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A note on $|\alpha p - q|$

by

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1. Introduction. Toward the question as to how small $|\alpha p-q|$ can be made, with given real α (>0) and primes p and q, infinitely often, Ramachandra [2] proved a theorem which, for instance, asserts that

$$\varliminf_{p\to\infty}\Big(\big(\min_{1\leqslant n\leqslant 4/\varepsilon}\ \min_{q\neq p}|2^n p-q|\big)p^{-\varepsilon}\Big)=0,$$

holds for every fixed ε, $0<\varepsilon<1$ (see Theorem 1 of [2]). In this note, we prove the following improvement of his result.

THEOREM 1. *There is a natural number N with the following property. Let $\beta_1, \ldots, \beta_N$ be any N distinct, positive real numbers given. Then there exist two of these numbers, β_i and β_j say ($i\neq j$), such that (with p, q primes)*

$$(1)\qquad \varliminf_{p\to\infty}\frac{\min_{q\neq p}|\beta_i p-\beta_j q|}{\log p}<\infty.$$

Remark. Ramachandra's theorem asserts this with $\log p$ replaced by p^{ε} and $N=[4/\varepsilon]$. Actually, we show that the ensuing stronger form of this result is true:

THEOREM 2. *Suppose that $\mathfrak{p}$ is a sequence of primes satisfying, for some $\delta>0$, as $x\to\infty$*

$$\sum_{\substack{p\leqslant x\\ p\in\mathfrak{p}}}1\geqslant \delta x/\log x.$$

Then there is a natural number $N=N(\delta)$, depending on δ, such that if $\{\beta_1, \ldots, \beta_N\}$ is any given set of N distinct, positive real numbers, then there are two of these numbers, β_i and β_j say ($i\neq j$), such that with p and q in $\mathfrak{p}$ one has (1).

In the last section, we include some corollaries of this result.

2. Proof of Theorem 2. Notation of Theorem 2 holds in this section. Set $d_p=\min_{q\in\mathfrak{p}}(q-p)$, where $p\in\mathfrak{p}$ and minimum is over $q>p$. By the prime number theorem, we have easily that for some $c=c(\delta)>1$ the number of $p \in \mathfrak{p}$ which lie in any interval (as $x \to \infty$) $(x, cx]$ is $\geqslant \delta'(c-1)x/\log x$ for some positive δ' depending on δ. Our proof depends on the following

LEMMA. *In the above notation, we have for some $K = K(\delta)$, as $x \to \infty$,*

$$\sum_{\substack{p \in (x, cx] \cap \mathfrak{p} \\ d_p \leqslant K \log x}} d_p \geqslant \varepsilon x(c-1) \tag{2}$$

for some $\varepsilon > 0$, depending on δ.

Proof. By the remark preceding the statement of the lemma, we have that the number of $p \in (x, cx] \cap \mathfrak{p}$ with $d_p \leqslant K \log x$ is $\geqslant \frac{1}{2}\delta'(c-1)x/\log x$ provided that K is suitably large. Next, by Brun's sieve (cf. [1], Cor. 2.4.1 on p. 80)

$$\sum_{\substack{p \in (x, cx] \cap \mathfrak{p} \\ d_p \leqslant \delta_1 \log x}} 1 \leqslant \frac{(c-1)x}{\log^2 x} \sum_{b \leqslant \delta_1 \log x} \frac{b}{\varphi(b)} \leqslant \frac{\delta_2(c-1)x}{\log x}$$

where $\delta_2 \to 0$ as $\delta_1 \to 0$. Thus, for small enough $\delta_1 > 0$, the left-hand side of (2) exceeds

$$(c-1)\,\delta_1 \log x(\tfrac{1}{2}\delta' - \delta_2)x/\log x \geqslant \varepsilon x(c-1),$$

provided we choose (as we can certainly do) δ_1 sufficiently small.

Proof of Theorem 2. Let us suppose that all our intervals of the form $(x, cx]$ which occur below are contained in (X, X^2) for sufficiently large X. Introducing $\partial(y) = 0$ or 1 according as $(y - 2K\log X, y + 2K\log X) \cap \mathfrak{p}$ is empty or not, we see that the above lemma yields

$$\int_x^{cx} \partial(y)\,dy \geqslant \varepsilon x(c-1).$$

Now using this with x replaced by $\beta_j^{-1}x$, where β_j's $(1 \leqslant j \leqslant N)$ are a given set of positive reals, we see that

$$\int_x^{cx} \Big(\sum_{j=1}^{N} \partial(y\beta_j^{-1})\,dy > N\varepsilon(c-1).$$

This shows that, if $N\varepsilon > 1$, there is a y in $(x, cx]$ with $\partial(y\beta_i^{-1}) = 1 = \partial(y\beta_j^{-1})$ for some $1 \leqslant i < j \leqslant N$. Thus we can ensure the existence of two primes p_1 and p_2 from $\mathfrak{p}$ for which

$$(|\beta_i| + |\beta_j|)^{-1}|\beta_i p - \beta_j q| \leqslant 8K\log X \leqslant 16K\log p_1.$$

Since there are only finitely many choices for i, j, Theorem 2 follows provided only $N\varepsilon > 1$; i.e., for some $N = N(\delta)$, which is effective, too.

3. Concluding remarks. In this section we give a few corollaries to Theorem 2. First, we have the consequence analogous to the Corollary in [2]. Indeed, we can prove a certain extension of that result as follows.

COROLLARY 1. *Let a be a positive real number. Consider the set*

$$A_a = \Big\{m:\ \varliminf_{p\to\infty} \frac{\min\limits_{q\neq p} |a^m p - q|}{\log p} < \infty;\ p, q \text{ in } \mathfrak{p}\Big\}.$$

Then we have

$$\varliminf_{x\to\infty} \Big(\frac{1}{x} \sum_{\substack{m \leqslant x \\ m \in A_a}} 1\Big) \geqslant \Delta > 0$$

with Δ depending only on δ.

Proof. By choosing $\beta_n = a^{nt}$, $1 \leqslant n \leqslant N$, where t is an arbitrary positive integer, we see that, by Theorem 2, a $jt \in A_a$ for some j $(1 \leqslant j \leqslant N)$ depending on t. Now consider $1 \leqslant t \leqslant x/N$. Obviously for $\geqslant x/2N^2$ values of t we get the same value of j, for sufficiently large x. And for these t the corresponding jt's are distinct. Thus we have this corollary with some $\Delta \geqslant 1/4N^2$ (say).

The next corollary is simpler (though ineffective).

COROLLARY 2. *There exists a finite set of positive reals $\{\beta_1, \ldots, \beta_M\}$ such that if a is any positive real number, then for a certain $j = j(a) \leqslant M$ we have*

$$\varliminf_{\substack{p\to\infty \\ p\in\mathfrak{p}}} \frac{\min\limits_{q\neq p} |ap - \beta_j q|}{\log p} < \infty.$$

Proof. This follows by an iterative construction of β's, in view of Theorem 2.

Finally we note that the method of proof of Theorem 2, in the case β's do not exceed 1, enables one to uphold the statement there with an $N = N(\delta, \varepsilon)$ and the "liminf" bounded by $(\delta^{-1} + \varepsilon)$, where $\varepsilon > 0$ is any preassigned number. Thus we can also state

COROLLARY 2*. *Let $\varepsilon > 0$ be given arbitrarily. Then there exists a finite set of positive reals $\{\beta_1, \ldots, \beta_M\}$ (with $M = M(\delta, \varepsilon)$) such that for every a, $0 < a < 1$, there is a $j = j(a) \leqslant M$ to fulfill*

$$\varliminf_{p\in\mathfrak{p}} \frac{\min\limits_{q\in\mathfrak{p}, q\neq p} |ap - \beta_j q|}{\log p} \leqslant (\delta^{-1} + \varepsilon).$$

Further it is possible to obtain, corresponding to Corollary 1, the following

COROLLARY 1*. *Let $\varepsilon > 0$ be given arbitrarily. Then for every α, $0 < \alpha < 1$, there is an infinity of natural numbers $j = j(\alpha, \varepsilon)$ to fulfill*

$$\varliminf_{p\in\mathfrak{p}} \frac{\min\limits_{q\in\mathfrak{p}, q\neq p} |\alpha^j p - q|}{\log p} \leqslant (\delta^{-1} + \varepsilon).$$

In particular, the inequality $|\alpha^j p - q| \leqslant (1+\varepsilon)\log p$ has infinitely many solutions in primes p and q.

References

[1] H. Halberstam and H.-E. Richert, *Sieve methods*, Academic Press, 1974.
[2] K. Ramachandra, *Two remarks in prime number theory*, Bull. Soc. Math. France 105 (1977), pp. 433–437.

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Generalizations of Ramanujan's formulae

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Ramanujan found the following formulae: For positive α, β with $\alpha\beta = \pi^2$ and an integer $\nu > 1$,

$$\alpha^{\nu}\left\{\frac{\zeta(1-2\nu)}{2} + \sum_{n=1}^{\infty} \sigma_{2\nu-1}(n) e^{-2n\alpha}\right\} = (-\beta)^{\nu}\left\{\frac{\zeta(1-2\nu)}{2} + \sum_{n=1}^{\infty} \sigma_{2\nu-1}(n) e^{-2n\beta}\right\}. \tag{1}$$

$$\alpha^{-(\nu-1)}\left\{\frac{\zeta(2\nu-1)}{2} + \sum_{n=1}^{\infty} \sigma_{1-2\nu}(n) e^{-2n\alpha}\right\} - (-\beta)^{-(\nu-1)}\left\{\frac{\zeta(2\nu-1)}{2} + \sum_{n=1}^{\infty} \sigma_{1-2\nu}(n) e^{-2n\beta}\right\} = -2^{2(\nu-1)} \sum_{k=0}^{\nu} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2\nu-2k}}{(2\nu-2k)!} \alpha^{\nu-k}\beta^k, \tag{2}$$

where $\zeta(s)$ is the Riemann zeta function, $\sigma_b(n) = \sum_{d|n} d^b$, and B_n are Bernoulli numbers defined by $\sum_{n=0}^{\infty} B_n x^n/n! = x/(e^x - 1)$. G. H. Hardy [3] gave two proofs of (1). E. Grosswald [2] proved a more general formula which contains both (1) and (2). Many variants of Ramanujan's formulae are known. The historical survey of the formula and its generalization are explained in [1].

Recently the author [4] presented as an analogue of (1) a formula for the values of $\zeta(s)$ at half integers. In this paper we shall extend further the Ramanujan's formulae (1) and (2) to rational numbers. Our method of the proof is similar to that used in [2].