On the invariants of the Hecke groups

by

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0. Introduction. In [4] K. Mahler defined the concept of an $S_p$-series. The primary example of an $S_p$-series is the modular invariant $j(q)$. In this article we provide additional examples of $S_p$-series by examining the invariants $j_q(a)$ of the Hecke groups $G_q$. Some of the arithmetic consequences for the Fourier coefficients of the invariants for $G(\sqrt{2})$ and $G(\sqrt{3})$ are then discussed.

1. $S_p$-series. Motivated by the behavior of Klein's modular invariant $j(q)$ which satisfies modular equations of order $p$ for every prime $p$, Kurt Mahler [4] considered solutions in formal Laurent series to functional equations of the form

\begin{equation}
(f(a^p))^{p+1} + f(a)^{p+1} + \sum_{r=-p}^{p} \sum_{s=0}^{p} c_{rs}f(a^p)^r f(a)^s = 0, \quad c_{rs} = c_{sr}.
\end{equation}

More specifically, formal series with the following property were studied.

Definition. Let $p$ be a fixed prime. Let $f(z) = \sum_{k=-m}^{\infty} a_k z^k$, $a_m \neq 0$, denote a nonconstant formal ascending Laurent series with complex coefficients. Let $R(f, p)$ denote the following set of $p+1$ derived Laurent series in $a^p$ and $a^{p^2}$:

\[ R(f, p) = \{ f(a^p), f(a^{2p}), f(a^{3p}), f(a^{4p}), \ldots, f(a^{p^2}) \} \]

where $a$ is a $p$th root of unity. Then $f(z)$ is an $S_p$-series of order $m$ if every elementary symmetric function of the elements of $R(f, p)$ can be expressed as a polynomial in $f(z)$.

Associated with each $S_p$-series is the polynomial $P_p(x, y)$ defined by

\[ P_p(x, y) = (x - f(a^p)) \prod_{j=0}^{p-1} (y - f(a^{j+p})) \]
When \( q = 4 \) or 6, the resulting groups are \( G(V^2) \) and \( G(V^3) \). These are the only Hecke groups which are commensurable with the modular group and therefore the only Hecke groups whose elements are completely characterized arithmetically. For notational convenience we let \( l = 2 \) or 3 and represent the transformation \( z' = (az + \beta)/(\gamma z + \delta) \) by a matrix

\[
\begin{pmatrix}
a & \beta \\
\gamma & \delta
\end{pmatrix}
\]

with \( ad - \beta \gamma = 1 \).

Note that both

\[
\begin{pmatrix}
a & \beta \\
\gamma & \delta
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-a & -\beta \\
-\gamma & -\delta
\end{pmatrix}
\]

represent the same linear fractional transformation. It is then well known that \( G(V^l) \) consists of the entirety of elements of the following two forms:

\[
\begin{pmatrix}
a & b \sqrt[l]{i} \\
\alpha \sqrt[l]{i} & d
\end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - \ell bc = 1,
\]

(2.1)

\[
\begin{pmatrix}
a \sqrt[l]{i} & b \\
c & d \sqrt[l]{i}
\end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad d \alpha - bc = 1.
\]

With this characterization of the elements of \( G(V^l) \), we prove

**Theorem 2.1.** \( j_4(\omega) \) and \( j_6(\omega) \) are \( S_p \)-series for all primes \( p \) except \( p = l \).

The proof of Theorem 2.1 relies on

**Lemma 2.2.** For \( q = 4 \) or 6 let \( j(\omega) = j_q(\omega) \) and \( \lambda = \lambda_q \). For \( \ell \neq 1 \), set

\[
T_\ell = \begin{bmatrix} p & 0 & 0 & \cdots & 0 \\ 0 & p & 0 & \cdots & 0 \\ 0 & 0 & \ell & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.
\]

For each \( M \) of \( T_\ell \), set \( j(\omega) = j(M, \omega) \), \( i = 1, \ldots, p + 1 \). Then for any \( \lambda \in \mathbb{C} \),

\[
\{ j(\omega) \} = \{ j(V \omega) \};
\]

in other words, replacing \( \omega \) by \( \lambda \omega \) merely permutes the elements of \( \{ j(\omega) \} \).

**Proof of Theorem 2.1.** We must show that the elementary symmetric functions of the elements of \( B(j, p) \) are polynomials in \( j(\omega) \). However, \( B(j, p) = \{ j_i(\omega) \} \). Since by Lemma 2.2 \( \{ j(V \omega) \} = \{ j(\omega) \} \) for any \( V \in G(\lambda) \), any symmetric combination of elements of \( B(j, p) \)
is invariant under $G(\lambda)$. In particular, the elementary symmetric functions are invariant. Since any function invariant under $G(\lambda)$ and analytic in $x$ is a polynomial in $j(\omega)$, the elementary symmetric functions of elements of $R(j, p)$ are indeed polynomials in $j(\omega)$.

Proof of Lemma 2.2. Since $j(\omega)$ is invariant under $G(\lambda)$ we need only show that for each $M_i \in T_p$, there exists an $M_j \in T_p$, $V_j \in G(\lambda)$ such that $M_jV_j = V_jM_j$ and that the resulting $M_j$ are distinct. In fact it suffices to verify this result for the two generators of $G(\lambda)$,

$$S = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For $S$ we have

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \lambda & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 \leq b \leq p - 2,$$

$$\begin{bmatrix} 1 & (b+1) \lambda \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix}.$$

For $T$ the corresponding identities are

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \quad 0 \leq b \leq p - 1,$$

$$\begin{bmatrix} 1 & (b+1) \lambda \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix}.$$

(2.2) $$\begin{bmatrix} 1 & b \lambda \\ 0 & p \end{bmatrix} = \begin{bmatrix} -b \lambda (1 + b^2 \lambda^2)/p \\ -b \lambda \end{bmatrix}, \quad 1 \leq b \leq p - 1,$$

where $b'$ is the solution to $\lambda^2b = -1 \pmod{p}$ with $1 \leq b' \leq p - 1$.

Before discussing the arithmetical consequences of Theorem 2.1 for the coefficients of $j_3$ and $j_4$, we look briefly at the question of whether any of the $j_q$ are $S_q$-series.

3. $j_q$ for $q \geq 4$. For $q = 5, 7, 8, \ldots$ is $j_q$ an $S_q$-series for some prime $p$?

The easy proof of the preceding section fails at the point in Lemma 2.2, formula (2.2), where we find $V_j \notin G(\sqrt{q})$ so that $M_jT = V_jM_j$. For $q \neq 3, 4, 6$ there is no quick way of determining whether $V_j \in G(\sqrt{q})$. To illustrate this difficulty, we take $p = 2$ and show that $j_q$ is an $S_q$-series if and only if $q = 3$ or 6. For the sake of notational convenience we drop the subscript $q$.

**Theorem 3.1.** $j(\omega)$ is an $S_q$-series if and only if

$$V = \begin{bmatrix} -\lambda & (1 + \lambda^2)/2 \\ -2 & \lambda \end{bmatrix} \in G(\lambda).$$

**Proof.** If $V \in G(\lambda)$, then the method of proof of Theorem 2.1 and Lemma 2.2 carries over to give that $j(\omega)$ is an $S_q$-series. On the other hand, if $j(\omega)$ is an $S_q$-series, then $F(\omega) = j(2\omega) + j(\omega/2) + j(\omega - 1/2 \omega)$ is invariant under $G(\lambda)$ since $F(\omega)$ is a polynomial in $j(\omega)$. In particular,

$$F(T\omega) = F(\omega) = j\left(-\frac{2\omega}{\omega} + j\left(-\frac{1-\omega}{2\omega} + j\left(\frac{\omega - 1}{2\omega}\right) = j\left(\frac{2\omega}{2}\right) + j(2\omega) + j\left(\frac{\omega - 1}{2\omega}\right)$$

which implies that $j\left(\frac{\omega + 1}{2}\right) = j\left(\frac{\omega - 1}{2\omega}\right)$ or, upon replacing $\omega$ by $\omega - 1$,

$$j(\omega) = j\left(\frac{2\omega - \lambda^2 - 1}{4\omega - 2\lambda}\right) = j(\lambda\omega).$$

Then since $G(\lambda)$ is the invariance group for $j(\omega)$, $V \in G(\lambda)$.

It is now clear that $j_3(\omega)$ is not an $S_3$-series since

$$V = \begin{bmatrix} -\sqrt{3} & 3/2 \\ -2 & \sqrt{2} \end{bmatrix}$$

is not in $G(\sqrt{2})$. To prove the same result for $q \geq 5$, $q \neq 6$, we use the following lemma.

**Lemma 3.2.** Suppose

$$V = \begin{bmatrix} a & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{and} \quad V' = \begin{bmatrix} a' & \beta' \\ \gamma' & \delta' \end{bmatrix}$$

are elements of $G(\lambda)$ with $a'/\gamma' = a/\gamma$. Then

$$V = \pm \begin{bmatrix} a & \beta + at \lambda \\ \gamma & \delta + t\gamma \lambda \end{bmatrix} \quad \text{for some} \ t \in \mathbb{Z}.$$

**Proof.** Since $V(\infty) = V'(\infty) = a/\gamma$, $V^{-1}V'(\infty) = \infty$ and $V^{-1}V' = S' \text{ for some} \ t \in \mathbb{Z}.$

**Theorem 3.3.** $j_q(\omega)$ is an $S_q$-series if and only if $q = 3, 6$.

**Proof.** By Theorem 3.1 it suffices to show that for $q \neq 3, 6$,

$$V = \begin{bmatrix} -\lambda & (1 + \lambda^2)/2 \\ -2 & \lambda \end{bmatrix}$$
is not in $G(\lambda)$. To do this we exhibit
\[
M = \begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix} \in G(\lambda) \quad \text{with} \quad \frac{\alpha}{\gamma} = \frac{\lambda}{2}, \quad \alpha \neq \pm \lambda.
\]
Then by Lemma 3.2, $V \notin G(\lambda)$. First note that
\[
ST = \begin{bmatrix}
\lambda & -1 \\
1 & 0
\end{bmatrix} \in G(\lambda).
\]
An induction proof gives that
\[
(ST)^n = \begin{bmatrix}
\frac{\sin(\pi(n+1)/q)}{\sin(\pi/q)} & -\frac{\sin(\pi n/q)}{\sin(\pi/q)} \\
\frac{\sin(\pi n/q)}{\sin(\pi/q)} & -\frac{\sin(\pi(n-1)/q)}{\sin(\pi/q)}
\end{bmatrix}.
\]
For $q$ even, set $n = q/2$ and look at
\[
M = (ST)^{q/2} = \begin{bmatrix}
\frac{\cos(\pi/q)}{\sin(\pi/q)} & -\frac{1}{\sin(\pi/q)} \\
\frac{1}{\sin(\pi/q)} & -\frac{\cos(\pi/q)}{\sin(\pi/q)}
\end{bmatrix}.
\]
Then $M \in G(\lambda)$ with $M \infty = \cos(\pi/q) = \lambda/2$. However, $\cos(\pi/q)/\sin(\pi/q) = \pm \lambda$ only if $\sin(\pi/q) = 1/2$ or $q = 6$. If $q$ is odd, set $n = (q-1)/2$ and consider
\[
M = (ST)^{(q-1)/2} = \begin{bmatrix}
\frac{A^2B + A^2D - ABC}{B^3} & * \\
\frac{A^2D}{B^3} & *
\end{bmatrix}
\]
where $A = \cos(\pi/2q)$, $B = \sin(\pi/q)$, $C = \cos(3\pi/2q)$, $D = \sin(2\pi/q)$. $M \in G(\lambda)$ and $M \infty = D/2B = \lambda/2$. However, $D^2/2B = A^2/2B = \pm \lambda$ if and only if $\sin(\pi/2q) = 1/2$ or $q = 3$. Thus for $q \neq 3, 6, V \notin G(\lambda)$. J. Lehner [3] and J. Raleigh [5] have examined the Fourier coefficients of $J_q(\omega)$ and have obtained the following interesting results. With a different normalization J. Lehner has proved that all the Fourier coefficients are rational. (For $q = 3, 4, 6$ it is well known that the coefficients are actually integers.) J. Raleigh normalized the invariants so that (in his notation) $J_q(\omega) = 1$. The Fourier expansion is then $J_q(\omega) = \sum_{n=1}^{\infty} a_n(q)q^n$ and our $J_q(\omega) = \frac{1}{a_q(q)} J_q(\omega)$. J. Raleigh then derived for all $q$ closed form expressions for $a_n(q)$, $n = -1, 0, 1, 2, 3$. In particular,
\[
a_{-1}(q) = 2^{-1+\frac{3}{2}\pi} \prod_{q=1}^{2\pi} \exp \left\{ 2(1 - \pi) \cos \frac{2\pi}{q} - \log \left( 2 - 3 \cos \frac{\pi}{q} \right) \right\}.
\]
Combining these results with Theorem B on $S_p$-series we have

**Theorem 3.4.** 1. $c_q(n)[a_{-1}(q)]^{n+1}$ is rational.

2. If $J_q(\omega)$ is an $S_p$-series for some prime $p$, $a_{-1}(q)$ is algebraic.

**Proof.** The first statement of the theorem comes from relating the coefficients of the three different normalizations. To get the second result we write $c_q(n) = r_n(a_{-1}(q))^{-n-1}$ where $r_n \in Q$ and use the fact that $c_q(p^n + p)$ can be expressed recursively over $Q$ in terms of the preceding coefficients.

**Corollary 3.5.** For all primes $p$, $J_q(\omega)$ is not an $S_p$-series if $q = 5, 8, 10, 12$.

**Proof.** For $q = 5, 8, 10, 12$, the value of $a_{-1}(q)$ is listed below; in all cases the number is transcendental:

- $a_{-1}(5) = \frac{\sqrt{5} + (2 + \sqrt{5})^{1/2}}{2^{1/2}}$,
- $a_{-1}(8) = \frac{(3 + 2\sqrt{5})^{1/2}}{2^{1/2}}$,
- $a_{-1}(10) = \frac{\sqrt{5} + (1 + \sqrt{5})^{1/2}}{2}$,
- $a_{-1}(12) = \frac{1}{2^{1/2}}(7 + 4\sqrt{3})^{1/2}.$

4. **Coefficients of $J_q(\omega)$ and $J_q(\omega)$.** We first note that $J_q(\omega)$ and $J_q(\omega)$ satisfy a uniqueness theorem analogous to Theorem A for all primes $p$ or which they are $S_p$-series.

**Theorem A'.** Let $p$ be a prime with $(p, q)/2 = 1$, $q = 4, 6$, and let $F_q(X, Y)$ be the polynomial associated with $J_q(\omega)$ when viewed as an $S_p$-series. Suppose $\phi(z) = 1 + \sum_{n=0}^{\infty} a_n z^n$ is a formal Laurent series such that $F_q(\phi(z))$ converges and defines an analytic function in $\{ z : 0 < |z| < 1 \}$ and $\phi(z) = \phi(\phi(z)) = J_q(\omega)$. Since $J_q(\omega)$, $q = 4, 6$, is related algebraically to the modular invariant $J(q)$, it has been known for quite some time that the coefficients $c_q(n)$ are integers. Now, since $J_q(\omega)$ is an $S_q$-series, by Theorem B, $c_q(n)$ for $n \geq 12$ can be computed in terms of $c_q(k)$, $k = 1, \ldots, 11$. Similarly, since $J_q(\omega)$ is an $S_q$-series, $c_q(n)$, $n \geq 6$, can be computed in terms of $c_q(k)$, $k = 1, \ldots, 5$. By way of example we have calculated the first thirteen
coefficients for \( j_4(\omega) \). To find the first six coefficients \( c_0(0), c_0(1), \ldots, c_0(6) \), we use the following identity [6] due to J. Raleigh:

\[
j_4(\omega)^3 - 2 \cdot 3^3 \cdot 7 j_4(\omega)^2 + 3^7 \cdot 23 \cdot j_4(\omega) = j(3^3 \cdot \omega) + j(\omega)^{3 \cdot \omega}).
\]

K. Mahler’s coefficient formulae ([4], p. 91) for \( S_\infty \)-series are then used to determine the other coefficients.

\[
c_0(0) = 42 = 2 \cdot 3 \cdot 7,
c_0(1) = 783 = 3^5 \cdot 29,
c_0(2) = 3,672 = 2^8 \cdot 271,
c_0(3) = 65,367 = 3^3 \cdot 269,
c_0(4) = 26,376 = 2^8 \cdot 3^3 \cdot 3 \cdot 43,
c_0(5) = 1,741,655 = 5 \cdot 163 \cdot 3137,
c_0(6) = 7,161,696 = 2^4 \cdot 3^5 \cdot 307,
c_0(7) = 26,587,946 = 2 \cdot 3^8 \cdot 53 \cdot 9283,
c_0(8) = 90,521,472 = 2^7 \cdot 3 \cdot 19^2 \cdot 653,
c_0(9) = 888,078,201 = 3^7 \cdot 157 \cdot 389,
c_0(10) = 864,924,480 = 2^4 \cdot 3^5 \cdot 5 \cdot 7 \cdot 227,
c_0(11) = 2,469,235,686 = 2 \cdot 3 \cdot 17 \cdot 97 \cdot 103 \cdot 423,
c_0(12) = 6,748,494,912 = 2^7 \cdot 3^4 \cdot 333,931.
\]

These numerical values suggest the following conjecture.

**CONJECTURE.** If \( 2^a \mid n \), then \( 2^{a+1} \mid c_0(n) \).

If \( 3^b \mid n \), then \( 3^{a+1} \mid c_0(n) \).

As the first step in verifying this conjecture we have the following result on the divisibility of two of the coefficients of an \( S_\infty \)-series.

**THEOREM 4.1.** Let \( f(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n \) be an \( S_\infty \)-series with integer coefficients. Let \( 2^n \) be the largest power of 2 dividing \( a_1, a_2 \) and \( a_n \). Then for \( n \geq 6 \)

\[
a_n = 0 \pmod{2^n} \quad \text{whenever} \quad n = 0 \pmod{2}.
\]

**Proof.** We make extensive use of Mahler’s formulae for the coefficients of \( S_\infty \)-series. For \( n \geq 6 \) they are:

\[
a_{2k} = a_{2k+1} + \sum_{j=1}^{k-1} j a_{2k-j} + (a_k^2 - a_k)/2, \quad (4.2)
\]

\[
a_{4k+2} = a_{4k-2} + \sum_{j=1}^{k} a_{2k-j-2} - \sum_{j=1}^{k} (-1)^{j-1} a_{2k-2j+1} + a_{2k-2j-1} - a_{2k} + (a_k^2 - a_k)/2 + (a_k^2 - a_k)/2, \quad (4.3)
\]

\[
a_{4k+4} = a_{4k+5} + \sum_{j=0}^{k} a_{2k-2j}, \quad (4.4)
\]

\[
a_{4k+6} = a_{4k+7} + \sum_{j=1}^{k} a_{2k-2j-1} + (a_k^2 - a_k)/2, \quad (4.5)
\]

\[
a_{4k+8} = a_{4k+9} + \sum_{j=1}^{k} a_{2k-2j-2} - \sum_{j=1}^{k} (-1)^{j-1} a_{2k-2j+1} + a_{2k-2j-1} - a_{2k} + (a_k^2 - a_k)/2. \quad (4.6)
\]

The proof is by induction. We note that \( a_0 = a_1 = a_2 = 0 \pmod{2^n} \), and \( a_{4k} = a_{4k-1} + a_k^2 - a_k = 0 \pmod{2^n} \). Now assume that (4.3) holds for all \( n \) even, \( n < m \), even. To show that (4.1) holds for all \( m \) even. Consider separately the cases \( m = 0 \pmod{8} \), \( m = 2 \pmod{4} \), and \( m = 4 \pmod{4} \).

We begin with the most difficult case, \( m = 0 \pmod{8} \), where \( m \geq 16 \). Then by an application of (4.2) followed by an application of (4.3) and the induction hypothesis, we have

\[
a_m = a_{m+1} + \sum_{j=1}^{m+1} a_{m-j+1} + (a_k^2 - a_k)/2
\]

\[
a_{m+1} + \sum_{j=1}^{k} a_{m-j+1} + (a_k^2 - a_k)/2 \pmod{2^n}.
\]

Applying (4.5) or (4.3) depending on whether or not \( k \) is even or odd, we find that when \( k \) is even

\[
a_{m} = a_{m+1} + \sum_{j=1}^{k} (-1)^{j-1} a_{2k-2j-2} + (a_k^2 - a_k)/2 + \sum_{j=1}^{k} a_{2k-2j-1} + (a_k^2 - a_k)/2 \pmod{2^n}
\]

\[
= 0 \pmod{2^n}
\]
and when \( k = 2k' + 1 \) is odd

\[
(4.7) \quad a_n = a_{2(k' + 1) + 1} + \sum_{j=1}^{k'} a_j a_{2(k' + 1) - j + 1} - \sum_{j=1}^{k'} (-1)^{j-1} a_j a_{k' + 1 - j} + \frac{a_{k'+1}^2 - a_{k'+2}}{2} + \frac{a_{k'+2}^2 - a_{k'+3}}{2} \equiv a_{2k'+1} \pmod{2^n},
\]

whereas if \( k = 2k'' \) is even,

\[
a_n = a_{2k'' + 1} + \sum_{j=1}^{k''} a_j a_{2k'' - j + 1} = a_{2k'' + 1} \pmod{2^n}.
\]

By (4.4) and the induction hypothesis.

Finally, if \( m = 4 \pmod{8}, m \geq 12, \)

\[
a_m = a_{4k+4} = a_{4k+4} + \sum_{j=1}^{k} a_j a_{4k+4-j} + \frac{a_{4k+3}^2 - a_{4k+2}}{2} \equiv 0 \pmod{2^n},
\]

by (4.3), (4.5) and the induction hypothesis.

**Corollary 4.2.** If \( n = 0 \pmod{2}, c_n(0) \equiv 0 \pmod{2^4} \) where \( c_n(0) \) is the \( n \)-th coefficient of \( j_3(\omega) \). If \( n = 0 \pmod{2} \), \( c_n(0) \equiv 0 \pmod{2^4} \) where \( c_n(0) \) is the \( n \)-th coefficient of the modular invariant \( j_3(\omega) = j_3(\omega) \).

Also if \( n = 0 \pmod{2} \), \( b_n(0) \equiv 0 \pmod{2^4} \) where \( b_n(0) \) is the \( n \)-th coefficient of \( j_3^* \omega = \sum_{n=1}^{\infty} b_n(0) \omega^{3n+1} \).

**Proof.** Since \( j_3(\omega) \) and \( j(\omega) \) are \( S_p \)-series, it is just a matter of checking that \( a = 5 \) for \( j_3(\omega) \) and \( a = 11 \) for \( j(\omega) \). The coefficient congruence for \( j(\omega) \) is already well known. In [4], K. Mahler verifies that \( j_3^* \omega = \sum_{n=1}^{\infty} b_n(0) \omega^{3n+1} \) is an \( S_p \)-series with \( b(2) = 2^9 \cdot 31, b(4) = 0, b(8) = 2^8 \cdot 3 \cdot 31 \).

**5. Conclusion.** The following interesting questions are as yet unanswered.

5.1. Is there a value for \( q \), other than 3, 4, or 6, for which \( j_3(q) \) is an \( S_p \)-series for some prime \( p \)?

5.2. (J. E. R. [5]) Is \( a_{-1}(q) \) transcendental for \( q \neq 3, 4, 6 \)?

5.3. Is the conjecture of § 4 for the coefficients of \( j_3(\omega) \) valid?

5.4. What is the analogous conjecture for the coefficients of \( j_3(q) \) and is it true?

5.5. (C. Pisot in [5]). Except for \( q = 3, 4, 6 \) is there a value of \( q \) for which there is a constant \( K_q \) so that \( K_q j_3(\omega) \) has integer coefficients? \( K_q \) certainly exists if \( a_{-1}(q) \) is rational and \( j_3(q) \) is an \( S_p \)-series for a prime \( p \).

**References**


[7] J. Young, On the group belonging to the sign (0, 3, 3, 4, 4, 1) and the functions belonging to it, ibid. 5 (1904), pp. 81-104.

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