Let $a \in \bigcap_{i=1}^{\infty} I_i$ then

$$\{d^{-1}a^n\} \in \left[\frac{a}{d}, \frac{a+1}{d}\right), \quad n = 1, 2, \ldots$$

There are uncountably many such numbers since at each stage in the construction there are two disjoint choices for I_{j+1}.

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On a result of Littlewood concerning prime numbers

by

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1. Introduction. We define

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) \tag{1.1}$$

where

$$\Lambda(n) = \begin{cases} \log p, & n = p^m, p \text{ prime}, m \text{ integer} \geqslant 1, \\ 0 & \text{otherwise}. \end{cases} \tag{1.2}$$

The prime number theorem is equivalent to

$$\psi(x) \sim x \quad (\text{as } x \to \infty). \tag{1.3}$$

Assuming the Riemann Hypothesis (the RH), we have the more precise result

$$\psi(x) - x = O(x^{1/2}\log^2 x) \tag{1.4}$$

and, on the other hand, we have (without hypothesis)

$$\psi(x) - x = \Omega_{\pm}(x^{1/2}\log\log\log x). \tag{1.5}$$

The result (1.4) is due to von Koch in 1901, while (1.5) was proved by Littlewood in 1914 (see [4], Chapters 4, 5). Presumably (1.5) is nearer to the truth. The basis for these results is the explicit formula for $\psi(x)$:

$$\frac{\psi(x+0)+\psi(x-0)}{2} = x - \sum_{\varrho} \frac{x^\varrho}{\varrho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}) \tag{1.6}$$

the summation being over the non-trivial zeros of the zeta function, $\varrho = \beta + i\gamma$. (The RH allows us to take $\beta = 1/2$.) The series in (1.6) is neither absolutely nor uniformly convergent, and is understood as

$$\sum_{\varrho} \frac{x^\varrho}{\varrho} = \lim_{T\to\infty} \sum_{|\gamma|<T} \frac{x^\varrho}{\varrho}.$$

For applications it is often useful to replace (1.6) by a formula due to Landau ([6], pp. 108–120, [1], Ch. 5): For k some absolute positive constant,

$$(1.7)\qquad \Big|\psi(x)-x+\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho}\Big| < k\Big(\frac{x\log^2 x}{y}+\frac{x\log y}{y}+\log x\Big)\qquad (x\geqslant 3,\ y\geqslant 3).$$

If $y\geqslant x^{1/2}\log x$, (1.7) implies (k' an absolute constant),

$$(1.8)\qquad \Big|\psi(x)-x+\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho}\Big| < k' x^{1/2}\log x\qquad (x\geqslant 5,\ y\geqslant x^{1/2}\log x).$$

Assuming the Riemann Hypothesis, Littlewood proved in [7] that the condition $y\geqslant x^{1/2}\log x$ in (1.8) can be replaced by $y\geqslant x^{1/2}$. In this paper we show, again assuming the Riemann Hypothesis, that we can take $y\geqslant x^{1/2}/\log x$.

This slight improvement allows us to give a simple proof of a result due to Cramér in 1919; assuming the RH, and letting p_n denote the nth prime,

$$(1.9)\qquad p_{n+1}-p_n = O(p_n^{1/2}\log p_n)$$

(see [2], [3]). Our proof is similar in principle to the proof given by Ingham ([5], p. 256), but proceeds more directly to the result. We also give a simple proof of the closely related result, assuming the RH, $h\leqslant x$,

$$(1.10)\qquad \pi(x+h)-\pi(x)\sim\frac{h}{\log x},\qquad \frac{h}{x^{1/2}\log x}\to\infty\quad \text{as } x\to\infty.$$

Here $\pi(x)$ is the number of primes less than or equal to x. This result was stated by Selberg [9].

2. A lemma. We need a lemma due to Littlewood [7].

LEMMA. *If* $|z|\leqslant 1/2$, $|mz|\leqslant 2$, *then*

$$|(1+z)^m-1-mz|\leqslant 2.6\,|m|(|m|+1)|z|^2.$$

Proof. Let $|z|=r$, $|m|=\mu$, and we may suppose $r>0$, $\mu>0$. We have

$$T=\Big|\frac{(1+z)^m-1-mz}{|m|(|m|+1)|z|^2}\Big|\leqslant\sum_{n=2}^{\infty}\frac{|m(m-1)\dots(m-n+1)|}{\mu(\mu+1)n!}r^{n-2}$$

$$\leqslant\sum_{n=2}^{\infty}\frac{\mu(\mu+1)\dots(\mu+n-1)}{\mu(\mu+1)n!}r^{n-2}=\frac{(1-r)^{-\mu}-1-\mu r}{\mu(\mu+1)r^2}.$$

With r fixed, the second to last expression clearly increases with μ, and so is maximum when $\mu=2/r$. Thus we have

$$T\leqslant\frac{(1-r)^{-2/r}-3}{2(2+r)}.$$

This last expression is strictly increasing for $0<r<1$. To see this, we differentiate and obtain

$$\frac{1}{2(2+r)^2(1-r)^{2/r}}\big[[2r^{-2}\log(1-r)+2r^{-1}(1-r)^{-1}](2+r)-1\big]+\frac{3}{2(2+r)^2}.$$

Expanding $\log(1-r)$ and $(1-r)^{-1}$ into power series (valid for $0<r<1$) and multipling out shows this expression is positive. Since $0<r\leqslant 1/2$, we conclude

$$T\leqslant\frac{(1-.5)^{-4}-3}{2(2.5)}=\frac{16-3}{5}=2.6.$$

3. The main theorem. Throughout the rest of this paper we will assume the Riemann Hypothesis. Thus the complex zeros of $\zeta(s)$ are $\varrho=\beta+i\gamma=\frac12+i\gamma$. We denote by θ a number satisfying $|\theta|\leqslant 1$. The number denoted will, in general, be different for different occurrences and may depend on variables. Most of our formulas will hold "for x sufficiently large", and we will denote this by "$x>A$", where A is some positive absolute constant which may differ on different occasions.

THEOREM 1. *Assuming the Riemann Hypothesis, we have*

$$(3.1)\qquad \Big|\psi(x)-x+\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho}\Big|<\frac{x}{2y}+2x^{1/2}\log y,\qquad x\geqslant 3,\ y>A.$$

In particular,

$$(3.2)\qquad \psi(x)-x=-\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho}+O(x^{1/2}\log x)$$

uniformly for $y\geqslant x^{1/2}/\log x$, $x>A$; *and*

$$(3.3)\qquad \Big|\psi(x)-x+\sum_{|\gamma|<x^{1/2}/\log x}\frac{x^{\varrho}}{\varrho}\Big|<1.5x^{1/2}\log x,\qquad x>A.$$

Proof. Let $\psi_1(x)=\int_0^x\psi(\tau)\,d\tau$. The explicit formula for $\psi_1(x)$ is, for $x\geqslant 1$,

$$(3.4)\qquad \psi_1(x)=\frac{x^2}{2}-\sum_{\varrho}\frac{x^{\varrho+1}}{\varrho(\varrho+1)}-x\frac{\zeta'}{\zeta}(0)+\frac{\zeta'}{\zeta}(-1)-\sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}$$

(see [4], p. 73). Let h be a function of x such that $1 \leqslant h \leqslant x/2$. Then

$$\frac{\psi_1(x\pm h)-\psi_1(x)}{\pm h} = x\pm\frac{h}{2}-\sum_{\varrho}\frac{(x\pm h)^{\varrho+1}-x^{\varrho+1}}{\varrho(\varrho+1)(\pm h)}-\frac{\zeta'}{\zeta}(0)\mp\frac{1}{h}\sum_{r=1}^{\infty}\frac{(x\pm h)^{1-2r}-x^{1-2r}}{2r(2r-1)}.$$

Now $\frac{\zeta'}{\zeta}(0)=\log 2\pi<2$; and for $x\geqslant 3$,

$$\left|\mp\frac{1}{h}\sum_{r=1}^{\infty}\frac{(x\pm h)^{1-2r}-x^{1-2r}}{2r(2r-1)}\right|\leqslant\sum_{r=1}^{\infty}\frac{x^{-1}(2+1)}{2r(2r-1)}\leqslant\sum_{r=1}^{\infty}\left[\frac{-1}{2r}+\frac{1}{2r-1}\right]=1.$$

Hence for $x\geqslant 3$, $1\leqslant h\leqslant x/2$,

$$\frac{\psi_1(x\pm h)-\psi_1(x)}{\pm h} = x\pm\frac{h}{2}-\sum_{|\gamma|<y}\frac{(x\pm h)^{\varrho+1}-x^{\varrho+1}}{\varrho(\varrho+1)(\pm h)}-\sum_{|\gamma|\geqslant y}\frac{(x\pm h)^{\varrho+1}-x^{\varrho+1}}{\varrho(\varrho+1)(\pm h)}+K, \tag{3.5}$$

where K depends on x and h, and $|K|<3$.

We have (without hypothesis)

$$\sum_{T\leqslant\gamma}\frac{1}{\gamma^2}=\frac{1}{2\pi}\frac{\log T}{T}+O\left(\frac{1}{T}\right)\quad\text{as}\quad T\to\infty \tag{3.6}$$

(see [4], Th. 25b; an argument like the one on p. 98 gives this result).

The second sum on the right of (3.5) is in absolute value

$$<\sum_{|\gamma|\geqslant y}\frac{\left((\tfrac{3}{2})^{3/2}+1\right)x^{3/2}}{\gamma^2 h}<\frac{6x^{3/2}}{h}\sum_{\gamma\geqslant y}\frac{1}{\gamma^2}=\frac{6x^{3/2}}{h}\frac{\log y}{2\pi y}\left(1+o(1)\right)<\frac{x^{3/2}\log y}{hy},$$

for $y>A$.

We have used here $h\leqslant x/2$, (3.6), and the fact that the zeros of $\zeta(s)$ are symmetric with the real axis. Next, the first sum in (3.5) is equal to

$$-\sum_{|\gamma|<y}\frac{x^\varrho}{\varrho}-\sum_{|\gamma|<y}\frac{(x\pm h)^{\varrho+1}-x^{\varrho+1}\mp h(\varrho+1)x^\varrho}{\varrho(\varrho+1)(\pm h)}.$$

Denote by w_ϱ the general term of the second sum. Thus

$$w_\varrho = x^{\varrho+1}\left[\frac{(1\pm h/x)^{\varrho+1}-1\mp h/x(\varrho+1)}{\varrho(\varrho+1)(\pm h)}\right].$$

We now apply the lemma taking $z=\pm h/x$, $m=\varrho+1$, and impose the condition

$$y\leqslant x/h. \tag{3.7}$$

The two conditions of the lemma are thus satisfied, for $|z|=h/x\leqslant 1/2$, and, since $|\gamma|<y$ in our sum,

$$|mz|=|\varrho+1|(h/x)\leqslant(3/2+|\gamma|)(h/x)\leqslant(3/2+y)(h/x)\leqslant 3/4+1<2.$$

Therefore,

$$|w_\varrho|\leqslant x^{3/2}\frac{2.6|\varrho+1|(|\varrho+1|+1)(h/x)^2}{|\varrho(\varrho+1)(\pm h)|}$$
$$=2.6x^{-1/2}h\frac{|\varrho+1|+1}{|\varrho|}\leqslant 2.6x^{-1/2}h\left(1+(2/|\varrho|)\right)$$
$$<2.6x^{-1/2}h(1+1/7)<3x^{-1/2}h,$$

since $1/|\varrho|<1/|\gamma|<1/14$.

Let $N(T)$ denote the number of zeros of $\zeta(s)$ with $0<\gamma\leqslant T$. Then (without hypothesis)

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T), \tag{3.8}$$

and consequently

$$N(T)<\frac{T}{2\pi}\log T\quad\text{for}\quad T>A. \tag{3.9}$$

Returning to our sum,

$$\left|-\sum_{|\gamma|<y}w_\varrho\right|\leqslant 6hx^{-1/2}\sum_{0<\gamma<y}1<6hx^{-1/2}\left(\frac{y}{2\pi}\log y\right)<x^{-1/2}hy\log y,$$

for $y>A$. Combining these results in (3.5) we obtain

$$\frac{\psi_1(x\pm h)-\psi_1(x)}{\pm h}=x\pm\frac{h}{2}+\theta_1\frac{x^{3/2}\log y}{hy}+\theta_2 x^{-1/2}hy\log y-\sum_{|\gamma|<y}\frac{x^\varrho}{\varrho} \tag{3.10}$$

for $y>A$, $x\geqslant 3$, $1\leqslant h\leqslant x/2$, and subject to (3.7). (The term K was absorbed into $\theta_1 x^{3/2}\log y/hy$ since we rounded up to obtain this estimate, and by (3.7) this term is $>\theta x^{1/2}$.) The θ's depend on x and h, and will be different in the cases $+h$ and $-h$.

Since $\psi(x)$ is nondecreasing, we have

$$\frac{\psi_1(x-h)-\psi_1(x)}{-h}=\frac{1}{h}\int_{x-h}^{x}\psi(\tau)\,d\tau\leqslant\psi(x)\leqslant\frac{1}{h}\int_{x}^{x+h}\psi(\tau)\,d\tau=\frac{\psi_1(x+h)-\psi_1(x)}{h}.$$

Hence from (3.10) we obtain, subject to the same conditions,

$$(3.11)\qquad \Big|\psi(x)-x+\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho}\Big|<\frac{x^{3/2}\log y}{hy}+x^{-1/2}hy\log y+\frac{h}{2}.$$

Comparing the first two terms on the right, we choose $hy = x$. Thus (3.7) is satisfied, and we have, for $y > A$, $x \geqslant 3$,

$$\Big|\psi(x)-x+\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho}\Big|<\frac{x}{2y}+2x^{1/2}\log y.$$

This proves (3.1).

We now pick $x^{1/2}/\log x \leqslant y \leqslant x$ and obtain

$$\Big|\psi(x)-x+\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho}\Big|<\frac{1}{2}x^{1/2}\log x+2x^{1/2}\log x = O(x^{1/2}\log x).$$

For $y \geqslant x$ Landau's result (1.7) implies

$$\psi(x)-x+\sum_{|\gamma|<y}\frac{x^{\varrho}}{\varrho} = O(\log^2 x)\qquad (y\geqslant x\geqslant 3).$$

Equation (3.2) now follows.

Finally, setting $y = x^{1/2}/\log x$ in (3.1), we have, for $x > A$,

$$\Big|\psi(x)-x+\sum_{|\gamma|<x^{1/2}/\log x}\frac{x^{\varrho}}{\varrho}\Big|<\tfrac{1}{2}x^{1/2}\log x+2x^{1/2}\log x^{1/2}-2x^{1/2}\log\log x<\tfrac{3}{2}x^{1/2}\log x.$$

4. Application to Cramér's theorem. As a simple consequence of our theorem we have

THEOREM 2 (Cramér). *Assuming the Riemann Hypothesis, we have*

$$(4.1)\qquad \pi(x+5x^{1/2}\log x)-\pi(x) > x^{1/2}\quad \textit{for } x > A,$$

and

$$(4.2)\qquad p_{n+1}-p_n < 4p_n^{1/2}\log p_n\quad \textit{for } n > A.$$

Proof. In what follows we suppose x is sufficiently large, and will not indicate it again.

Let $1 \leqslant h \leqslant x/5$. Replacing x by $x+h$ in (3.1) and taking $y = x^{1/2}/\log x$, we have

$$\Big|\psi(x+h)-(x+h)+\sum_{|\gamma|<x^{1/2}/\log x}\frac{(x+h)^{\varrho}}{\varrho}\Big|<\frac{(x+h)\log x}{2x^{1/2}}+2(x+h)^{1/2}\log(x^{1/2})$$
$$< x^{1/2}\log x+[\tfrac{6}{5}]^{1/2}x^{1/2}\log x < 1.7x^{1/2}\log x.$$

Combining this with (3.3) we have

$$(4.3)\qquad \psi(x+h)-\psi(x) = h-\sum_{|\gamma|<x^{1/2}/\log x}\frac{(x+h)^{\varrho}-x^{\varrho}}{\varrho}+3.2\theta x^{1/2}\log x.$$

Since

$$\Big|\frac{(x+h)^{\varrho}-x^{\varrho}}{\varrho}\Big| = \Big|\int_x^{x+h}\tau^{\varrho-1}d\tau\Big|\leqslant x^{-1/2}h,$$

$$(4.4)\qquad \psi(x+h)-\psi(x)$$
$$= h+\theta x^{-1/2}h\sum_{|\gamma|<x^{1/2}/\log x}1+3.2\theta x^{1/2}\log x$$
$$= h+2\theta x^{-1/2}h\Big[\frac{1}{2\pi}\frac{x^{1/2}}{\log x}\log(x^{1/2}/\log x)\Big]+3.2\theta x^{1/2}\log x,$$

by (3.9),

$$= h+\frac{\theta h}{2\pi}+3.2\theta x^{1/2}\log x.$$

Thus, taking $h = 5x^{1/2}\log x$, we have

$$(4.5)\quad \psi(x+5x^{1/2}\log x)-\psi(x) > 5x^{1/2}\log x-\Big(\frac{5}{2\pi}+3.2\Big)x^{1/2}\log x > x^{1/2}\log x.$$

Finally, we have for $1\leqslant h\leqslant x$ ($[w]$ = integer part of w),

$$(4.6)\qquad \psi(x+h)-\psi(x) = \sum_{x<p\leqslant x+h}\log p+O\Big(\sum_{p^2\leqslant x+h}\log p\Big[\frac{\log(x+h)}{\log p}\Big]\Big)$$
$$= \sum_{x<p\leqslant x+h}(\log x+O(1))+O\Big(\sum_{p^2\leqslant 2x}\log 2x\Big)$$
$$= \{\pi(x+h)-\pi(x)\}(\log x+O(1))+O(x^{1/2}).$$

Combining (4.5) and (4.6) proves (4.1). Next, taking $h = 4x^{1/2}\log x$ in (4.4) gives

$$(4.7)\qquad \psi(x+4x^{1/2}\log x)-\psi(x) > .1x^{1/2}\log x > 0.$$

Equation (4.6) now implies $\pi(x+4x^{1/2}\log x)-\pi(x) > .1x > 0$. Taking $x = p_n$, we see $p_{n+1}-p_n < 4x^{1/2}\log x = 4p_n^{1/2}\log p_n$.

The constants in (4.1) and (4.2) can be decreased. The 5 in (4.1) may be replaced by a number less than 4 and the 4 in (4.2) by a number less than 2. It is interesting to compare this with the conjectured result ([8])

$$(4.8)\qquad \psi(x+h)-\psi(x) = h+O(h^{1/2}x^{\varepsilon}),\qquad 1\leqslant h\leqslant x.$$

We can give an easy proof of the best result known in this direction, assuming the RH. It is stated in [9].

THEOREM 3. *Assume the Riemann Hypothesis. Let h be a function of x such that* (i) $h \leqslant x$, (ii) *h is monotonically increasing, and* (iii) $h/(x^{1/2}\log x) \to \infty$ *as* $x \to \infty$. *Then*

$$\psi(x+h)-\psi(x) \sim h \tag{4.9}$$

and

$$\pi(x+h)-\pi(x) \sim h/\log x. \tag{4.10}$$

Proof. The two assertions are equivalent by (4.6) and (iii). Thus we shall prove (4.9). Let $\varphi(x)$ be any function such that $\varphi(x)\to\infty$ as $x\to\infty$ and $\varphi(x) = O(\log x)$. Then by (4.3)

$$\psi(x+h)-\psi(x) = h - \sum\nolimits_1 \frac{(x+h)^\varrho - x^\varrho}{\varrho} - \sum\nolimits_2 \frac{(x+h)^\varrho - x^\varrho}{\varrho} + O(x^{1/2}\log x),$$

where $\sum_1$ is summed over $|\gamma| < x^{1/2}/(\log x)\varphi(x)$, and $\sum_2$ is summed over $x^{1/2}/(\log x)\varphi(x) \leqslant |\gamma| < x^{1/2}/\log x$. Handling $\sum_1$ as before,

$$\psi(x+h)-\psi(x) = h + O\left(hx^{-1/2}\left(\frac{x^{1/2}}{(\log x)\varphi(x)}\right)\left(\log\left(\frac{x^{1/2}}{(\log x)\varphi(x)}\right)\right)\right) + \\ + O\left(x^{1/2}\sum\nolimits_2 \frac{1}{\gamma}\right) + O(x^{1/2}\log x).$$

Since (see [4], p. 98)

$$\sum_{0<\gamma<T} \frac{1}{\gamma} = \frac{1}{4\pi}\log^2 T + O(\log T), \tag{4.11}$$

we obtain

$$\psi(x+h)-\psi(x) = h + O\left(\frac{h}{\varphi(x)}\right) + \\ + O\left(x^{1/2}\left\{\frac{1}{4\pi}(\log x^{1/2} - \log\log x)^2 - \frac{1}{4\pi}(\log x^{1/2} - \log\log x - \log\varphi(x))^2 + \\ + O(\log x)\right\}\right) + O(x^{1/2}\log x) \\ = h + O\left(\frac{h}{\varphi(x)}\right) + O\big(x^{1/2}(\log x)(\log\varphi(x))\big) + O(x^{1/2}\log x).$$

Hence,

$$\psi(x+h)-\psi(x) = h + O\left(\frac{h}{\varphi(x)}\right) + O\big(x^{1/2}(\log x)(\log\varphi(x))\big). \tag{4.12}$$

We obtain the theorem by picking h larger than the last order term, i.e. $h \geqslant x^{1/2}(\log x)\varphi(x)$.

We note that by (4.4), K any positive constant, and $x > A$,

$$[K(1-1/2\pi)-3.2]x^{1/2}\log x < \psi(x+Kx^{1/2}\log x)-\psi(x) \\ < [K(1+1/2\pi)+3.2]x^{1/2}\log x. \tag{4.13}$$

It seems to require new ideas to replace (4.13) by an asymptotic result. The above proof shows how Theorem 1 must be improved in order to obtain new results on primes in short intervals. Let $\varphi(x)$ be *any* function monotonically increasing to infinity. Then the result

$$\psi(x)-x = -\sum_{|\gamma|<y} \frac{x^\varrho}{\varrho} + O\big(E(x)\big) \text{ uniformly for } y \geqslant \frac{x^{1/2}}{(\log x)\varphi(x)}, \tag{4.14}$$

$x > A$, implies (with RH) $\psi(x+h)-\psi(x) = h + O\big(h/\varphi(x)\big) + O\big(E(x)\big)$, $1 \leqslant h \leqslant x$, $x > A$. This gives (i) $\psi(x+h)-\psi(x) \sim h$ if $h/E(x)\to\infty$ as $x\to\infty$ and (ii) $p_{n+1}-p_n = O\big(E(p_n)\big)$. When $\gamma \sim x^{1/2}/\log x$ the terms in the sum in (4.14) are $O(\log x)$. This, together with the cancellation between terms in the sum makes it seem reasonable that $E(x)$ is smaller than in Theorem 1.

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