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On a certain infinite series for a periodic arithmetical function

by

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1. Introduction. Let $q \geqslant 2$ be an integer and let f be a function defined on the ring of integers $\boldsymbol{Z}$ with period q. Then Baker, Birch and Wirsing proved that if f satisfies the three conditions (A), (B) and (C) below, then $f = 0$ ([2], Theorem 1).

(A) $$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0.$$

(B) $f(1), \ldots, f(q)$ are algebraic and Φ_q is irreducible over $\boldsymbol{Q}(f(1), \ldots \ldots, f(q))$, where Φ_q denotes the qth cyclotomic polynomial and $\boldsymbol{Q}$ denotes the field of rationals.

(C) $f(r) = 0$ if $1 < (r, q) < q$.

This resolved in the negative a well-known problem of Chowla as to whether there exists a rational-valued function f periodic with prime period for which (A) holds.

The main purpose of this note is to prove a result which provides a description of all functions f such that (A) and (B) hold. It can be stated as follows: If f satisfies (B), then (A) holds if and only if $(f(1), \ldots, f(q))$ is a solution of a certain system of $\varphi(q)+t(q)$ homogeneous linear equations with rational coefficients, where $t(q)$ denotes the number of primes dividing q (see Theorem 10 for the precise statement). Thus, in particular, it reveals that if $2\varphi(q)+1 > q$ and $f(n) \in \{1, -1\}$ when $n = 1, \ldots, q-1$ and $f(q) = 0$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent (Corollary 16). This gives a partial answer for a conjecture of Erdös ([4], p. 430).

Our argument is a slight modification of that of Baker, Birch and Wirsing and depends on a combination of the basic result on the linear independence of the logarithms of algebraic numbers due to Baker [1], Theorem 1 with the nonvanishing of the Dirichlet L-series at $s = 1$.

2. Notations and definitions. As in Section 1, we denote by q any natural number $\geqslant 2$ and by f a function defined on $\boldsymbol{Z}$ with period q. We denote by $D = D_q$ the set of all Dirichlet characters to the modulus q. We put $\zeta = e^{2\pi i/q}$. We denote by P the set of all primes dividing q. For $p \in P$ and $n \in \boldsymbol{Z}$ we denote by $\nu_p(n)$ the exponent to which p divides n. For simplicity we write

$$J = \{a \in \boldsymbol{Z}\colon 1 \leqslant a \leqslant q \text{ and } (a, q) = 1\},$$
$$L = \{r \in \boldsymbol{Z}\colon 1 \leqslant r \leqslant q \text{ and } 1 < (r, q) < q\},$$

and

$$L' = L \cup \{q\}.$$

For $a \in J$ we denote by $\bar{a}$ the integer for which $\bar{a} \in J$ and $\bar{a}a \equiv 1 \pmod q$.

We define for $r \in L'$

$$\Delta(r) = \sum_{p \in P(r)} \frac{\log p}{p-1} + \log(r, q), \tag{1}$$

where

$$P(r) = \{p \in P\colon \nu_p(r) \geqslant \nu_p(q)\}.$$

Note that if we define for $r \in L'$ and $p \in P$

$$s(r, p) = \begin{cases} \nu_p(q) + 1/(p-1) & \text{if } p \in P(r), \\ \nu_p(r) & \text{otherwise,} \end{cases}$$

then we have

$$\Delta(r) = \sum_{p \in P} s(r, p) \log p. \tag{2}$$

We define further for $r \in L$ and $a \in J$

$$A(r, a) = \frac{1}{(r, q)} \prod_{p \in P(r)} \left(1 - \frac{1}{p^{\varphi(q)}}\right)^{-1} \sum_{n \in S(r)} \frac{\delta(r, a, n)}{n},$$

where

$$S(r) = \Big\{\prod_{p \in P(r)} p^{a(p)}\colon\ 0 \leqslant a(p) < \varphi(q)\Big\}$$

and

$$\delta(r, a, n) = \begin{cases} 1 & \text{if } r \equiv a(r, q)n \pmod q, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. As usual we adopt the convention that the sum (resp. the product) of an empty set of numbers is 0 (resp. 1). Therefore we have $S(r) = \{1\}$ when $P(r)$ is empty.

3. Preliminary results. We define

$$H(n) = H_q(n) = -\frac{1}{q} \sum_{r=1}^{q-1} \zeta^{-rn} \log(1 - \zeta^r) \quad (n \in \boldsymbol{Z}).$$

The function $H(n)$ arose essentially in Lehmer's work [3] (cf. also [5]) and was used to evaluate the infinite series $\sum f(n)/n$. We note that $H(n) = \gamma(n, q) - \gamma/q$, where $\gamma(n, q)$ is the Euler constant for the arithmetical progression $n + mq$ $(m = 1, 2, \ldots)$ and γ is Euler's constant ([3], Theorem 1).

Lemma 1 ([3], Theorem 8). *We have*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{r=1}^{q} f(r) H(r)$$

provided $\sum_{r=1}^{q} f(r) = 0$; *which is a necessary and sufficient condition for convergence of the infinite series* $\sum_{n=1}^{\infty} f(n)/n$.

Corollary 2. (A) *holds if and only if* $(f(1), \ldots, f(q))$ *is a solution of the system*:

$$\begin{cases} \sum_{r=1}^{q} f(r) H(r) = 0, \\ \sum_{r=1}^{q} f(r) = 0. \end{cases}$$

We set

$$\hat{H}(\chi) = \frac{1}{\varphi(q)} \sum_{a \in J} H(a) \bar{\chi}(a) \quad (\chi \in D). \tag{3}$$

Clearly (3) is inverted by

$$H(a) = \sum_{\chi \in D} \hat{H}(\chi) \chi(a) \quad (a \in \boldsymbol{Z},\ (a, q) = 1). \tag{4}$$

Lemma 3. *We have*

$$\hat{H}(\chi_0) = \frac{1}{q} \sum_{p \in P} \frac{\log p}{p-1}, \tag{5}$$

where χ_0 *is the principal character to the modulus* q. *For* $\chi \neq \chi_0$ *we have*

$$\hat{H}(\chi) = \frac{1}{\varphi(q)} L(1, \bar{\chi}). \tag{6}$$

Proof. (5) follows from the formulas (6) and (16) in [3] and (6) follows from Lemma 1 and (3).

Lemma 4 ([3], Theorem 2). *Let $1 \leqslant d < q$ be a common divisor of r and q. Then*

$$H_q(r) = \frac{1}{d} H_{q/d}\left(\frac{r}{d}\right) - \frac{1}{q}\log d. \tag{7}$$

Remark. If we define $H_1(n) = 0$, then (7) is also true for $d = q$, since we have

$$H_q(q) = -\frac{1}{q}\log q \tag{8}$$

by (2) in [3].

By Lemma 3 we have that $\hat{H}(\chi) \neq 0$ for all $\chi \in D$, which enables us to define the function

$$K(a) = \frac{1}{\varphi(q)^2} \sum_{\chi \in D} \frac{\chi(a)}{\hat{H}(\chi)} \qquad (a \in \mathbf{Z}, (a, q) = 1). \tag{9}$$

The following is easily proved by using (1), (4), (8), (9), Lemmas 3 and 4 and the orthogonality properties of the characters and we omit its proof.

Lemma 5. (i) *If $a, c \in J$, then*

$$\sum_{b \in J} K(\bar{a}b) H(\bar{b}c) = \begin{cases} 1 & \text{if } a = c, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) $$\sum_{a \in J} K(a) = \frac{1}{\varphi(q)\hat{H}(\chi_0)}.$$

(iii) *If $r \in L$, then*

$$\sum_{a \in J} H(ar) = \varphi(q)\left\{\hat{H}(\chi_0) - \frac{1}{q}\Delta(r)\right\}.$$

(iv) $$H(q)\sum_{a \in J} K(a) = \frac{1}{\varphi(q)}\left\{1 - \frac{\Delta(q)}{q\hat{H}(\chi_0)}\right\}.$$

Lemma 6. *If $a \in J$ and $r \in L$, then*

$$\sum_{b \in J} K(\bar{a}b) H(\bar{b}r) = A(r, a) - \frac{\Delta(r)}{q\varphi(q)\hat{H}(\chi_0)}.$$

Proof. By Lemma 4 the left-hand side of the above is equal to

$$\frac{1}{d}\sum_{b \in J} K(\bar{a}b) H'\left(\bar{b}\frac{r}{d}\right) - \frac{1}{q}\log d \sum_{b \in J} K(\bar{a}b), \tag{10}$$

where $d = (r, q)$ and $H'(n) = H_{q/d}(n)$. By Lemma 5 (ii) the second sum in (10) is equal to

$$\frac{\log d}{q\varphi(q)\hat{H}(\chi_0)}.$$

Applying (4), (9) and the orthogonality properties of the characters to the first sum in (10) reduces it to

$$\frac{1}{d\varphi(q)} \sum_{\psi \in D'} \frac{\hat{H}'(\psi)}{\hat{H}(\chi_0\psi)} \psi\left(\bar{a}\frac{r}{d}\right), \tag{11}$$

where $D' = D_{q/d}$. Since for any nonprincipal character $\psi \in D'$ we have by (6)

$$\frac{\hat{H}'(\psi)}{\hat{H}(\chi_0\psi)} = \frac{\varphi(q)}{\varphi(q/d)} \prod_{p \in P(d)} \left(1 - \frac{\psi(p)}{p}\right)^{-1}$$

and we have

$$\frac{\varphi(q)}{\varphi(q/d)} = d \prod_{p \in P(d)} \left(1 - \frac{1}{p}\right), \tag{12}$$

(11) becomes

$$\frac{1}{d\varphi(q/d)} \sum_{\psi \in D'} \psi\left(\bar{a}\frac{r}{d}\right) \prod_{p \in P(d)} \left(1 - \frac{\bar{\psi}(p)}{p}\right)^{-1} - \frac{1}{\varphi(q)} + \frac{\hat{H}'(\psi_0)}{d\varphi(q)\hat{H}(\chi_0)}, \tag{13}$$

where ψ_0 is the principal character to the modulus q/d. If we put

$$R(d) = \left\{\prod_{p \in P(d)} p^{\alpha(p)} \colon 0 \leqslant \alpha(p) < \infty\right\},$$

then the first sum in (13) becomes

$$\frac{1}{d\varphi(q/d)} \sum_{n \in R(d)} \frac{1}{n}\left\{\sum_{\psi \in D'} \psi\left(\bar{a}\frac{r}{d}\right)\bar{\psi}(n)\right\} = \frac{1}{d}\sum_{n \in R(d)} \frac{\delta(r, a, n)}{n} = A(r, a),$$

since

$$\sum_{\psi \in D'} \psi\left(\bar{a}\frac{r}{d}\right)\bar{\psi}(n) = \varphi\left(\frac{q}{d}\right)\delta(r, a, n)$$

and

$$n^{\varphi(q)} \equiv 1 \left(\operatorname{mod} \frac{q}{d}\right)$$

for any $n \in R(d)$. Lastly we have

$$\frac{1}{\varphi(q)}\left\{-1+\frac{\hat{H}'(\varphi_0)}{d\hat{H}(\chi_0)}-\frac{\log d}{q\hat{H}(\chi_0)}\right\} = -\frac{\Delta(r)}{q\varphi(q)\hat{H}(\chi_0)}$$

in view of (5) and (1). Combining these results, we get the lemma.

The following lemma plays a crucial role in the proof of our main theorem (Theorem 10) and is a reformulation of Lemmas 2 and 4 in [2], whose proofs rest on an application of Theorem 1 of [1] relating to the linear forms in the logarithms of algebraic numbers.

LEMMA 7. *Let $a_1, \ldots, a_q$ be algebraic numbers such that Φ_q is irreducible over $Q(a_1, \ldots, a_q)$. If*

$$\sum_{r=1}^{q} a_r H(r) = 0,$$

then

$$\sum_{r=1}^{q} a_r H(ar) = 0$$

for any integer a with $(a, q) = 1$.

THEOREM 8. *Let K be an algebraic number field such that Φ_q is irreducible over K. Then the numbers $H(a)$, $a \in J$ are linearly independen over K.*

Proof. Suppose that there exist $a_c \in K$ such that

$$\sum_{c\in J} a_c H(c) = 0.$$

Then by Lemma 7 we get

$$\sum_{c\in J} a_c H(\bar{b}c) = 0 \quad (b \in J).$$

Multiplying both members of the above by $K(\bar{a}b)$ and summing over $b \in J$ gives us

$$a_a = 0 \quad (a \in J)$$

in view of Lemma 5 (i). This proves the theorem.

The following is a slight generalization of [2], Corollary 1 to Theorem 1.

COROLLARY 9. *Let $(q, \varphi(q)) = 1$ and let χ run through the nonprincipal characters to the modulus q. Then the numbers $\sum_{p\in P}\log p/(p-1)$ and $L(1, \chi)$ are linearly independent over Q.*

Proof. By Lemma 3 it suffices to prove that $\hat{H}(\chi)$, $\chi \in D$ are linearly independent over Q and this follows immediately from (3) and Theorem 8 on noting that Φ_q is irreducible over the $\varphi(q)$-th cyclotomic number field and the matrix $[\chi(a)]$ $(\chi \in D, a \in J)$ is nonsingular.

4. Results. Our main theorem is as follows.

THEOREM 10. *If f satisfies (B), then (A) holds if and only if $(f(1), \ldots, f(q))$ is a solution of the following system of $\varphi(q)+t(q)$ homogeneous linear equations with rational coefficients:*

$$\begin{cases} f(a)+\sum_{r\in L} f(r)\Delta(r, a)+\frac{1}{\varphi(q)}f(q) = 0 & (a \in J), \\ \sum_{r\in L'} f(r)\varepsilon(r, p) = 0 & (p \in P). \end{cases}$$

Proof. Since f satisfies (B), we have by Corollary 2 and Lemma 7 that (A) holds if and only if $(f(1), \ldots, f(q))$ is a solution of the system:

$$(14)_1 \qquad \sum_{r=1}^{q} x_r H(\bar{b}r) = 0 \quad (b \in J),$$

$$(14)_2 \qquad \sum_{r=1}^{q} x_r = 0.$$

Hence the following two lemmas lead to the proof of the theorem.

LEMMA 11. *The complete solution of the system* (14) *is given by*

$$(15) \qquad x_a = -\sum_{r\in L} x_r \Delta(r, a) - \frac{1}{\varphi(q)} x_q \quad (a \in J),$$

$$(16) \qquad x_q = -\frac{1}{\Delta(q)}\sum_{r\in L} x_r \Delta(r).$$

Proof. If we sum both sides of $(14)_1$ over the $\varphi(q)$ numbers $b \in J$, we obtain

$$\sum_{c\in J} x_c \sum_{b\in J} H(\bar{b}c) + \sum_{r\in L} x_r \sum_{b\in J} H(\bar{b}r) + x_q \sum_{b\in J} H(\bar{b}q)$$
$$= \Big(\sum_{c\in J} x_c\Big)\varphi(q)\hat{H}(\chi_0) + \sum_{r\in L} x_r\varphi(q)\Big\{\hat{H}(\chi_0) - \frac{1}{q}\Delta(r)\Big\} + x_q\varphi(q)H(q)$$
$$= \Big(\sum_{c\in J} x_c + \sum_{r\in L} x_r\Big)\varphi(q)\hat{H}(\chi_0) - \frac{\varphi(q)}{q}\sum_{r\in L} x_r\Delta(r) + x_q\varphi(q)H(q)$$
$$= -\frac{\varphi(q)}{q}\sum_{r\in L'} x_r\Delta(r) = 0$$

in view of Lemma 5 (iii) and $(14)_2$ and the fact that

$$H(q)-\hat{H}(\chi_0) = -\frac{1}{q}\,\Delta(q).$$

Solving for x_q we get (16).

We next multiply both members of $(14)_1$ by $K(\bar{a}b)$ and sum over $b \in J$ to obtain

$$\sum_{c\in J} x_c \sum_{b\in J} K(\bar{a}b)H(\bar{b}c) + \sum_{r\in L} x_r \sum_{b\in J} K(\bar{a}b)H(\bar{b}r) + x_q H(q) \sum_{b\in J} K(\bar{a}b)$$

$$= x_a + \sum_{r\in L} x_r A(r, a) - \frac{1}{q\varphi(q)\hat{H}(\chi_0)} \sum_{r\in L} x_r \Delta(r) + \frac{1}{\varphi(q)}\, x_q \left\{1 - \frac{\Delta(q)}{q\hat{H}(\chi_0)}\right\}$$

$$= x_a + \sum_{r\in L} x_r A(r, a) + \frac{1}{\varphi(q)}\, x_q = 0,$$

where we have used Lemma 5 (i), (iv), Lemma 6 and (16). Solving for x_a we get (15) and this completes the proof of the lemma.

LEMMA 12. *Let $x_1, \ldots, x_q$ be algebraic numbers. Then $(x_1, \ldots, x_q)$ is a solution of the system (14) if and only if it satisfies the following system:*

$$\begin{cases} x_a + \sum\limits_{r\in L} x_r A(r, a) + \dfrac{1}{\varphi(q)}\, x_q = 0 & (a \in J), \\ \sum\limits_{r\in L'} x_r \varepsilon(r, p) = 0 & (p \in P). \end{cases}$$

Proof. By (16) and (2) we have

$$\sum_{r\in L'} x_r \Delta(r) = \sum_{p\in P} \Big\{\sum_{r\in L'} x_r \varepsilon(r, p)\Big\} \log p = 0,$$

which implies that

$$\sum_{r\in L'} x_r \varepsilon(r, p) = 0 \qquad (p \in P),$$

since $\log p$, $p \in P$ are linearly independent over the field of all algebraic numbers by the fundamental theorem of arithmetic and Theorem 1 of [1]. This together with Lemma 11 proves the lemma.

The following two corollaries follow immediately from Theorem 10 and the theory of homogeneous linear equations on noting that the $\varphi(q)+t(q)$ linear forms in Theorem 10 are linearly independent.

COROLLARY 13. *Let K be an algebraic number field such that Φ_q is irreducible over K. Denote by F_q the set of all functions $f\colon \mathbf{Z} \to K$ with*

period q such that (A) holds. Then F_q is a vector space of the dimension $q-\varphi(q)-t(q)$ over K and has a basis of functions $h\colon \mathbf{Z} \to \mathbf{Z}$.

COROLLARY 14 (cf. [2], Theorem 1). *Let*

$$q = p_1^{a_1} \ldots p_t^{a_t}$$

be the prime power decomposion of q and let $r_1, \ldots, r_t \in L'$ be such that

$$\det[\varepsilon(r_j, p_k)] \neq 0 \qquad (j, k = 1, \ldots, t).$$

Assume that f satisfies (A), (B) and

(C′) $\quad f(r) = 0$ *if* $r \in L' \setminus \{r_1, \ldots, r_t\}$.

Then $f = 0$.

EXAMPLE. Let $q = p_1^{a_1} \ldots p_t^{a_t}$ and let $0 \leqslant \beta_{jk} \leqslant a_j$ $(j, k = 1, \ldots, t)$ be integers such that $\beta_{jk} = 0$ if $j > k$ and $\beta_{jj} > 0$. Put $r_j = \prod\limits_{k=1}^{t} p_k^{\beta_{jk}}$. Then

$$\det[\varepsilon(r_j, p_k)] = \prod_{j=1}^{t} \varepsilon(r_j, p_j) \geqslant \prod_{j=1}^{t} \beta_{jj} > 0.$$

COROLLARY 15. *If f satisfies (A) and (B), then*

$$|f(a)| \leqslant \left(\frac{q-1}{\varphi(q)} - 1\right) M + \frac{1}{\varphi(q)}|f(q)| \qquad (a \in J),$$

where

$$M = \max_{r\in L} |f(r)|.$$

Proof. By Theorem 10 we have for $a \in J$

$$f(a) = -\sum_{r\in L} f(r) A(r, a) - \frac{1}{\varphi(q)} f(q) \tag{17}$$

$$= -\sum_{\substack{d|q \\ 1<d<q}} \sum_{\substack{m=1 \\ (m, q/d)=1}}^{q/d} f(dm) A(dm, a) - \frac{1}{\varphi(q)} f(q)$$

$$= -\sum_{\substack{d|q \\ 1<d<q}} \frac{1}{d} \prod_{p\in P(d)} \left(1 - \frac{1}{p^{\varphi(q)}}\right)^{-1} \cdot \sum_{n\in S(d)} \frac{f(a\bar{d}n)}{n} - \frac{1}{\varphi(q)} f(q).$$

From this we obtain

$$|f(a)| \leqslant M \sum_{\substack{d|q \\ 1<d<q}} \frac{1}{d} \prod_{p\in P(d)} \left(1-\frac{1}{p^{\varphi(d)}}\right)^{-1} \cdot \sum_{n\in S(d)} \frac{1}{n} + \frac{1}{\varphi(q)} |f(q)|$$

$$= M \sum_{\substack{d|q \\ 1<d<q}} \frac{1}{d} \prod_{p\in P(d)} \left(1-\frac{1}{p}\right)^{-1} + \frac{1}{\varphi(q)} |f(q)|$$

$$= \frac{M}{\varphi(q)} \sum_{\substack{d|q \\ 1<d<q}} \varphi\left(\frac{q}{d}\right) + \frac{1}{\varphi(q)} |f(q)|$$

$$= \frac{M}{\varphi(q)} \big(q-\varphi(q)-1\big) + \frac{1}{\varphi(q)} |f(q)|,$$

where we have used (12) and the well-known fact that

$$\sum_{d|q} \varphi(d) = q.$$

Thus the proof of Corollary 15 is complete.

EXAMPLE. In case $q = p^{\alpha}$, a power of prime p, (17) becomes

$$f(a) = -\sum_{\nu=1}^{\alpha-1} \frac{f(p^{\nu}a)}{p^{\nu}} - \frac{1}{p^{\alpha-1}(p-1)} f(p^{\alpha}) \qquad (a \in J),$$

since $d = p^{\nu}$ $(\nu = 1, \ldots, \alpha-1)$, $P(d)$ is empty and $S(d) = \{1\}$.

Erdös conjectured ([4], p. 430) that if $f(n) \in \{1, -1\}$ when $n = 1, \ldots$ $\ldots, q-1$ and $f(q) = 0$; then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

We can give a partial answer for the conjecture as a direct consequence of Corollary 15.

COROLLARY 16. *If $2\varphi(q)+1 > q$, then the conjecture is true for q.*

EXAMPLE. Assume that $p_0 \geqslant \frac{t}{\log 2}+1$, where $p_0 = \min P$ and $t = t(q)$. Then we have

$$\frac{q-1}{\varphi(q)} < \frac{q}{\varphi(q)} = \prod_{p\in P} \left(1+\frac{1}{p-1}\right) < \exp\left(\sum_{p\in P} \frac{1}{p-1}\right) \leqslant \exp\left(\frac{t}{p_0-1}\right) \leqslant 2.$$

Therefore, if $p_0 \geqslant \frac{t}{\log 2}+1$, the conjecture is true for q.

Remark. If f is even, then Theorem 10 and consequently Corollaries 14 and 15 hold without the assumption that Φ_q is irreducible over $Q(f(1), \ldots$ $\ldots, f(q))$. This follows immediately from [2] (Section 5) and Lemma 12.

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