n, and with 1 in place of the 1/4 in (20R). We find that the central inequality is equivalent to

(34)

\[ \left(\log(n-1) - \frac{1}{4}\right)^2 \]

\[ < (c-\frac{1}{4} + \alpha)\log(n-1)\log\log(n-1) + \frac{1}{4}\log(n-1) + \alpha(c+\frac{1}{4})\log\log(n-1)^2 + f(n-1), \]

where t is the function in (25): t(n−1) = O(log log n) and t(n−1) > 0 for n ≥ 4. With ε = 1/4 − α (cf. the discussion following (25)), (34) holds for n ≥ 20 (indeed for n ≥ 4); the proof of this last is as follows.

Since \( \frac{1}{4}\log(n-1) < (34R), \) and the functions tabulated below are increasing, the values listed in the table show that (34) holds for 20 ≤ n ≤ 645.

\[
\begin{array}{c|c|c}
 n & g(n) = (\log(n-1) - \frac{1}{4})^2 & h(n) = \frac{1}{2}\log(n-1) \\
\hline
 20 & 0.68... & 1.47... \\
 30 & 1.43... & 2.12... \\
 40 & 2.00... & 2.64... \\
 645 & 2.61... & \\
\end{array}
\]

Beyond n = 645, g(n) increases more slowly than does h(n), since

\[ g'(645) = 0.0077634... < 0.0077639... = h'(645) \]

and g'(n) increases more slowly than does h'(n) for n ≥ 645 (in fact for n ≥ 55).

Theorem 4 now follows.

Remark 3. Replacing α by its supremum 1 in (32) would give

\[ p_n < U(y(n-1), 1/4), \]

which would imply the inequality in Theorem 4 without the ε. Our computations reveal that for 105 ≤ n ≤ 200000,

\[ p_n < U(y(n-1), 1/4). \]
with differences $d > 0$ that tend to increase with $n$. Some sample values follow; the values of $U$ in the table are truncated at one decimal place.

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<th>$n$</th>
<th>$p_n$</th>
<th>$d$</th>
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References


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