COROLLARY 1*. *Let $\varepsilon > 0$ be given arbitrarily. Then for every a, $0 < a < 1$, there is an infinity of natural numbers $j = j(a, \varepsilon)$ to fulfill*

$$\varliminf_{p\in\mathfrak{p}} \frac{\min\limits_{q\in\mathfrak{p},q\neq p} |a^j p - q|}{\log p} \leqslant (\delta^{-1}+\varepsilon).$$

In particular, the inequality $|a^j p - q| \leqslant (1+\varepsilon)\log p$ has infinitely many solutions in primes p and q.

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SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Homi Bhabha Roal
Bombay 400 005, India

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Generalizations of Ramanujan's formulae

by

YASUSHI MATSUOKA (Nishinagano, Japan)

Ramanujan found the following formulae: For positive α, β with $\alpha\beta = \pi^2$ and an integer $\nu > 1$,

$$(1)\qquad \alpha^{\nu}\left\{\frac{\zeta(1-2\nu)}{2} + \sum_{n=1}^{\infty} \sigma_{2\nu-1}(n) e^{-2n\alpha}\right\} = (-\beta)^{\nu}\left\{\frac{\zeta(1-2\nu)}{2} + \sum_{n=1}^{\infty} \sigma_{2\nu-1}(n) e^{-2n\beta}\right\}.$$

$$(2)\qquad \alpha^{-(\nu-1)}\left\{\frac{\zeta(2\nu-1)}{2} + \sum_{n=1}^{\infty} \sigma_{1-2\nu}(n) e^{-2n\alpha}\right\} - (-\beta)^{-(\nu-1)}\left\{\frac{\zeta(2\nu-1)}{2} + \sum_{n=1}^{\infty} \sigma_{1-2\nu}(n) e^{-2n\beta}\right\} = -2^{2(\nu-1)} \sum_{k=0}^{\nu} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2\nu-2k}}{(2\nu-2k)!} \alpha^{\nu-k}\beta^k,$$

where $\zeta(s)$ is the Riemann zeta function, $\sigma_b(n) = \sum_{d|n} d^b$, and B_n are Bernoulli numbers defined by $\sum_{n=0}^{\infty} B_n x^n/n! = x/(e^x-1)$. G. H. Hardy [3] gave two proofs of (1). E. Grosswald [2] proved a more general formula which contains both (1) and (2). Many variants of Ramanujan's formulae are known. The historical survey of the formula and its generalization are explained in [1].

Recently the author [4] presented as an analogue of (1) a formula for the values of $\zeta(s)$ at half integers. In this paper we shall extend further the Ramanujan's formulae (1) and (2) to rational numbers. Our method of the proof is similar to that used in [2].

THEOREM 1. *Let a be a positive integer and ν be an integer greater than* 1, *and define for $x > 0$*

$$G_{a,\nu}(x) = x^{a\nu-\frac{a-1}{2}}\Big\{\sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)\,e^{-2a\sqrt[a]{nx}}+$$

$$+(-1)^{a\nu}\sum_{r=0}^{a-1}a(2\pi a)^{-2a\nu+a-1-r}\,\Gamma(2a\nu-a+1+r)\times$$

$$\times\prod_{\substack{k=-a+1+r\\ k\neq 0}}^{r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-a+1+r}^{r}\zeta\Big(2\nu+\frac{l}{a}\Big)\,\pi^{-r}x^{r}\Big\},$$

where

$$\sigma_{a,b}(n) = \sum_{\prod_{k=0}^{a-1} n_k m_k = n,\, n_k\in N,\, m_k\in N}\ \prod_{k=0}^{a-1} n_k^{-k/a}\, m_k^{b-(a-1-k)/a}.$$

Then for any positive α, β with $\alpha\beta = \pi^2$ we have

$$G_{a,\nu}(\alpha) = (-1)^{a\nu}G_{a,\nu}(\beta). \tag{3}$$

Remark 1. Ramanujan's formula (1) and the theorem in [4] follows from (3) with $a = 1$ and $a = 2$, respectively.

Remark 2. The function $\sigma_{a,b}(n)$ coincides with the ordinary divisor function $\sigma_b(n)$ when $a = 1$; i.e.

$$\sigma_{1,b}(n) = \sigma_b(n) = \sum_{d|n} d^b.$$

Equation (3) implies especially

$$G_{a,\nu}(2^{2t}\pi) = (-1)^{a\nu}G_{a,\nu}(2^{-2t}\pi) \qquad (t = 1, 2, \ldots, a),$$

which leads to the following

COROLLARY 1. *Let a be a positive integer, ν be an integer greater than* 1 *and r be an integer with $0 \leqslant r \leqslant a-1$. Then*

$$\prod_{\substack{k=-a+1+r\\ k\neq 0}}^{r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-a+1+r}^{r}\zeta\Big(2\nu+\frac{l}{a}\Big)$$

$$=\pi^{2a\nu-a+1+r}\sum_{t=1}^{a}\Big\{b_{a,\nu,r,t}\sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)\,e^{-2^{2t+1}a\sqrt[a]{n}\pi}+$$

$$+c_{a,\nu,r,t}\sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)\,e^{-2^{-2t+1}a\sqrt[a]{n}\pi}\Big\}$$

where $b_{a,\nu,r,t}$ and $c_{a,\nu,r,t}$ are rational numbers.

Theorem 1 is equivalent to the following

THEOREM 1′. *Let a be a positive integer, ν be an integer greater than* 1, *and define for* $\operatorname{Im} z > 0$

$$E_{a,\nu}(z) = \frac{2}{\zeta(1-2\nu)}\Big\{\sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)\,e^{2\pi i a\sqrt[a]{nz}}+$$

$$+(-1)^{a\nu}\sum_{r=0}^{a-1}a(2\pi a)^{-2a\nu+a-1-r}\,\Gamma(2a\nu-a+1+r)\times$$

$$\times\prod_{\substack{k=-a+1+r\\ k\neq 0}}^{r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-a+1+r}^{r}\zeta\Big(2\nu+\frac{l}{a}\Big)\,i^{-r}z^{r}\Big\}.$$

Then the function satisfies the transformation equation

$$E_{a,\nu}(-1/z) = z^{2a\nu-a+1}i^{a-1}E_{a,\nu}(z).$$

Remark 3. $E_{1,\nu}(z)$ is the normalized Eisenstein series of weight 2ν.

$$E_{1,\nu}(z) = E_{2\nu}(z) = 1-\frac{4\nu}{B_{2\nu}}\sum_{n=1}^{\infty}\sigma_{2\nu-1}(n)\,e^{2\pi i n z},$$

which can be found in [5].

Proof of Theorem 1. If we write

$$\Phi_{a,\nu}(s) = a(2\pi a)^{-as}\,\Gamma(as)\prod_{k=0}^{a-1}\zeta\Big(s+\frac{k}{a}\Big)\,\zeta\Big(s+\frac{a-1-k}{a}+1-2\nu\Big), \tag{4}$$

we have the following functional equation

$$\Phi_{a,\nu}(2\nu-1+1/a-s) = (-1)^{a\nu}\Phi_{a,\nu}(s). \tag{5}$$

To show this we put

$$\varphi_k(s) = (2\pi)^{-s}\,\Gamma\Big(s+\frac{k}{a}\Big)\,\zeta\Big(s+\frac{k}{a}\Big)\,\zeta\Big(s+\frac{a-1-k}{a}+1-2\nu\Big)$$

$$(k = 0, 1, \ldots, a-1).$$

From the functional equation of the zeta function, we have

$$\varphi_k\Big(2\nu-1+\frac{1}{a}-s\Big) = (-1)^{\nu}\frac{\cos\frac{\pi}{2}\Big(s+\frac{k}{a}\Big)}{\cos\frac{\pi}{2}\Big(s+\frac{a-1-k}{a}\Big)}\,\varphi_k(s),$$

and thus

$$\prod_{k=0}^{a-1}\varphi_k(2\nu-1+1/a-s) = (-1)^{a\nu}\prod_{k=0}^{a-1}\varphi_k(s).$$

Using Gauss' multiplication formula for the gamma function, we get

$$\prod_{k=0}^{a-1}\varphi_k(s) = (2\pi)^{(a-1)/2}a^{-1/2}\Phi_{a,\nu}(s),$$

which yields (5).

We next consider the function

$$g_{a,\nu}(t) = \sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)e^{-2\pi a\sqrt[a]{nt}} \qquad (t > 0).$$

The series converges absolutely in $t > 0$ and uniformly in any interval $\delta \leqslant t < \infty$ with $\delta > 0$, since

$$(6) \qquad \sigma_{a,2\nu-1}(n) \leqslant \sum_{\prod_{k=0}^{a-1} n_k m_k = n}\prod_{k=0}^{a-1} m_k^{2\nu-1} \leqslant n^{a(2\nu-1)}\sum_{\prod_{k=0}^{a-1} n_k m_k = n} 1$$
$$\leqslant n^{a(2\nu-1)+2a-1} \leqslant n^{a(2\nu+1)-1},$$

so that

$$\sum_{n=1}^{\infty}|\sigma_{a,2\nu-1}(n)e^{-2\pi a\sqrt[a]{nt}}| \leqslant \sum_{n=1}^{\infty} n^{a(2\nu+1)-1}e^{-2\pi a\sqrt[a]{n\delta}} < \infty.$$

Thus we have

$$\int_0^{\infty} g_{a,\nu}(t)t^{s-1}dt = \int_0^{\infty}\sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)e^{-2\pi a\sqrt[a]{nt}}t^{s-1}dt$$
$$= \sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)\int_0^{\infty} e^{-2\pi a\sqrt[a]{nt}}t^{s-1}dt.$$

The inversion of the order of integration and summation can be justified by the uniform convergence. Substituting $u = 2\pi a\sqrt[a]{nt}$ in the last integral, we get

$$(7) \qquad \int_0^{\infty} g_{a,\nu}(t)t^{s-1}dt = \sum_{n=1}^{\infty}\sigma_{a,2\nu-1}(n)\int_0^{\infty} e^{-u}\left(\frac{u^a}{(2\pi a)^a n}\right)^{s-1}\frac{au^{a-1}}{(2\pi a)^a n}du$$
$$= a(2\pi a)^{-as}\Gamma(as)\sum_{n=1}^{\infty} n^{-s}\sigma_{a,2\nu-1}(n).$$

Taking account of the inequality (6), the last series is absolutely convergent in the half-plane $\mathrm{Re}\,s > a(2\nu+1)$. Thus

$$(8) \qquad \sum_{n=1}^{\infty} n^{-s}\sigma_{a,2\nu-1}(n)$$
$$= \sum_{n_0=1}^{\infty}\cdots\sum_{n_{a-1}=1}^{\infty}\sum_{m_0=1}^{\infty}\cdots\sum_{m_{a-1}=1}^{\infty}\prod_{k=0}^{a-1} n_k^{-s-\frac{k}{a}} m_k^{-s-\frac{a-1-k}{a}-1+2\nu}$$
$$= \prod_{k=0}^{a-1}\zeta\left(s+\frac{k}{a}\right)\zeta\left(s+\frac{a-1-k}{a}+1-2\nu\right)$$

for $\mathrm{Re}\,s > a(2\nu+1)$ and so for all s (by the identity theorem). By (4), (7) and (8), we obtain

$$\Phi_{a,\nu}(s) = \int_0^{\infty} g_{a,\nu}(t)t^{s-1}dt.$$

The definition (4) shows immediately that $\Phi_{a,\nu}(s)$ is regular in $\sigma > 2\nu$. We note further that

$$(9) \qquad \Phi_{a,\nu}(\sigma+it) = O(e^{-\frac{a\pi}{2}|t|}|t|^{\Delta}) \qquad (b \leqslant \sigma \leqslant c,\ |t| \geqslant 1),$$

where b and c are any fixed real number, and $\Delta > 0$ is a constant independent of t, which can easily be verified. Thus we can apply Mellin's inversion formula and obtain

$$(10) \qquad g_{a,\nu}(t) = \frac{1}{2\pi i}\int_{2\nu+1/2a-i\infty}^{2\nu+1/2a+i\infty}\Phi_{a,\nu}(s)t^{-s}ds.$$

By means of (9) we can shift the line of integration to any position $(\sigma_0 - i\infty, \sigma_0 + i\infty)$. Taking $\sigma_0 = -1+1/2a$, we obtain

$$(11) \qquad g_{a,\nu}(t) = \sum_{r=0}^{a-1}\left\{\operatorname*{Res}_{s=2\nu-r/a}\left(\Phi_{a,\nu}(s)t^{-s}\right) + \operatorname*{Res}_{s=-r/a}\left(\Phi_{a,\nu}(s)t^{-s}\right)\right\} +$$
$$+\frac{1}{2\pi i}\int_{-1+1/2a-i\infty}^{-1+1/2a+i\infty}\Phi_{a,\nu}(s)t^{-s}ds.$$

If we substitute $s = 2\nu-1+1/a-S$ and use the functional equation (5), we get

$$(12) \qquad \frac{1}{2\pi i}\int_{-1+1/2a-i\infty}^{-1+1/2a+i\infty}\Phi_{a,\nu}(s)t^{-s}ds$$
$$= (-1)^{a\nu}t^{-2\nu+1-1/a}\frac{1}{2\pi i}\int_{2\nu+1/2a-i\infty}^{2\nu+1/2a+i\infty}\Phi_{a,\nu}(S)\left(\frac{1}{t}\right)^{-S}dS = (-1)^{a\nu}t^{-2\nu+1-1/a}g_{a,\nu}\left(\frac{1}{t}\right).$$

The residues in (11) are as follows:

$$\operatorname*{Res}_{s=2\nu-r/a}\big(\Phi_{a,\nu}(s)t^{-s}\big) = a(2\pi a)^{-2a\nu+r}\Gamma(2a\nu-r)\prod_{\substack{k=-r\\k\neq 0}}^{a-1-r}\zeta\Big(1+\frac{k}{a}\Big)\times$$

$$\times\prod_{l=-r}^{a-1-r}\zeta\Big(2\nu+\frac{l}{a}\Big)t^{-2\nu+r/a} \quad (0\leqslant r\leqslant a-1).$$

To calculate the residue at $s=-r/a$ we need the functional equation of the zeta function, Gauss' multiplication formula for the gamma function, and the equation

$$\prod_{k=1}^{a-1}\sin\frac{k\pi}{a} = 2^{1-a}a. \tag{13}$$

Thus

$$\operatorname*{Res}_{s=-r/a}\big(\Phi_{a,\nu}(s)t^{-s}\big) = (2\pi a)^r\frac{(-1)^r}{r!}\prod_{\substack{k=-r\\k\neq 0}}^{a-1-r}\zeta\Big(\frac{k}{a}\Big)\prod_{l=-r}^{a-1-r}\zeta\Big(1-2\nu+\frac{l}{a}\Big)t^{r/a}$$

$$=(-1)^{a\nu}a(2\pi a)^{-2a\nu+a-1-r}\Gamma(2a\nu-a+1+r)\prod_{\substack{k=-a+1+r\\k\neq 0}}^{r}\zeta\Big(1+\frac{k}{a}\Big)\times$$

$$\times\prod_{l=-a+1+r}^{r}\zeta\Big(2\nu+\frac{l}{a}\Big)t^{r/a} \quad (0\leqslant r\leqslant a-1).$$

These calculations as well as (11) and (12) imply

$$g_{a,\nu}(t) = \sum_{r=0}^{a-1}a(2\pi a)^{-2a\nu+r}\Gamma(2a\nu-r)\prod_{\substack{k=-r\\k\neq 0}}^{a-1-r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-r}^{a-1-r}\zeta\Big(2\nu+\frac{l}{a}\Big)t^{-2\nu+r/a} -$$

$$-(-1)^{a\nu}\sum_{r=0}^{a-1}a(2\pi a)^{-2a\nu+a-1-r}\Gamma(2a\nu-a+1+r)\times$$

$$\times\prod_{\substack{k=-a+1+r\\k\neq 0}}^{r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-a+1+r}^{r}\zeta\Big(2\nu+\frac{l}{a}\Big)t^{r/a}+$$

$$+(-1)^{a\nu}t^{-2\nu+1-1/a}g_{a,\nu}\Big(\frac{1}{t}\Big).$$

Replacing r by $a-1-r$ in the first sum, we get

$$g_{a,\nu}(t) = \sum_{r=0}^{a-1}a(2\pi a)^{-2a\nu+a-1-r}\Gamma(2a\nu-a+1+r)\times$$

$$\times\prod_{\substack{k=-a+1+r\\k\neq 0}}^{r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-a+1+r}^{r}\zeta\Big(2\nu+\frac{l}{a}\Big)t^{-2\nu+(a-1-r)/a} -$$

$$-(-1)^{a\nu}\sum_{r=0}^{a-1}a(2\pi a)^{-2a\nu+a-1-r}\Gamma(2a\nu-a+1+r)\times$$

$$\times\prod_{\substack{k=-a+1+r\\k\neq 0}}^{r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-a+1+r}^{r}\zeta\Big(2\nu+\frac{l}{a}\Big)t^{r/a}+$$

$$+(-1)^{a\nu}t^{-2\nu+1-1/a}g_{a,\nu}\Big(\frac{1}{t}\Big).$$

Setting $t=(a/\pi)^a$, $1/t=(\beta/\pi)^a$, we obtain the equation (3).

THEOREM 2. *Let a be a positive integer, ν be an integer greater than 1, and define for $x>0$*

$$F_{a,\nu}(x) = x^{-a\nu+(a+1)/2}\Big\{\sum_{n=1}^{\infty}\sigma_{a,1-2\nu}(n)e^{-2a\sqrt[a]{n}x} -$$

$$-\sum_{r=0}^{a-1}(2\pi a)^r\frac{(-1)^r}{r!}\prod_{k=-r}^{a-1-r}\zeta\Big(\frac{k}{a}\Big)\prod_{l=-r}^{a-1-r}\zeta\Big(2\nu-1+\frac{l}{a}\Big)x^r\Big\}.$$

Then for any positive a,β with $a\beta=\pi^2$ we have

$$F_{a,\nu}(a)-(-1)^{a(\nu-1)}F_{a,\nu}(\beta) \tag{14}$$

$$=\sum_{r=0}^{a-1}a(2\pi a)^{-a+r}(a-1-r)!\prod_{\substack{k=-r\\k\neq 0}}^{a-1-r}\zeta\Big(1+\frac{k}{a}\Big)\prod_{l=-r}^{a-1-r}\zeta\Big(2\nu+\frac{l}{a}\Big)a^{-a\nu+\frac{1}{2}+\frac{r}{2}}\beta^{\frac{a}{2}-\frac{r}{2}}+$$

$$+\sum_{b=1}^{\nu}\sum_{r=0}^{a-1}(2\pi a)^{2ab-a+r}\frac{(-1)^{a-r}}{(2ab-a+r)!}\prod_{k=-r}^{a-1-r}\zeta\Big(1-2b+\frac{k}{a}\Big)\times$$

$$\times\prod_{l=-r}^{a-1-r}\zeta\Big(2\nu-2b+\frac{l}{a}\Big)a^{-a\nu+\frac{1}{2}+\frac{r}{2}}\beta^{-ab+\frac{a}{2}-\frac{r}{2}}.$$

Remark 4. Ramanujan's formula (2) follows from (14) with $a = 1$.

The proof of Theorem 2 shall be done in the same way as that of Theorem 1, using the functional equation

$$\Psi_{a,\nu}\left(1-2\nu+\frac{1}{a}-s\right) = (-1)^{a(\nu-1)}\Psi_{a,\nu}(s),$$

with

$$\Psi_{a,\nu}(s) = a(2\pi a)^{-as}\Gamma(as)\prod_{k=0}^{a-1}\zeta\left(s+\frac{k}{a}\right)\zeta\left(s+\frac{a-1-k}{a}-1+2\nu\right).$$

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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
SHINSHU UNIVERSITY
Nishinagano, Nagano 380, Japan

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Об одной теореме А. Шаркози

А. В. Соколовский (Ташкент)

В статье [3] А. Шаркози (A. Sárközy) с помощью разработанного им „аналога по модулю p" одного неравенства К. Рота (см. [2]) получил результат, интересным следствием которого является

Теорема 1. *Пусть p — произвольное нечётное простое число, χ_p — любой характер по модулю p. Тогда существует целое x, такое что*

(1) $$\left|\sum_{n=x}^{x+(p-3)/2}\chi_p(n)\right| \geqslant c(\sqrt{p}-1/\sqrt{p})$$

и $c \geqslant 1/\pi$.

Эту теорему можно усилить лишь за счёт увеличения значения c. Поэтому в работе [3] ставился вопрос о наилучшей постоянной c в неравенстве (1).

В статье [4] мы доказали равенство, из которого следует, что в (1) можно взять $c = 1/2$, и это значение, вообще говоря, наилучшее. А именно, в [4] доказана

Теорема 2. *Пусть p — любое нечётное простое число, t_m ($m = 0; \pm 1; \pm 2; \ldots$) — периодическая последовательность комплексных чисел с периодом p. Тогда для любого $0 \leqslant z \leqslant p-1$ имеем*

(2) $$\sum_{r=1}^{p-1}\sum_{n=1}^{p}\left|\sum_{j=0}^{z}t_{n+jr}\right|^2 = (z+1)(p-z-1)\sum_{m=1}^{p}|t_m|^2+z(z+1)\left|\sum_{m=1}^{p}t_m\right|^2.$$

Заметим, что в случае $t_m = \chi_p(m)$ равенство (2) превращается в равенство

$$\sum_{k=1}^{p}\left|\sum_{j=0}^{z}\chi_p(k+j)\right|^2 = (p-z-1)(z+1),$$

которое имеется в книге И. М. Виноградова *Основы теории чисел* (вопрос 10 „β" к главе 6).

При $z = (p-3)/2$ и $t_m = \chi_p(m)$ отсюда следует (1) с $c = 1/2$.

Доказательство основано на использовании конечных сумм Фурье взамен интегралов в рассуждениях, аналогичных проводимым в [2] и [3].