ACTA ARITHMETICA
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On generalised arithmetic and geometric progressions

by

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1. Introduction. In [2] Erdös asked the following question.

Let $a > 1$ and β be real numbers. We call the sequence $[at+\beta]$, $t = 1, 2, \ldots$, a *generalised arithmetic progression*. Let (n_k) be a sequence of integers tending to infinity sufficiently fast. Is it true that the complement of (n_k) contains an infinite generalised arithmetic progression?

Here $[x]$ denotes the greatest integer $\leqslant x$, and $\{x\} = x-[x]$.

We answer the question in the affirmative by showing that, given any sequence of integers (n_k) for which $n_{k+1}/n_k \geqslant \delta > 1$, for all k, that is (n_k) is a lacunary sequence, we can always find a generalised arithmetic progression which does not meet the sequence (n_k). We shall also show that, if (n_k) grows so slowly that the sequence $(n_k\theta)$ is dense mod 1 for all irrationals θ, then there is no irrational a and real number β for which the sequence $[at+\beta]$ lies in the complement of (n_k). A consequence of this second result is that, given any such sequence of integers (n_k), we can construct another sequence of integers (t_k), containing (n_k) as a subsequence, such that (t_k) has the same asymptotic density as (n_k) but (t_k) meets *every* generalised arithmetic progression infinitely often. By combining these two results we see that, if (n_k) is a lacunary sequence, then there is an irrational θ for which $\{n_k\theta\}$ is not dense in $[0, 1]$. This answers another question of Erdös [2].

By defining a generalised geometric progression in an analogous fashion, namely as $[a^n]$, $n = 1, 2, \ldots$, where $a > 1$ is a real number, we shall show that, given any natural numbers a and d, there are uncountably many generalised geometric progressions for which every term of the progression lies in the residue class congruent to a mod d.

2. Generalised arithmetic progressions. In this section we prove the results about generalised arithmetic progressions mentioned in the introduction.

THEOREM 1. *If $\delta > 1$ and (n_j) is a sequence of positive integers with*

$$n_{j+1}/n_j \geqslant \delta \quad \textit{for} \quad j = 1, 2, \ldots$$

then, given any $0 < s_0 < 1$, *we can construct a set of real numbers* $S = S(s_0)$ *such that, if* $\alpha \in S$, *then*

$$[\alpha t] \quad \text{for} \quad t = 1, 2, \ldots \tag{1}$$

is contained in the complement of the sequence (n_j), *and the Hausdorff dimension of* S *is greater than or equal to* s_0.

COROLLARY. *The set of numbers* T, *such that if* $\alpha \in T$ *then* $[t\alpha]$ *lies in the complement of* (n_j), *has Hausdorff dimension equal to* 1.

Proof. Put $T = \bigcup_{n=1}^{\infty} S(1-1/n)$. Then

$$\text{H.dim.}T \geqslant 1-1/n \quad \text{for all } n$$

and so

$$\text{H.dim.}T = 1.$$

Proof of Theorem 1. Since $\delta > 1$, we can choose a real number $d < 2$ and an integer r so that $1 \leqslant d \leqslant \delta$ and d^r is an integer with

$$d^r > d^{rs_0} + (r+2). \tag{2}$$

Clearly

$$n_{k+1}/n_k \geqslant d \quad \text{for} \quad k = 1, 2, \ldots \tag{3}$$

Now choose $l > 1$ so large that

$$l > d(d^r-1)/(d-1). \tag{4}$$

We next choose a_1 so that

$$n_{t-1}+1 \leqslant a_1 < n_t - l \tag{5}$$

but

$$2a_1 > n_t + 1 \quad \text{for some } t \tag{6}$$

and

$$a_1 > d^r l + 1. \tag{7}$$

These choices are all possible since the sequence n_k grows exponentially. Put $b_1 = a_1 + l$.

To construct a particular α, our method will be to construct a nested sequence of closed intervals

$$I_1 \supset I_2 \supset \ldots$$

so that if $I_j = [a_j, b_j]$, then

$$[a_j, b_j] \cup [2a_j, 2b_j] \cup \ldots \cup [d^{r(j-1)}a_j, d^{r(j-1)}b_j] \tag{8}$$

contains no elements x with $[x] = n_k$, $k = 1, 2, \ldots$

Then $\alpha \in \bigcap_{j=1}^{\infty} I_j$ satisfies (1).

We next construct the intervals I_j. Put $I_1 = [a_1, b_1]$. Suppose that $I_1 \supset I_2 \supset \ldots \supset I_k$ have been constructed to satisfy (8), and that

$$l(I_k) = b_k - a_k = d^{-r(k-1)} l, \tag{9}$$

where $l(I_k)$ denotes the length of I_k. We now construct $I_{k+1} \subset I_k$ so that (8) and (9) hold. Consider the intervals $[ja_k, jb_k]$, $d^{r(k-1)}+1 \leqslant j \leqslant d^{rk}$. These are disjoint and the distance between them is at least 1 for

$$\begin{aligned}(j+1)a_k - jb_k &= a_k - j(b_k - a_k)\\ &= a_k - jd^{-r(k-1)} l\\ &\geqslant a_k - d^{rk} d^{-r(k-1)} l \geqslant d^r l + 1 - d^r l \quad \text{by (7)}\\ &= 1.\end{aligned}$$

By (8) there is a $u = u(k)$ such that

$$n_{u-1} + l + 1 < d^{r(k-1)} b_k < n_u. \tag{10}$$

Suppose that $x \in jI_k$ for some $d^{r(k-1)} < j \leqslant d^{rk}$ and $[x] = n_v$ for some v. Then clearly $jb_k \geqslant n_v$ and so by (10)

$$j > d^{r(k-1)} d^{v-u}. \tag{11}$$

But $j \leqslant d^{rk}$ and so $u \leqslant v < u + r$. Clearly

$$\{x\colon\, x \in jI_k,\ [x] = n_v\}$$

is a sub-interval of jI_k with length at most 1.

Put

$$T_v = \{x\colon\, x \in d^{rk} I_k,\ [xj/d^{rk}] = n_v\}.$$

Then T_v is an interval and has length

$$l(T_v) \leqslant d^r/j \leqslant d^{r-v+u}$$

since the intervals jI_k have mutual distance at least 1.

Put $T = \bigcup_v T_v$. Then T is the union of at most r intervals, and the Lebesgue measure of T is at most

$$\sum_{v=u}^{u+r-1} d^{r-v+u} = \sum_{t=1}^{r} d^t = \frac{d(d^r-1)}{d-1}.$$

Hence the complement of T in $d^{rk} I_k$ is the union of at most $(r+1)$ intervals, $K_1, \ldots, K_{r+1}$ say, of length $m_i l$ respectively. Then

$$\sum_{i=1}^{r+1} m_i l = d^r l - m(T) \geqslant d^r l - \frac{d(d^r-1)}{d-1}.$$

Thus

$$\sum_{i=1}^{r+1} m_i \geq d^r - \frac{d(d^r-1)}{l(d-1)}$$

and so

$$\sum_{i=1}^{r+1} [m_i] \geq d^r - \frac{d(d^r-1)}{l(d-1)} - (r+1) \geq d^r - (r+2) \quad \text{by (4)}.$$

Hence we can find at least $d^r-(r+2)$ disjoint sub-intervals of $d^{rk}I_k$ of length l which do not meet T. Choose one of these, J, arbitrarily, and put

$$I_{k+1} = \frac{J}{d^{rk}}.$$

The construction is now complete and clearly (8) and (9) hold.

At each stage in this construction we have $d^r-(r+2)$ distinct choices for each interval I_{k+1}. Let S be the set of all possible numbers obtained in the construction above. We employ the following result due to Eggleston [1] to show that the H. dimension of S is at least s_0.

THEOREM (Eggleston). *Suppose A_k $(k = 1, 2, \ldots)$ is a linear set consisting of N_k closed intervals each of length δ_k. Let each interval of A_k contain $m_{k+1} > 0$ disjoint intervals of A_{k+1}.*

Suppose that $0 < s_0 \leq 1$ and that for all $s < s_0$ the sum

$$\sum_k \frac{\delta_{k-1}}{\delta_k} (N_k(\delta_k)^s)^{-1} \text{ converges}.$$

Then $P = \bigcap_{k=1}^{\infty} A_k$ has dimension greater than or equal to s_0.

We apply this theorem with $N_k = (d^r-(r+2))^{k-1}$, A_k = {possible intervals at the kth stage in the construction} and $\delta_k = ld^{-r(k-1)}$.

Then

$$\sum_k \frac{\delta_{k-1}}{\delta_k} (N_k(\delta_k)^s)^{-1} = d^r \sum_k [(d^r-(r+2))^{k-1}(ld^{-r(k-1)})^s]^{-1}$$

$$= d^r l^s \sum_k [(d^r-r-2)^{-1} d^{rs}]^{k-1}$$

which converges if and only if $d^{rs}/(d^r-r-2) < 1$.

But by (2) $d^r-r-2 > d^{r_0}$ and so

$$d^{rs}/(d^r-r-2) < d^{r(s-s_0)}$$

and so the sum will converge for all $s < s_0$ and hence by Eggleston's Theorem

$$\text{H.dim. } S \geq s_0.$$

THEOREM 2. *If (n_k) is a sequence of integers for which $(n_k\theta)$ is dense in the unit interval $[0, 1]$ for all irrationals θ, then every generalised arithmetic progression $[at+\beta]$, $t = 1, 2, \ldots$, for which a is irrational and β is any real number, meets the sequence (n_k) infinitely often.*

Unfortunately Theorem 2 gives us no information about what happens if a is rational. For example $[4t+4]$, $t = 1, 2, \ldots$, is contained in the complement of the sequence p_k, where p_k denotes the kth prime, but $\{p_k\theta\}$ is dense in the unit interval for all irrationals θ. (See, for example, Vinogradov [7] or Vaughan [6].) However by adding points to the sequence (n_k) we obtain the following

COROLLARY 1. *If $\{n_k\theta\}$ is dense for all irrationals θ, then we can construct a sequence (t_k) with the same asymptotic density as and containing (n_k) such that (t_k) meets every generalised arithmetic progression infinitely often.*

Proof. We obtain (t_k) by adding points to the sequence (n_k). By Theorem 2 there are at most countably many generalised arithmetic progressions which do not meet (n_k). Order these in such a way that each generalised arithmetic progression appears infinitely often in this ordering. Let A_n represent the nth element of this ordering. Let $f(n)$ be a function growing as quickly as we like. To each integer n insert the first element of A_n which is larger than $f(n)$ into the sequence (n_k). This gives a new sequence (t_k) and by choosing f to grow sufficiently fast we can satisfy all of the conditions necessary to prove the corollary.

Proof of Theorem 2. We are required to prove that given any irrational $a > 0$ and any real β there are

$$t_i = t_i(a, \beta) \quad \text{and} \quad k_i = k_i(a, \beta), \quad i = 1, 2, \ldots,$$

with

$$[at_i+\beta] = n_{k_i}, \quad i = 1, 2, \ldots$$

Since a is irrational so is $1/a$ and thus by the hypothesis of the theorem $\{n_k(1/a)\}$ and hence

$$\{n_k(1/a)+(1/a)-(\beta/a)\} = \{(n_k+1)(1/a)-(\beta/a)\}$$

is dense in $[0, 1]$.

Then given any $0<\varepsilon<1/(2a)$ we can find natural numbers $k_1<k_2<\ldots$ such that

$$\varepsilon < \{(n_{k_i}+1)(1/a)-(\beta/a)\} < 2\varepsilon, \quad i = 1, 2, \ldots$$

Hence there are natural numbers $t_1 < t_2 < \ldots$ such that

$$n_{k_i}+1-2a\varepsilon < at_i+\beta < n_{k_i}+1-a\varepsilon.$$

Now $0 < 2a\varepsilon < 1$ and consequently

$$[at_i+\beta] = n_{k_i}, \quad i = 1, 2, \ldots$$

as required.

COROLLARY 2 (of Theorem 2). *If (n_k) is a lacunary sequence there is an uncountable set of real numbers U with Hausdorff dimension equal to 1 such that if $\theta \in U$ then $\{n_k\theta\}$ are not dense in $[0,1]$.*

A similar result has recently been obtained independently by B. de Mathan [3], [4].

Proof. Put $U = \{x: x = 1/\alpha,\ \alpha \in T\}$ where T is the set of the corollary to Theorem 1.

Now suppose that $\theta \in U$ and $\{n_k\theta\}$ are dense in $[0,1]$. Then as in the proof of Theorem 2 we can find integers k and t so that $[(1/\theta)t] = n_k$. But this contradicts the fact that $1/\theta \in T$. Hence $\{n_k\theta\}$ are not dense in $[0,1]$. It now remains to show that

$$\text{H. dim. } U = 1.$$

We use the following theorem (Rogers [5], p. 53):

THEOREM. *Let f: $E \to R$, where $E \subseteq R$, and satisfy the condition*

$$|f(x_1)-f(x_2)| \leqslant C_1|x_1-x_2|$$

for all x_1, x_2 in E where C_1 is a positive constant. Then for all $s > 0$

$$\Lambda^s(f(E)) \leqslant C_2\Lambda^s(E)$$

where Λ^s is the s-dimensional Hausdorff measure and C_2 is a real positive constant.

We apply this theorem with

$$f(x) = 1/x, \quad E_n = \{x \in U\colon (1/x) \in T \cap S(1-1/n)\}.$$

Then $U = \bigcup_n E_n$ and $f(E_n) = S(1-1/n)$.

Suppose $x_1, x_2 \in E_n$. Then

$$1/b_1 \leqslant x_1, x_2 \leqslant 1/a_1$$

where $[a_1, b_1] = I_1$ is the first interval in the construction of $S(1-1/n)$. Hence

$$|f(x_1)-f(x_2)| = \frac{1}{x_1x_2}|x_1-x_2| \leqslant b_1^2|x_1-x_2|$$

and so by the theorem above

$$\Lambda^s(E_n) \geqslant C_2\Lambda^s(F(E_n)) = C_2\Lambda^s(S(1-1/n)).$$

But $\Lambda^s(S(1-1/n)) > 0$ for all $s < 1-1/n$, and hence $\Lambda^s(E_n) > 0$ for all $s < 1-1/n$ and so H. dim. $E_n \geqslant 1-1/n$. Hence

$$\text{H. dim. } U = \text{H. dim.}\Big(\bigcup_n E_n\Big) = 1$$

as required.

3. Generalised geometric progressions. Here we prove the result about generalised geometric progressions mentioned in the introduction.

THEOREM 3. *Suppose that $d > 1$ and $0 \leqslant a < d$ are integers. Then there are uncountably many real numbers α for which*

$$[\alpha^n] \equiv a \pmod d, \quad n = 1, 2, \ldots$$

Proof. To prove this theorem we note that it is sufficient to show that there are uncountably many α for which

$$\{d^{-1}\alpha^n\} \in [a/d, (a+1)/d), \quad n = 1, 2, \ldots,$$

for then, $d^{-1}\alpha^n \in [(a/d)+k, (a+1)/d+k)$, and so

$$\alpha^n \in [a+kd, a+1+kd),$$

i.e., $[\alpha^n] \equiv a \pmod d$.

We will construct intervals $I_1 \supset I_2 \supset \ldots$ as follows: Put $I_1 = [a+ +k_1d, a+1+k_1d)$ where $k_1 \geqslant 3$ is an integer. Suppose that I_j has been constructed so that if $I_j = [a_j, b_j]$, then

$$a_j^j = a+dk_j, \quad b_j^j = a+1+dk_j.$$

We now construct I_{j+1}. Clearly $b_j^j - a_j^j = 1$, whereas

$$b_j^{j+1}-a_j^{j+1} \geqslant b_j(b_j^j-a_j^j) = b_j > a_1 \geqslant 3d.$$

Therefore there are at least two closed intervals of length 1 in $[a_j^{j+1}, b_j^{j+1}]$ with integer end points and with left end point congruent to $a \pmod d$. We choose one of these arbitrarily and define $I_{j+1} \subset I_j$ as follows.

Let

$$a_{j+1}^{j+1} = a+dk_{j+1} \quad \text{and} \quad b_{j+1}^{j+1} = a+1+dk_{j+1}$$

where $[a_{j+1}^{j+1}, b_{j+1}^{j+1}] \subset [a_j^{j+1}, b_j^{j+1})$ and k_{j+1} is an integer.

Put $I_{j+1} = [a_{j+1}, b_{j+1})$.

Let $a \in \bigcap_{i=1}^{\infty} I_i$ then

$$\{d^{-1}a^n\} \in \left[\frac{a}{d}, \frac{a+1}{d}\right), \quad n = 1, 2, \ldots$$

There are uncountably many such numbers since at each stage in the construction there are two disjoint choices for I_{j+1}.

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On a result of Littlewood concerning prime numbers

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1. Introduction. We define

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) \tag{1.1}$$

where

$$\Lambda(n) = \begin{cases} \log p, & n = p^m, p \text{ prime}, m \text{ integer} \geqslant 1, \\ 0 & \text{otherwise}. \end{cases} \tag{1.2}$$

The prime number theorem is equivalent to

$$\psi(x) \sim x \quad (\text{as } x \to \infty). \tag{1.3}$$

Assuming the Riemann Hypothesis (the RH), we have the more precise result

$$\psi(x) - x = O(x^{1/2}\log^2 x) \tag{1.4}$$

and, on the other hand, we have (without hypothesis)

$$\psi(x) - x = \Omega_{\pm}(x^{1/2}\log\log\log x). \tag{1.5}$$

The result (1.4) is due to von Koch in 1901, while (1.5) was proved by Littlewood in 1914 (see [4], Chapters 4, 5). Presumably (1.5) is nearer to the truth. The basis for these results is the explicit formula for $\psi(x)$:

$$\frac{\psi(x+0)+\psi(x-0)}{2} = x - \sum_{\varrho} \frac{x^\varrho}{\varrho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}) \tag{1.6}$$

the summation being over the non-trivial zeros of the zeta function, $\varrho = \beta + i\gamma$. (The RH allows us to take $\beta = 1/2$.) The series in (1.6) is neither absolutely nor uniformly convergent, and is understood as

$$\sum_{\varrho} \frac{x^\varrho}{\varrho} = \lim_{T\to\infty} \sum_{|\gamma|<T} \frac{x^\varrho}{\varrho}.$$