COROLLARY 1. Let $\varepsilon > 0$ be given arbitrarily. Then for every $\alpha, 0 < \alpha < 1$, there is an infinity of natural numbers $j = j(\alpha, \varepsilon)$ to fulfill

$$\min |\alpha^j p - q| \leqslant \frac{\delta^{-1} + \varepsilon}{\log p}.$$

In particular, the inequality $|\alpha^j p - q| \leqslant (1 + \varepsilon)\log p$ has infinitely many solutions in primes $p$ and $q$.

References


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Generalizations of Ramanujan’s formulae

by

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Ramanujan found the following formulae: For positive $a, \beta$ with $a\beta = \pi^2$ and an integer $v > 1$,

\begin{align*}
(1) \quad & a^v \left\{ \frac{\zeta(1-2v)}{2} + \sum_{n=1}^{\infty} a_{2v-1}(n) e^{-2\pi n} \right\} \\
& = (-\beta)^v \left\{ \frac{\zeta(1-2v)}{2} + \sum_{n=1}^{\infty} a_{2v-1}(n) e^{-2\pi n} \right\}.
\end{align*}

\begin{align*}
(2) \quad & a^{v-1} \left\{ \frac{\zeta(2v-1)}{2} + \sum_{n=1}^{\infty} a_{2v-1}(n) e^{-2\pi n} \right\} \\
& = (-\beta)^{-v} \left\{ \frac{\zeta(2v-1)}{2} + \sum_{n=1}^{\infty} a_{2v-1}(n) e^{-2\pi n} \right\} \\
& = -2^{2v}\sum_{k=0}^{v} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2v-2k}}{(2v-2k)!} \alpha^{v-k} \beta^k,
\end{align*}

where $\zeta(s)$ is the Riemann zeta function, $a_n = \sum_{d|n} a^d$, and $B_n$ are Bernoulli numbers defined by $\sum_{n=0}^{\infty} B_n x^n/n! = x/(e^x - 1)$. G. H. Hardy [3] gave two proofs of (1). E. Grosswald [2] proved a more general formula which contains both (1) and (2). Many variants of Ramanujan’s formulae are known. The historical survey of the formula and its generalization are explained in [1].

Recently the author [4] presented as an analogue of (1) a formula for the values of $\zeta(s)$ at half integers. In this paper we shall extend further the Ramanujan’s formulae (1) and (2) to rational numbers. Our method of the proof is similar to that used in [2].
Theorem 1. Let \( a \) be a positive integer and \( n \) be an integer greater than 1, and define for \( x > 0 \)
\[
G_{a,n}(x) = \frac{1}{a} \sum_{k=2}^{\infty} \sigma_{a,n-1}(a) e^{-\frac{a^2}{n} \frac{k^2}{x^2}} + \sum_{r=1}^{\infty} \left( \frac{a}{2\pi a} \right)^{2a+1-r} \Gamma(2av-a+1+r) \times \nonumber
\times \prod_{k=-a+1+r}^{r} \zeta \left( 1 + \frac{k}{a} \right) \left( 2v + \frac{1}{a} \right) \pi^{-\frac{1}{2}x^2}.
\]
where
\[
\sigma_{a,n}(k) = \sum_{n=1}^{\infty} \frac{1}{\zeta \lambda a \cdots n \lambda a}.
\]

Then for any positive \( a, \beta \) with \( \beta = \pi^2 \) we have
\[
(3) \quad G_{a,n}(a) = (-1)^n G_{a,n}(\beta).
\]

Remark 1. Ramanujan's formula (1) and the theorem in [4] follows from (3) with \( a = 1 \) and \( a = 2 \), respectively.

Remark 2. The function \( \sigma_{a,n}(n) \) coincides with the ordinary divisor function \( \sigma_{a}(n) \) when \( a = 1 \); i.e.
\[
\sigma_{a,n}(n) = \sigma_{n}(n) = \sum_{d|n} d^a.
\]

Equation (3) implies especially
\[
G_{a,n}(2\pi x) = (-1)^n G_{a,n}(2\pi \beta x) \quad (t = 1, 2, \ldots, a),
\]
which leads to the following

Corollary 1. Let \( a \) be a positive integer, \( n \) be an integer greater than 1 and \( r \) be an integer with \( 0 \leq r \leq a - 1 \). Then
\[
\prod_{k=-a+1+r}^{r} \zeta \left( 1 + \frac{k}{a} \right) \prod_{l=-a+1+r}^{r} \zeta \left( 2v + \frac{1}{a} \right)
\]
\[
= \pi^{-\frac{a^2}{a} - a + 1 + r} \sum_{a=1}^{\infty} \sigma_{a,n-1}(a) e^{-\frac{a^2}{n} \frac{k^2}{x^2}} + \sum_{r=1}^{\infty} \left( \frac{a}{2\pi a} \right)^{2a+1-r} \Gamma(2av-a+1+r) \times \nonumber
\times \prod_{k=-a+1+r}^{r} \zeta \left( 1 + \frac{k}{a} \right) \left( 2v + \frac{1}{a} \right) \pi^{-\frac{1}{2}x^2}.
\]
where \( b_{a,n,r} \) and \( c_{a,n,r} \) are rational numbers.

Theorem 1 is equivalent to the following

Theorem 1'. Let \( a \) be a positive integer, \( n \) be an integer greater than 1, and define for \( \text{Im} \, z > 0 \)
\[
E_{a,z}(x) = \frac{2}{\zeta(1-2v)} \left( \sum_{n=1}^{\infty} \sigma_{a,n-1}(a) e^{\frac{a}{n} \frac{k^2}{x^2}} + \right.
onumber
\times \prod_{k=-a+1+r}^{r} \zeta \left( 1 + \frac{k}{a} \right) \left( 2v + \frac{1}{a} \right) \pi^{-\frac{1}{2}x^2}.
\]
\[
\left. + \left( -1 \right)^n \sum_{r=0}^{\infty} a(2\pi a)^{-2ar+1-r} \Gamma(2av-a+1+r) \times \nonumber
\times \prod_{k=-a+1+r}^{r} \zeta \left( 1 + \frac{k}{a} \right) \left( 2v + \frac{1}{a} \right) i^{-\frac{1}{2}x^2} \right).
\]

Then the function satisfies the transformation equation
\[
E_{a,z}(-1/z) = \frac{a^{-\frac{1}{2}z}}{\zeta(1/2)} E_{a,z}(x).
\]

Remark 3. \( E_{1,v}(x) \) is the normalized Eisenstein series of weight \( 2v \).
\[
E_{1,v}(x) = E_{2,v}(x) = 1 - \frac{4\pi}{B_{2v}} \sum_{n=1}^{\infty} \sigma_{2v-1}(n) e^{\frac{2\pi inz}{a}}
\]
which can be found in [5].

Proof of Theorem 1. If we write
\[
(4) \quad \Phi_{a,v}(x) = a(2\pi a)^{-\frac{a}{2}} \Gamma(2v) \prod_{k=0}^{a-1} \zeta \left( s + \frac{k}{a} \right) \zeta \left( s + \frac{a-1-k}{a} + 1 - 2v \right),
\]
we have the following functional equation
\[
(5) \quad \Phi_{a,v}(2v-1/\alpha - s) = (-1)^{\frac{a}{2}} \Phi_{a,v}(s).
\]
To show this we put
\[
\varphi_{a}(s) = (2\pi)^{-\frac{a}{2}} \Gamma \left( s + \frac{k}{a} \right) \zeta \left( s + \frac{a-1-k}{a} + 1 - 2v \right)
\]
\[
(\alpha = 0, 1, \ldots, a-1).
\]
From the functional equation of the zeta function, we have
\[
\varphi_{a}(2v-1/\alpha - s) = (-1)^{\frac{a}{2}} \frac{\cos \frac{\pi}{2} \left( s + \frac{k}{a} \right)}{\cos \frac{\pi}{2} \left( s + \frac{a-1-k}{a} \right)} \varphi_{a}(s),
\]
and thus
\[
\prod_{k=0}^{a-1} \varphi_k(2v-1+1/a-s) = (-1)^{av} \prod_{k=0}^{a-1} \varphi_k(s).
\]

Using Gauss' multiplication formula for the gamma function, we get
\[
\prod_{k=0}^{a-1} \varphi_k(s) = (2\pi)^{(a-1)/2} a^{-1/2} \Phi_{a,v}(s),
\]
which yields (5).

We next consider the function
\[
g_{a,v}(t) = \sum_{n=1}^{\infty} \sigma_{a,2v-1}(n) e^{-\frac{\pi}{4} an^2 t} \quad (t > 0).
\]

The series converges absolutely in \( t > 0 \) and uniformly in any interval \( \delta < t < \infty \) with \( \delta > 0 \), since
\[
\left(6\right) \quad \sigma_{a,2v-1}^2(n) \leq \sum_{n=1}^{\infty} \prod_{k=0}^{a-1} m_{k}^{-2v-1} \sum_{\substack{n = n_1, \ldots, n_a = n \atop \sum_{k=0}^{a-1} n_k = n}} 1
\]
\[
\leq \sum_{n=1}^{\infty} \sigma_{2v-1}^{a-1} \quad \text{for} \quad s > a(2v-1)+1,
\]
so that
\[
\sum_{n=1}^{\infty} \left| \sigma_{a,2v-1}(n) e^{-\frac{\pi}{4} an^2 t} \right| \leq \sum_{n=1}^{\infty} \sigma_{a,2v-1}(n) e^{-\frac{\pi}{4} an^2 t} < \infty.
\]

Thus we have
\[
\int_{0}^{\infty} g_{a,v}(t) e^{-st} dt = \int_{0}^{\infty} \sum_{n=1}^{\infty} \sigma_{a,2v-1}(n) e^{-\frac{\pi}{4} an^2 t} e^{-st} dt
\]
\[
= \sum_{n=1}^{\infty} \sigma_{a,2v-1}(n) \int_{0}^{\infty} e^{-\frac{\pi}{4} an^2 t} e^{-st} dt.
\]

The inversion of the order of integration and summation can be justified by the uniform convergence. Substituting \( u = 2\pi an^2 t \) in the last integral, we get
\[
\int_{0}^{\infty} g_{a,v}(t) e^{-st} dt = \sum_{n=1}^{\infty} \sigma_{a,2v-1}(n) \int_{0}^{\infty} e^{-u} \frac{u^{s-1}}{(2\pi)^{a/2} n} \frac{a an^{-1}}{(2\pi)^{a/2} n} du
\]
\[
= a(2\pi)^{-av} \Phi_{a,v}(s) \sum_{n=1}^{\infty} \sigma_{a,2v-1}(n).
\]

Taking account of the inequality (6), the last series is absolutely convergent in the half-plane \( \text{Re} s > a(2v+1) \). Thus
\[
\sum_{n=1}^{\infty} \frac{\sigma_{a,2v-1}}{n^s} = \sum_{n=1}^{\infty} \sum_{\substack{n_1, \ldots, n_a = n \atop \sum_{k=0}^{a-1} n_k = n}} \prod_{k=0}^{a-1} \frac{1}{n_k} \frac{e^{-\frac{\pi}{4} an^2 t}}{an^2 t}
\]
\[
= \prod_{k=0}^{a-1} \frac{\zeta(s+k)}{a^k} \frac{\zeta(s+a-1)}{a} + 1 - 2v
\]
for \( \text{Re} s > a(2v+1) \) and so for all \( s \) (by the identity theorem). By (4), (7) and (8), we obtain
\[
\Phi_{a,v}(s) = \int_{0}^{\infty} g_{a,v}(t) t^{-s} dt.
\]

The definition (4) shows immediately that \( \Phi_{a,v}(s) \) is regular in \( s > 2v \).

We note further that
\[
\Phi_{a,v}(s+it) = O(e^{-\frac{\pi}{2}t^2/\sigma}) \quad (b \leq s \leq c, \quad |t| \geq 1),
\]
where \( b \) and \( c \) are any fixed real number, and \( d > 0 \) is a constant independent of \( t \), which can easily be verified. Thus we can apply Mellin's inversion formula and obtain
\[
\int_{0}^{\infty} g_{a,v}(t) e^{-st} dt = \frac{1}{2\pi i} \int_{2v+i\infty}^{2v+1/2+\infty} \Phi_{a,v}(s) e^{st} ds.
\]

By means of (9) we can shift the line of integration to any position \( (\sigma_0 - i\infty, \sigma_0 + i\infty) \). Taking \( \sigma_0 = -1+1/2a \), we obtain
\[
\int_{0}^{\infty} g_{a,v}(t) e^{-st} dt = \frac{1}{2\pi i} \int_{-1+1/2a+i\infty}^{1/2+1/2a+i\infty} \Phi_{a,v}(s) e^{st} ds.
\]

If we substitute \( s = 2v-1+1/a-S \) and use the functional equation (5), we get
\[
\int_{0}^{\infty} g_{a,v}(t) e^{-st} dt = \frac{1}{2\pi i} \int_{-1+1/2a+i\infty}^{1/2+1/2a+i\infty} \Phi_{a,v}(s) e^{st} ds
\]
\[
= (-1)^{av} (-2v+1-a-S) \frac{1}{2\pi i} \int_{-1+1/2a+i\infty}^{1/2+1/2a+i\infty} \Phi_{a,v}(s) e^{st} ds = (-1)^{av} t^{2v+1-a} \Phi_{a,v}(\frac{1}{t}).
\]
The residues in (11) are as follows:

\[
\text{Res}_{s=-r/a} \left( \Phi_{a,r}(s) t^{-s} \right) = a(2\pi a)^{-2ar+r} \Gamma(2av-a) \prod_{k=0}^{a-1-r} \xi \left( 1 + \frac{k}{a} \right) \times \\
\times \prod_{m=0}^{a-1-r} \xi \left( 2r + \frac{1}{a} \right) t^{-2r+a} (0 \leq r \leq a-1).
\]

To calculate the residue at \( s = -r/a \) we need the functional equation of the zeta function, Gauss' multiplication formula for the gamma function, and the equation

\[
\prod_{k=1}^{a-1} \sin \frac{k\pi}{a} = \phi^{1-a}.
\]

Thus

\[
\text{Res}_{s=-r/a} \left( \Phi_{a,r}(s) t^{-s} \right) = (2\pi a)^{-r} \prod_{k=0}^{a-1-r} \xi \left( 1 + \frac{k}{a} \right) \times \\
\prod_{m=0}^{a-1-r} \xi \left( 2r + \frac{1}{a} \right) t^{-2r+a} (0 \leq r \leq a-1).
\]

These calculations as well as (11) and (12) imply

\[
g_{a,r}(t) = \sum_{r=0}^{a-1} a(2\pi a)^{-2ar+r} \Gamma(2av-a) \prod_{k=0}^{a-1-r} \xi \left( 1 + \frac{k}{a} \right) \times \\
\prod_{m=0}^{a-1-r} \xi \left( 2r + \frac{1}{a} \right) t^{-2r+a} (0 \leq r \leq a-1).
\]

Replacing \( r \) by \( a-1-r \) in the first sum, we get

\[
g_{a,r}(t) = \sum_{r=0}^{a-1} a(2\pi a)^{-2ar+a-1-r} \Gamma(2av-a+1+r) \times \\
\times \prod_{k=0}^{a-1-r} \xi \left( 1 + \frac{k}{a} \right) \prod_{m=0}^{a-1-r} \xi \left( 2r + \frac{1}{a} \right) t^{2r+a} (0 \leq r \leq a-1).
\]

Setting \( t = (a/\pi)^a, \ 1/t = (\beta/\pi)^a, \) we obtain the equation (3).

**Theorem 2.** Let \( a \) be a positive integer, \( r \) be an integer greater than 1, and define for \( x > 0 \)

\[
F_{a,r}(x) = x^{-ar+(a-1)b} \left\{ \sum_{n=1}^{\infty} \sigma_{n-1-2r}(n) e^{-2\pi anx} - \\
- \sum_{r=0}^{a-1} (2\pi a)^{-r} \prod_{k=0}^{a-1-r} \xi \left( \frac{k}{a} \right) \prod_{m=0}^{a-1-r} \xi \left( 2r + 1 + \frac{1}{a} \right) x^r \right\}.
\]

Then for any positive \( a, \beta \) with \( a\beta = \pi^2 \) we have

\[
F_{a,r}(a) = (-1)^{a-1} F_{a,r}(\beta)
\]

\[
= \sum_{r=0}^{a-1} a(2\pi a)^{-a+r} (a-1-r)! \prod_{k=0}^{a-1-r} \xi \left( 1 + \frac{k}{a} \right) \prod_{m=0}^{a-1-r} \xi \left( 2r + 1 + \frac{1}{a} \right) x^{a-1-2r} + \\
+ \sum_{b=1}^{a} \sum_{r=0}^{a-1} (2\pi a)^{ab-a+r} (2ab-a-r)! \prod_{k=0}^{a-1-r} \xi \left( 1-2b + \frac{k}{a} \right) \times \\
\prod_{m=0}^{a-1-r} \xi \left( 2r - 2b + 1 + \frac{1}{a} \right) x^{a-1-2r}.
\]
Remark 4. Ramanujan’s formula (2) follows from (14) with \(a = 1\).

The proof of Theorem 2 shall be done in the same way as that of Theorem 1, using the functional equation

\[\psi_{a,s}(1 - 2s + \frac{1}{a} - s) = (-1)^{a(1 - s)}\psi_{a,s}(s),\]

with

\[\psi_{a,s}(s) = a(2\pi a)^{-s-1}\Gamma(s) \prod_{k=0}^{a-1} \frac{s + \frac{k}{a}}{\zeta(s + \frac{a-1-k}{a}) - 1 + 2s}\]

References


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