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On the counting function for sums of two squares

by

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1. Introduction. For each nonnegative integer n, $r(n) = r_2(n)$ denotes the number of representations of n as a sum of two squares. In any such representation $n = x^2+y^2$, (x, y) is to be regarded as an ordered pair of integers. The function δ is then defined for each positive integer n by: $r(n) = 4\delta(n)$. J. W. L. Glaisher derived in [1] a recursive formula for δ. His result we here state as

THEOREM 1. *For each positive integer n,*

$$(1)\quad \sum_{k=0} (-1)^{k(k+1)/2}\delta\big(n-k(k+1)/2\big) = \begin{cases} (-1)^{[m/2]}[(m+1)/2], & \text{if } n = m(m+1)/2, \\ 0, & \text{otherwise}. \end{cases}$$

Here, $[x]$ denotes for any real number x the largest integer not exceeding x; and, summation extends as far as the arguments of δ remain positive.

The major objective of this note is to give an easy proof of a theorem equivalent to Glaisher's Theorem 1. This result we state as

THEOREM 2. *For each nonnegative integer n,*

$$(2)\quad \sum_{k=0} (-1)^{k(k+1)/2} r\big(n-k(k+1)/2\big) = \begin{cases} (-1)^{m(m+3)/2}(2m+1), & \text{if } n = m(m+1)/2, \\ 0, & \text{otherwise}. \end{cases}$$

Here, summation extends as far as the arguments of r remain nonnegative.

We observe that $r(0) = 1$, and then establish the equivalence of recurrences (1) and (2) for positive arguments by use of the identity

$$4(-1)^{[m/2]}[(m+1)/2]+(-1)^{[(m+1)/2]} = (-1)^{m(m+3)/2}(2m+1).$$

Section 2 is devoted to the proof of Theorem 2. However, in our concluding remarks we mention another type of recursive formula for r, which, though not very efficient for tabulation of values, has some theoretical interest. Before embarking on technical development, we state four well-known identities to be used in our proof.

$$\prod_{n=1}^{\infty}(1-x^{2n-1})(1+x^n)=1, \tag{3}$$

$$\prod_{n=1}^{\infty}(1-x^n)(1-x^{2n-1}) = \sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}, \tag{4}$$

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^n) = \sum_{n=0}^{\infty} x^{n(n+1)/2}, \tag{5}$$

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{n(n+1)/2}. \tag{6}$$

(3) is due to Euler, (4) and (5) to Gauss, and (6) to Jacobi. For proofs, see [2], pp. 277–285.

2. Proof of Theorem 2. By use of (3), we express (5) as follows:

$$\prod_{n=1}^{\infty}(1-x^n)(1-x^{2n-1})^{-2} = \sum_{n=0}^{\infty} x^{n(n+1)/2}.$$

We now multiply the foregoing identity by the square of identity (4) to get

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \Big\{\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}\Big\}^2\Big\{\sum_{n=0}^{\infty}x^{n(n+1)/2}\Big\}.$$

(6) and the last identity then imply

$$\Big\{\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}\Big\}^2\Big\{\sum_{n=0}^{\infty}x^{n(n+1)/2}\Big\} = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{n(n+1)/2},$$

whence

$$\Big\{\sum_{n=0}^{\infty}r(n)x^n\Big\}\Big\{\sum_{n=0}^{\infty}(-x)^{n(n+1)/2}\Big\} = \sum_{n=0}^{\infty}(-1)^{n(n+3)/2}(2n+1)x^{n(n+1)/2}.$$

Expanding the left side of this identity and equating coefficients of like powers in the resulting identity, we thus prove our theorem.

Remarks. For large n, we observe that the left side of (2) has about $\sqrt{2n}$ terms. Hence, our recursive scheme is indeed efficient.

For each positive integer n, $\sigma(n)$ denotes the sum of the positive divisors of n. In the statement of our final result we shall also use the representation of an arbitrary positive integer n as $n = 2^{b(n)}O(n)$, where $b(n)$ is a nonnegative integer and $O(n)$ is odd.

THEOREM 3. *For each positive integer* n,

$$nr(n) = 4\sum_{j=1}^{n}(-1)^{j-1}r(n-j)2^{b(j)}\sigma\big(O(j)\big).$$

The theorem is easily proved by using identity (4) and the technique of logarithmic differentiation. Moreover, the theorem remains valid when r_2 is everywhere replaced by r_{2k} and the factor of 4 is replaced by $4k$, where k is an arbitrary positive integer, and for each nonnegative integer n, $r_{2k}(n)$ denotes the number of representations of n as a sum of $2k$ squares.

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References

[1] J. W. L. Glaisher, *On the function which denotes the difference between the number of* $(4m+1)$*-divisors and the number of* $(4m+3)$*-divisors of a number*, Proc. London Math. Soc. (1) 15 (1884), pp. 104–122.

[2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford University Press, 1960.

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