Let \( a \in \bigcap_{i=1}^{\infty} I_i \) then
\[
\{d^{-1}a^n \} \in \left[ \frac{a}{d^i}, \frac{a+1}{d^i} \right], \quad n = 1, 2, \ldots
\]

There are uncountably many such numbers since at each stage in the construction there are two disjoint choices for \( I_{i+1} \).

References


IILLINOIS STATE UNIVERSITY
Normal, Illinois 61761, USA

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**On a result of Littlewood concerning prime numbers**

by

D. A. GOLDSTON (Berkeley, Calif.)

1. Introduction. We define
\[
\psi(x) = \sum_{n \leq x} A(n)
\]
where
\[
A(n) = \begin{cases} 
\log p, & n = p^m, p \text{ prime, } m \text{ integer } \geq 1, \\
0 & \text{otherwise}.
\end{cases}
\]
The prime number theorem is equivalent to
\[
\psi(x) \sim x \quad (\text{as } x \to \infty).
\]
Assuming the Riemann Hypothesis (the RH), we have the more precise result
\[
\psi(x) - x = O(x^{1/2} \log^2 x)
\]
and, on the other hand, we have (without hypothesis)
\[
\psi(x) - x = \Omega_{\pm}(x^{1/2} \log \log \log x).
\]
The result (1.4) is due to von Koch in 1901, while (1.5) was proved by Littlewood in 1914 (see [4], Chapters 4, 5). Presumably (1.5) is nearer to the truth. The basis for these results is the explicit formula for \( \psi(x) \):
\[
\psi(x) = x - \sum_{\ell} \frac{a_{\ell}^2}{\ell} \log\frac{x}{\ell} - \frac{1}{2} \log(1-x^{-2})
\]
the summation being over the non-trivial zeros of the zeta function, \( \ell = \beta + i \gamma \). (The RH allows us to take \( \beta = 1/2 \)). The series in (1.6) is neither absolutely nor uniformly convergent, and is understood as
\[
\sum_{\ell} \frac{a_{\ell}^2}{\ell} = \lim_{x \to \infty} \sum_{|\gamma|<\delta} \frac{a_{\ell}^2}{\ell}.
\]
For applications it is often useful to replace (1.6) by a formula due to Landau [9], pp. 108-120, [1], Ch. 5: For \( k \) some absolute positive constant,

\[
(1.7) \quad \psi(x) - x - \sum_{|\theta| < \sqrt{y}} \frac{\vartheta(x)}{\theta} < k \left( \frac{x \log^2 x}{y} + \frac{x \log y}{y} + \frac{\psi(x)}{x} \right) \quad (x \geq 3, y \geq 3).
\]

If \( y \geq x^{1/3} \log x \), (1.7) implies (1.6) an absolute constant),

\[
(1.8) \quad \psi(x) - x - \sum_{|\theta| < \sqrt{y}} \frac{\vartheta(x)}{\theta} < k' x^{1/3} \log x \quad (x \geq 5, y \geq x^{1/3} \log x).
\]

Assuming the Riemann Hypothesis, Littlewood proved in [7] that the condition \( y \geq x^{1/3} \log x \) in (1.8) can be replaced by \( y \geq x^{1/8} \). In this paper we show, again assuming the Riemann Hypothesis, that we can take \( y \geq x^{1/8} \).

This slight improvement allows us to give a simple proof of a result due to Cramér in 1919; assuming the RH, and letting \( p_n \) denote the \( n \)th prime,

\[
(1.9) \quad p_{n+1} - p_n = O(p_n^{1/3} \log p_n)
\]

(see [2], [3]). Our proof is similar in principle to the proof given by Ingham ([5], p. 256), but proceeds more directly to the result. We also give a simple proof of the closely related result, assuming the RH, \( h \ll x \),

\[
(1.10) \quad \pi(x + h) - \pi(x) \sim \frac{h}{\log x}, \quad \frac{h}{x^{1/3} \log x} \to \infty \quad \text{as} \quad x \to \infty.
\]

Here \( \pi(x) \) is the number of primes less than or equal to \( x \). This result was stated by Selberg [9].

2. A lemma. We need a lemma due to Littlewood [7].

**Lemma.** If \( \sigma \leq 1/2, \ |m| \leq 2 \), then

\[
|\{(1+\sigma)^m - 1 - mx\} | \leq 2.6 |m| (|m| + 1) \sigma^2.
\]

**Proof.** Let \( |u| = \rho, \ |m| = \mu \), and we may suppose \( \sigma > 0, \mu > 0 \). We have

\[
T = \left| \frac{(1+\sigma)^m - 1 - mx}{|m|(|m|+1)\sigma^2} \right| \leq \sum_{|n| \leq 2} \frac{|m(m-1) \cdots (m-n+1)|}{\mu(n+1)|n|!} \leq \sum_{|n| \leq 2} \frac{\mu(n+1) \cdots (n-n+1)}{\mu(n+1)|n|!} \rho^{n-1} = (1-\rho)^{\mu-1} - \mu \rho,
\]

uniformly for \( y \geq x^{1/2} \log x, \quad x > A \); and

\[
(3.1) \quad \left| \psi(x) - x - \sum_{|\theta| < \sqrt{y}} \frac{\vartheta(x)}{\theta} \right| < \frac{x}{2y} + 2x^{1/2} \log y, \quad x \geq 3, \ y > A.
\]

In particular,

\[
(3.2) \quad \psi(x) - x = - \sum_{|\theta| < \sqrt{y}} \frac{\vartheta(x)}{\theta} + O(x^{1/2} \log x)
\]

uniformly for \( x > A \). And

\[
(3.3) \quad \left| \psi(x) - x - \sum_{|\theta| < x^{1/2} \log x} \frac{\vartheta(x)}{\theta} \right| < 1.5x^{1/2} \log x, \quad x > A.
\]

**Proof.** Let \( \psi_1(x) = \int_x^A \psi(x) \, dx \). The explicit formula for \( \psi_1(x) \) is, for \( x > 1 \),

\[
(3.4) \quad \psi_1(x) = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho + 1}}{\rho (\rho + 1)} - x^2 \int_0^{\infty} \left( \frac{1}{\xi^2} \right) \left( \frac{1}{\xi^2} \right) \left( -1 \right) - \sum_{\rho} \frac{x^2 - x^{2\rho}}{2\rho (2\rho - 1)}
\]

With \( r \) fixed, the second to last expression clearly increases with \( \mu \), and so is maximum when \( \mu = 2/r \). Thus we have

\[
T \leq \frac{(1-r^{-2})r - 3}{2(1+r)}.
\]

This last expression is strictly increasing for \( 0 < r < 1 \). To see this, we differentiate and obtain

\[
\frac{1}{2(1+r)^2} \left( \frac{(2r^{-2}\log(1-r) + 2r^{-1}(1-r)^{-1})(2+r)-1}{2(1+r)^2} \right) + \frac{3}{2(1+r)^2}.
\]

Expanding \( \log(1-r) \) and \( (1-r)^{-1} \) into power series (valid for \( 0 < r < 1 \)) and multiplying out shows this expression is positive. Since \( 0 < r < 1/2 \), we conclude

\[
T \leq \frac{(1-0.5)^{-1}}{2(2.5)} = 16 - \frac{3}{5} = 2.6.
\]

3. The main theorem. Throughout the rest of this paper we will assume the Riemann Hypothesis. Thus the complex zeros of \( \zeta(s) \) are

\[
\rho = \beta + i\gamma = \frac{1}{2} + iv.
\]

We denote by \( \theta \) a number satisfying \( |\theta| \leq 1 \). The number denoted in general, will be different for different occurrences and may depend on variables. Most of our formulas will hold "for sufficiently large" \( x \), and we will denote this by "\( x > A \)" , where \( A \) is some positive absolute constant which may differ on different occasions.

**Theorem 1.** Assuming the Riemann Hypothesis, we have

\[
(3.1) \quad \psi(x) - x - \sum_{|\theta| < \sqrt{y}} \frac{\vartheta(x)}{\theta} < \frac{x}{2y} + 2x^{1/2} \log y, \quad x \geq 3, \ y > A.
\]
(see [4], p. 73). Let \( h \) be a function of \( x \) such that \( 1 \leq h \leq x/2 \). Then

\[
\psi_1(x \pm h) - \psi_1(x) \pm \frac{h}{2} = \sum_{\rho \leq \sqrt{x}} \frac{(x \pm h)^{\rho+1} - x^{\rho+1}}{\rho x^{\rho-1}} - \frac{\log x}{h} \sum_{\rho \leq \sqrt{x}} \frac{(x \pm h)^{\rho-1} - x^{\rho-1}}{2 \rho \rho^{-1}} - \frac{h}{2} \sum_{\rho \leq \sqrt{x}} \frac{(x \pm h)^{\rho-1} - x^{\rho-1}}{2 \rho \rho^{-1}}
\]

Now \( \frac{\log x}{h} \leq 2 \pi < \frac{h}{2} \); and for \( x \geq 3 \),

\[
\left| \sum_{\rho \leq \sqrt{x}} \frac{(x \pm h)^{\rho-1} - x^{\rho-1}}{2 \rho \rho^{-1}} \right| \leq \sum_{\rho \leq \sqrt{x}} \frac{x^{\rho-1} (2 + 1)}{2 \rho \rho^{-1}} \leq \sum_{\rho \leq \sqrt{x}} \left[ \frac{-1}{2 \rho} + \frac{1}{2 \rho - 1} \right] = 1.
\]

Hence for \( x \geq 3 \), \( 1 \leq h \leq x/2 \),

\[
\psi_1(x \pm h) - \psi_1(x) = \sum_{\rho \leq \sqrt{x}} \frac{(x \pm h)^{\rho-1} - x^{\rho-1}}{\rho x^{\rho-1}} - \sum_{\rho \leq \sqrt{x}} \frac{(x \pm h)^{\rho-1} - x^{\rho-1}}{\rho (x + 1)^{\rho-1} (h \pm h)} + K,
\]

where \( K \) depends on \( x \) and \( h \), and \( |K| < 3 \).

We have (without hypothesis)

\[
\sum_{T < \rho < \infty} \frac{1}{\rho^s} = \frac{\log T}{2 \pi T} + O \left( \frac{1}{T} \right) \quad \text{as} \quad T \to \infty
\]

(see [4], Th. 25b; an argument like the one on p. 98 gives this result). The second sum on the right of (3.5) is in absolute value

\[
< \sum_{\rho \leq \sqrt{x}} \frac{1}{\rho^s} \frac{\log x}{2 \pi} (1 + o(1)) \left( \frac{x}{2 \pi \rho} \right)^{\rho(\rho-1)/2} \leq \frac{6 \pi^2}{h} \frac{\log y}{2 \pi y} \left( 1 + o(1) \right) \leq \frac{a^2 \log x}{hy},
\]

for \( y > A \).

We have used here \( h \leq x/2 \), (3.6), and the fact that the zeros of \( \zeta(x) \) are symmetric with the real axis. Next, the first sum in (3.5) is equal to

\[
- \sum_{\rho \leq \sqrt{x}} \frac{x^{\rho-1} \rho^2}{\rho (x + 1)^{\rho-1}} - \sum_{\rho \leq \sqrt{x}} \frac{(x \pm h)^{\rho-1} - x^{\rho-1}}{\rho (x + 1)^{\rho-1} (h \pm h)}.
\]

Denote by \( \omega_\rho \) the general term of the second sum. Thus

\[
\omega_\rho = \frac{(x \pm h)^{\rho-1} - x^{\rho-1}}{\rho (x + 1)^{\rho-1} (h \pm h)}.
\]

We now apply the lemma taking \( s = \frac{1}{2} \), \( m = q+1 \), and impose the condition

\[
y \leq \frac{x}{h}.
\]

The two conditions of the lemma are thus satisfied, for \( |s| = h/x \leq 1/2 \), and, since \( |y| < y \) in our sum,

\[
|m| = |q+1| = (3/2 + |y|)(h/x) \leq (3/2 + y)(h/x) \leq 3/4 + 1 < 2.
\]

Therefore,

\[
|m| \leq \frac{1}{2} \frac{2h}{2 \pi} \frac{|y+1|}{h} \frac{2/2}{2 \pi} \leq \frac{2.6 \pi}{1/2} \frac{|y+1|}{h} \frac{2/2}{2 \pi} \leq \frac{2.6 \pi}{1/2} \frac{|y+1|}{h} \frac{2/2}{2 \pi} < \frac{2.6 \pi}{1/2} \frac{|y+1|}{h} \frac{2/2}{2 \pi} < 3a^{-1/2} h
\]

since \( 1/|y| < 1/|y| < 1/14 \).

Let \( N(T) \) denote the number of zeros of \( \zeta(x) \) with \( 0 < y \leq T \). Then (without hypothesis)

\[
N(T) = \frac{T}{2 \pi} \log T - \frac{T}{2 \pi} + O(\log T),
\]

and consequently

\[
N(T) \leq \frac{T}{2 \pi} \log T \quad \text{for} \quad T > A.
\]

Returning to our sum,

\[
\left| \sum_{\rho \leq \sqrt{x}} \frac{\omega_\rho}{\rho^s} \right| \leq \sum_{\rho \leq \sqrt{x}} \frac{6 a \rho^2 \log x}{2 \pi \rho} (1 + o(1)) \leq \frac{6 a \rho^2 \log x}{2 \pi \rho} (1 + o(1)) \leq \frac{a^2 \rho^2 \log x}{hy},
\]

for \( y > A \). Combining these results in (3.5) we obtain

\[
\psi_1(x \pm h) - \psi_1(x) = x \pm h - \frac{2 \pi}{h} \frac{a \rho^2 \log x}{hy} + \frac{\theta_x a^{-1/2} hy \log y - \sum_{\rho \leq \sqrt{x}} \omega_\rho}{h}
\]

for \( y > A \), \( x \geq 3 \), \( 1 \leq h \leq x/2 \), and subject to (3.7). (The term \( K \) was absorbed into \( \theta_x a^{-1/2} hy \log y \) since we rounded up to obtain this estimate, and by (3.7) this term is \( \geq a \).) The \( \theta_x \)'s depend on \( x \) and \( h \), and will be different in the cases \( +h \) and \( -h \).

Since \( \psi(x) \) is nondecreasing, we have

\[
\psi_1(x \pm h) - \psi_1(x) = \frac{1}{h} \int_{x-h}^{x+h} \psi(x) \, dx \leq \psi(x) \leq \frac{1}{h} \int_{x-h}^{x+h} \psi(x) \, dx = \psi_1(x \pm h) - \psi_1(x).
\]
Hence from (3.10) we obtain, subject to the same conditions,

\[\psi(x) - x + \sum_{\nu < x} \frac{\alpha^\nu}{\nu} < \frac{x^{1/2} \log y}{hy} + x^{-1/2} hy \log y + \frac{h}{2}.\]

Comparing the first two terms on the right, we choose \(hy = y\). Thus (3.7) is satisfied, and we have, for \(y > A\), \(x \geq 3\),

\[\left| \psi(x) - x + \sum_{\nu < x} \frac{\alpha^\nu}{\nu} \right| < \frac{x^{1/2} \log x}{2y} + 2x^{1/2} \log y .\]

This proves (3.1).

We now pick \(a^{1/2} / \log x \leq y \leq x\) and obtain

\[\left| \psi(x) - x + \sum_{\nu < x} \frac{\alpha^\nu}{\nu} \right| < \frac{1}{2} a^{1/2} \log x + 2a^{1/2} \log x = O(a^{1/2} \log x).\]

For \(y > x\) Landau's result (1.7) implies

\[\psi(x) - x + \sum_{\nu < x} \frac{\alpha^\nu}{\nu} = O(\log^4 x) \quad (y > x \geq 3).\]

Equation (3.2) now follows.

Finally, setting \(y = a^{1/2} / \log x\) in (3.1), we have, for \(x > A\),

\[\left| \psi(x) - x + \sum_{\nu \leq a^{1/2} \log x} \frac{\alpha^\nu}{\nu} \right| < \frac{1}{2} a^{1/2} \log x + 2a^{1/2} \log x = \frac{5}{2} a^{1/2} \log x .\]

4. Application to Cramér's theorem. As a simple consequence of our theorem we have

**Theorem 2** (Cramér). Assuming the Riemann Hypothesis, we have

\[\pi(x + 5a^{1/2} \log x) - \pi(x) > a^{1/2}\]

for \(x > A\),

and

\[P_{n+1} - P_n < 4a^{1/2} \log P_n\]

for \(n > A\).

**Proof.** In what follows we suppose \(x\) is sufficiently large, and will not indicate it again.

Let \(1 \leq h \leq a/5\). Replacing \(x\) by \(x + h\) in (3.1) and taking \(y = a^{1/2} / \log x\), we have

\[\psi(x + h) - (x + h) + \sum_{\nu \leq a^{1/2} \log x} \frac{\alpha^{\nu+h}}{\nu} = (x + h)^2 + (x + h)^{1/2} \log (x^{1/2}) + x^{1/2} \log x + \frac{1}{2} x^{1/2} \log x .\]

Combining this with (3.3) we have

\[\psi(x + h) - \psi(x) = h - \sum_{\nu \leq a^{1/2} \log x} \frac{(x + h) - x}{\nu} + 3.2 \nu x^{1/2} \log x .\]

Since

\[\left| \frac{(x + h) - x}{\nu} \right| = \int_{x}^{x+h} \frac{1}{t^2} \, dt \leq x^{-1/2} h ,\]

we have

\[\psi(x + h) - \psi(x) = h + 2 \nu x^{-1/2} h \left[ \frac{1}{2} \log (x^{1/2} \log x) \right] + 3.2 \nu x^{1/2} \log x ,\]

by (3.9),

\[h + 2 \nu x^{-1/2} h + 3.2 \nu x^{1/2} \log x .\]

Thus, taking \(h = 5x^{1/2} \log x\), we have

\[\psi(x + 5x^{1/2} \log x) - \psi(x) > 5x^{1/2} \log x + \left( \frac{5}{2} \right) x^{1/2} \log x ,\]

Finally, we have for \(1 \leq h \leq a\) ([\(w\]) = integer part of \(w\)),

\[\psi(x + h) - \psi(x) = \sum_{\nu < x} \log p + O \left( \sum_{\nu < x} \log p \left[ \frac{\log (x + h)}{\log p} \right] \right) .\]

Combining (4.5) and (4.6) proves (4.1). Next, taking \(h = 4x^{1/2} \log x\) in (4.4),

\[\psi(x + 4a^{1/2} \log x) - \psi(x) > 1.5x^{1/2} \log x ,\]

Equation (4.6) now implies \(\pi(x + 4a^{1/2} \log x) - \pi(x) > 1.5x > 0\). Taking \(x = P_n\), we see \(P_{n+1} - P_n < 4x^{1/2} \log x = 4P_n^{1/2} \log P_n\).

The constants in (4.1) and (4.2) can be decreased. The 5 in (4.1) may be replaced by a number less than 4 and the 4 in (4.2) by a number less than 2. It is interesting to compare this with the conjectured result

\[\psi(x + h) - \psi(x) = \frac{h}{2} + O(h^{1/2} \log x) ,\]

for \(1 \leq h \leq x\).
We can give an easy proof of the best result known in this direction, assuming the RH. It is stated in [9].

**Theorem 3.** Assume the Riemann Hypothesis. Let \( h \) be a function of \( x \) such that (i) \( h \ll x \), (ii) \( h \) is monotonically increasing, and (iii) \( h/(\alpha^{12}\log x) \rightarrow 0 \) as \( x \rightarrow \infty \). Then

\[
(4.9) \quad \psi(x+h) - \psi(x) \sim h
\]

and

\[
(4.10) \quad \chi(x+h) - \chi(x) \sim h/\log x.
\]

**Proof.** The two assertions are equivalent by (4.6) and (iii). Thus we prove (4.9). Let \( \varphi(x) \) be any function such that \( \varphi(x) \rightarrow \infty \) as \( x \rightarrow \infty \) and \( \varphi(x) = O(\log x) \). Then by (4.3)

\[
(4.11) \quad \psi(x+h) - \psi(x) = h - \sum_{1} \frac{(x+h)^{\varphi(x)}}{\varphi(x)} - \sum_{2} \frac{(x+h)^{\varphi(x)}}{\varphi(x)} + O(\alpha^{12}\log x),
\]

where \( \sum_{1} \) is summed over \(|\gamma| < \alpha^{12}/(\log x)\varphi(x)\), and \( \sum_{2} \) is summed over \( \alpha^{12}/(\log x)\varphi(x) \leq |\gamma| < \alpha^{12}/\log x \). Handling \( \sum_{1} \) as before,

\[
(4.12) \quad \psi(x+h) - \psi(x) = h + O \left( \frac{\alpha^{12}}{\log x} \left[ \log \frac{\alpha^{12}}{\log x} \varphi(x) \right] \right) + O(\alpha^{12}\log x).
\]

Since (see [4], p. 98)

\[
(4.13) \quad \sum_{1} \frac{1}{|\gamma|} = \frac{1}{4\pi} \log^{2} T + O(\log T),
\]

we obtain

\[
\psi(x+h) - \psi(x) = h + O \left( \frac{h}{\varphi(x)} \right) + O \left( \frac{1}{4\pi} \left( \log \alpha^{12} - \log \log x \right)^{2} \right. \]

\[
- \frac{1}{4\pi} \left( \log \alpha^{12} - \log \log x - \log \varphi(x) \right)^{2} + O(\log x) \right) + O(\alpha^{12}\log x).
\]

Hence,

\[
(4.14) \quad \psi(x+h) - \psi(x) = h + O \left( \frac{h}{\varphi(x)} \right) + O(\alpha^{12}\log x) \log \varphi(x) + O(\alpha^{12}\log x).
\]

We obtain the theorem by picking \( h \) larger than the last order term, i.e. \( h \geq \alpha^{12}/(\log x)\varphi(x) \).

We note that by (4.4), \( K \) any positive constant, and \( x > A \),

\[
(4.15) \quad \left[ K(1-1/2\pi) - 3.2 \right] \alpha^{12}/\log x < \psi(x + K\alpha^{12}/\log x) - \psi(x) \]

\[< \left[ K(1+1/2\pi) + 3.2 \right] \alpha^{12}/\log x.\]

It seems to require new ideas to replace (4.13) by an asymptotic result. The above proof shows how Theorem 1 must be improved in order to obtain new results on primes in short intervals. Let \( \varphi(x) \) be any function monotonically increasing to infinity. Then the result

\[
(4.16) \quad \psi(x+h) - \psi(x) = - \sum_{\varphi(x)} \frac{\alpha^{12}}{\log x} + O(E(x)) \text{ uniformly for } x \geq \frac{\alpha^{12}}{\log x} \varphi(x),
\]

\[x > A, \text{ implies (with RH)} \psi(x+h) - \psi(x) = h + O(h \varphi(x)) + O(E(x)), \quad 1 \ll h \ll x, x > A. \]

This gives (i) \( \psi(x+h) - \psi(x) \sim h \) if \( h \alpha^{12}/\log x \rightarrow \infty \) as \( x \rightarrow \infty \) and (ii) \( \psi(x+h) - \psi(x) = O(E(x)) \). When \( \gamma \sim \alpha^{12}/\log x \) the terms in the sum in (4.14) are \( O(\log x) \). This, together with the cancellation between terms in the sum makes it seem reasonable that \( E(x) \) is smaller than in Theorem 1.

**References**


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
Berkeley, California 94720, USA

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