ACTA ARITHMETICA
XL (1982)

On the invariants of the Hecke groups

by

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0. Introduction. In [4] K. Mahler defined the concept of an S_p-series. The primary example of an S_p-series is the modular invariant $j(\omega)$. In this article we provide additional examples of S_p-series by examining the invariants $j_q(\omega)$ of the Hecke groups $G(\lambda_q)$. Some of the arithmetic consequences for the Fourier coefficients of the invariants for $G(\sqrt{2})$ and $G(\sqrt{3})$ are then discussed.

1. S_p-series. Motivated by the behavior of Klein's modular invariant $j(\omega)$ which satisfies modular equations of order p for every prime p, Kurt Mahler [4] considered solutions in formal Laurent series to functional equations of the form

$$f(z^p)^{p+1}+f(z)^{p+1}+\sum_{r=0}^{p}\sum_{s=0}^{p}c_{rs}f(z^p)^r f(z)^s=0, \quad c_{rs}=c_{sr}. \tag{1.1}$$

More specifically, formal series with the following property were studied.

DEFINITION. Let p be a fixed prime. Let $f(z)=\sum_{h=m}^{\infty}a_h z^h$, $a_m\neq 0$, denote a nonconstant formal ascending Laurent series with complex coefficients. Let $R(f,p)$ denote the following set of $p+1$ derived Laurent series in z^p and $z^{1/p}$:

$$R(f,p)=\{f(z^p), f(z^{1/p}), f(\varepsilon z^{1/p}), f(\varepsilon^2 z^{1/p}), \ldots, f(\varepsilon^{p-1}z^{1/p})\}$$

where ε is a pth root of unity. Then $f(z)$ is an *S_p-series of order m* if every elementary symmetric function of the elements of $R(f,p)$ can be expressed as a polynomial in $f(z)$.

Associated with each S_p-series is the polynomial $F_p(x, y)$ defined by

$$F_p\big(x, f(z)\big)=\big(x-f(z^p)\big)\prod_{j=0}^{p-1}\big(x-f(\varepsilon^j z^{1/p})\big).$$

* Supported in part by NSF Grant MCS 78-02135.

$F_p(x, y)$ is necessarily symmetric; and the equation $F_p(f(z^p), f(z)) = 0$ is of the form (1.1). In the special case when $f(z) = f(e^{2\pi i\omega}) = j(\omega)$, $f(z)$ is an S_p-series for every prime p and $F_p(x, j(\omega)) = 0$ is the modular equation of order p.

The following two results which are consequences of the general theory of S_p-series (see [4]) are of particular interest. The first indicates the extent to which $j(\omega)$ is determined by its modular equation of any prime order p.

THEOREM A. *Let p be a prime and let $F_p(x, y)$ be the modular polynomial of order p. Let $\varphi(z) = 1/z + \sum_{h=0}^{\infty} a_h z^h$ be any formal Laurent series with*

$$F_p(\varphi(z^p), \varphi(z)) = 0.$$

Then $\varphi(z)$ is analytic in $\{z\colon 0 < |z| < 1\}$ and $\varphi(z) = \varphi(e^{2\pi i\omega}) = j(\omega)$.

THEOREM B. *If $f(z) = 1/z + \sum_{h=0}^{\infty} a_h z^h$ is an S_p-series, then the coefficients a_h with $h \geqslant p^2+p$ can be expressed recursively as polynomials in $a_0, a_1, \ldots, a_{p^2+p-1}$.*

For $p = 2$ or 3 the recursive formulae of Theorem B are given explicitly in [4]. For $p \geqslant 5$ the formulae become excessively complicated. However it scarcely needs emphasizing that the formulae for $p = 2, 3$ are extremely useful for calculating the coefficients and for studying their arithmetic properties.

2. The invariants of the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$ are S_p-series. Since $j(\omega)$ is the canonical example of an S_p-series, it is natural to ask whether there are other groups with invariants which are also S_p-series. In [1], in connection with the correspondence between Dirichlet series and automorphic forms, E. Hecke introduced the class of properly discontinuous groups $G(\lambda_q)$ which are generated by $T(\omega) = -1/\omega$ and $S(\omega) = \omega+\lambda_q$ where $\lambda_q = 2\cos(\pi/q)$, $q = 3, 4, 5, \ldots$ When $q = 3$, $\lambda_3 = 1$ and $G(1)$ is the modular group. Associated with each of these groups is an invariant $j_q(\omega)$ which has a simple pole at $i\infty$ and maps

$$F = \{-\lambda_q/2 \leqslant \operatorname{Re}\omega \leqslant 0,\ |\omega| \geqslant 1\} \cup \{0 < \operatorname{Re}\omega < \lambda_q/2,\ |\omega| > 1\}$$

univalently onto the upper half plane $\mathscr{H}$. Normalizing the Fourier expansion at $i\infty$ we have

$$j_q(\omega) = \frac{1}{z} + \sum_{n=0}^{\infty} c_q(n) z^n \quad \text{with} \quad z = \exp(2\pi i\omega/\lambda_q).$$

When $q = 4$ or 6, the resulting groups are $G(\sqrt{2})$ and $G(\sqrt{3})$. These are the only Hecke groups which are commensurable with the modular group and therefore the only Hecke groups whose elements are completely characterized arithmetically. For notational convenience we let $l = 2$ or 3 and represent the transformation $z' = (\alpha z+\beta)/(\gamma z+\delta)$ by a matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{with} \quad \alpha\delta-\beta\gamma = 1.$$

Note that both

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{bmatrix}$$

represent the same linear fractional transformation. It is then well known ([2], [7]) that $G(\sqrt{l})$ consists of the entirety of elements of the following two forms:

$$(2.1)\qquad \begin{aligned} &\begin{bmatrix} a & b\sqrt{l} \\ c\sqrt{l} & d \end{bmatrix}, \quad a, b, c, d \in \mathbf{Z},\ ad-lbc = 1, \\ &\begin{bmatrix} a\sqrt{l} & b \\ c & d\sqrt{l} \end{bmatrix}, \quad a, b, c, d \in \mathbf{Z},\ lad-bc = 1. \end{aligned}$$

With this characterization of the elements of $G(\sqrt{l})$, we prove

THEOREM 2.1. *$j_4(\omega)$ and $j_6(\omega)$ are S_p-series for all primes p except $p = l$.*

The proof of Theorem 2.1 relies on

LEMMA 2.2. *For $q = 4$ or 6 let $j(\omega) = j_q(\omega)$ and $\lambda = \lambda_q$. For $p \neq l$, set*

$$T_p = \left\{\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}, \begin{bmatrix} 1 & \lambda \\ 0 & p \end{bmatrix}, \ldots, \begin{bmatrix} 1 & (p-1)\lambda \\ 0 & p \end{bmatrix}\right\}.$$

For each $M_i \in T_p$, set $j_i(\omega) = j(M_i\omega)$, $i = 1, \ldots, p+1$. Then for any $V \in G(\lambda)$,

$$\{j_i(\omega)\} = \{j_i(V\omega)\};$$

in other words, replacing ω by $V\omega$ merely permutes the elements of $\{j_i(\omega)\}$.

Proof of Theorem 2.1. We must show that the elementary symmetric functions of the elements of $R(j, p)$ are polynomials in $j(\omega)$. However, $R(j, p) = \{j_i(\omega)\}$. Since by Lemma 2.2 $\{j_i(V\omega)\} = \{j_i(\omega)\}$ for any $V \in G(\lambda)$, any symmetric combination of elements of $R(j, p)$

is invariant under $G(\lambda)$. In particular, the elementary symmetric functions are invariant. Since any function invariant under $G(\lambda)$ and analytic in $\mathscr{H}$ is a polynomial in $j(\omega)$, the elementary symmetric functions of elements of $R(j,p)$ are indeed polynomials in $j(\omega)$. ■

Proof of Lemma 2.2. Since $j(\omega)$ is invariant under $G(\lambda)$ we need only show that for each $M_i \in T_p$, there exists an $M_j \in T_p$, $V_j \in G(\lambda)$ such that $M_i V = V_j M_j$ and that the resulting M_j are distinct. In fact it suffices to verify this result for the two generators of $G(\lambda)$,

$$S = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For S we have

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p\lambda \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & b\lambda \\ 0 & p \end{bmatrix}\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (b+1)\lambda \\ 0 & p \end{bmatrix}, \quad 0 \leqslant b \leqslant p-2,$$

$$\begin{bmatrix} 1 & (p-1)\lambda \\ 0 & p \end{bmatrix}\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}.$$

For T the corresponding identities are

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & b\lambda \\ 0 & p \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -b\lambda & (1+bb'\lambda^2)/p \\ -p & b'\lambda \end{bmatrix}\begin{bmatrix} 1 & b'\lambda \\ 0 & p \end{bmatrix}, \quad 1 \leqslant b \leqslant p-1, \tag{2.2}$$

where b' is the solution to $\lambda^2 bx \equiv -1 \pmod p$ with $1 \leqslant b' \leqslant p-1$. ■

Before discussing the arithmetical consequences of Theorem 2.1 for the coefficients of j_4 and j_6, we look briefly at the question of whether any of the other j_q are S_p-series.

3. j_q for $q \geqslant 4$. For $q = 5, 7, 8, 9, \ldots$ is j_q an S_p-series for some prime p? The easy proof of the preceding section fails at the point in Lemma 2.2, formula (2.2), where we find $V_j \in G(\sqrt{l})$ so that $M_i T = V_j M_j$. For $q \neq 3, 4, 6$ there is no quick way of determining whether $V_j \in G(\lambda_q)$. To illustrate this difficulty, we take $p = 2$ and show that j_q is an S_2-series if and only if $q = 3$ or 6. For the sake of notational convenience we drop the subscript q.

THEOREM 3.1. *$j(\omega)$ is an S_2-series if and only if*

$$V = \begin{bmatrix} -\lambda & (1+\lambda^2)/2 \\ -2 & \lambda \end{bmatrix} \in G(\lambda).$$

Proof. If $V \in G(\lambda)$, then the method of proof of Theorem 2.1 and Lemma 2.2 carries over to give that $j(\omega)$ is an S_2-series. On the other hand, if $j(\omega)$ is an S_2-series, then $F(\omega) = j(2\omega)+j(\omega/2)+j((\omega+\lambda)/2)$ is invariant under $G(\lambda)$ since $F(\omega)$ is a polynomial in $j(\omega)$. In particular, $F(T\omega) = F(\omega)$. However,

$$F(T\omega) = j\left(\frac{-2}{\omega}\right)+j\left(\frac{-1}{2\omega}\right)+j\left(\frac{\omega\lambda-1}{2\omega}\right) = j\left(\frac{\omega}{2}\right)+j(2\omega)+j\left(\frac{\omega\lambda-1}{2\omega}\right)$$

which implies that $j\left(\frac{\omega+\lambda}{2}\right) = j\left(\frac{\omega\lambda-1}{2\omega}\right)$ or, upon replacing ω by $2\omega-\lambda$, $j(\omega) = j\left(\frac{2\lambda\omega-\lambda^2-1}{4\omega-2\lambda}\right) = j(V\omega)$. Then since $G(\lambda)$ is the invariance group for $j(\omega)$, $V \in G(\lambda)$. ■

It is now clear that $j_4(\omega)$ is not an S_2-series since

$$V = \begin{bmatrix} -\sqrt{2} & 3/2 \\ -2 & \sqrt{2} \end{bmatrix}$$

is not in $G(\sqrt{2})$. To prove the same result for $q \geqslant 5$, $q \neq 6$, we use the following lemma.

LEMMA 3.2. *Suppose*

$$V = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \text{and} \quad V' = \begin{bmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{bmatrix}$$

are elements of $G(\lambda)$ with $\alpha/\gamma = \alpha'/\gamma'$. Then

$$V = \pm\begin{bmatrix} \alpha & \beta+\alpha t\lambda \\ \gamma & \delta+\gamma t\lambda \end{bmatrix} \quad \text{for some } t \in \mathbf{Z}.$$

Proof. Since $V(\infty) = V'(\infty) = \alpha/\gamma$, $V^{-1}V'(\infty) = \infty$ and $V^{-1}V' = S^t$ for some $t \in \mathbf{Z}$. ■

THEOREM 3.3. *$j_q(\omega)$ is an S_2-series if and only if $q = 3, 6$.*

Proof. By Theorem 3.1 it suffices to show that for $q \neq 3, 6$,

$$V = \begin{bmatrix} -\lambda & (1+\lambda^2)/2 \\ -2 & \lambda \end{bmatrix}$$

is not in $G(\lambda)$. To do this we exhibit

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in G(\lambda) \quad \text{with} \quad \frac{\alpha}{\gamma} = \frac{\lambda}{2}, \qquad \alpha \neq \pm\lambda.$$

Then by Lemma 3.2, $V \notin G(\lambda)$.

First note that

$$ST = \begin{bmatrix} \lambda & -1 \\ 1 & 0 \end{bmatrix} \in G(\lambda).$$

An induction proof gives that

$$(ST)^n = \begin{bmatrix} \dfrac{\sin(\pi(n+1)/q)}{\sin(\pi/q)} & -\dfrac{\sin(\pi n/q)}{\sin(\pi/q)} \\ \dfrac{\sin(\pi n/q)}{\sin(\pi/q)} & -\dfrac{\sin(\pi(n-1)/q)}{\sin(\pi/q)} \end{bmatrix}.$$

For q even, set $n = q/2$ and look at

$$M = (ST)^{q/2} = \begin{bmatrix} \dfrac{\cos(\pi/q)}{\sin(\pi/q)} & -\dfrac{1}{\sin(\pi/q)} \\ \dfrac{1}{\sin(\pi/q)} & -\dfrac{\cos(\pi/q)}{\sin(\pi/q)} \end{bmatrix}.$$

Then $M \in G(\lambda)$ with $M\infty = \cos(\pi/q) = \lambda/2$. However, $\cos(\pi/q)/\sin(\pi/q) = \pm\lambda$ only if $\sin(\pi/q) = 1/2$ or $q = 6$. If q is odd, set $n = (q-1)/2$ and consider

$$M = (ST)^{(q-1)/2} S (ST)^{(q-1)/2} = \begin{bmatrix} \dfrac{A^2D}{B^3} & * \\ \dfrac{A^2B + A^2D - ABC}{B^3} & * \end{bmatrix}$$

where $A = \cos(\pi/2q)$, $B = \sin(\pi/q)$, $C = \cos(3\pi/2q)$, $D = \sin(2\pi/q)$. $M \in G(\lambda)$ and $M(\infty) = D/2B = \lambda/2$. However, $\dfrac{A^2D}{D^3} = \dfrac{A^2}{B^2}\lambda = \pm\lambda$ if and only if $\sin(\pi/2q) = 1/2$ or $q = 3$. Thus for $q \neq 3, 6$, $V \notin G(\lambda)$. ∎

J. Lehner [3] and J. Raleigh [5] have examined the Fourier coefficients of $j_q(\omega)$ and have obtained the following interesting results. With a different normalization J. Lehner has proved that all the Fourier coefficients are rational. (For $q = 3, 4, 6$ it is well known that the coefficients are actually integers.) J. Raleigh normalized the invariants so that (in his notation) $J_q(i) = 1$. The Fourier expansion is then $J_q(\omega) = \sum_{n=-1}^{\infty} a_n(q) z^n$ and our $j_q(\omega) = \dfrac{1}{a_{-1}(q)} J_q(\omega)$. J. Raleigh then derived for all q closed form expressions for $a_n(q)$, $n = -1, 0, 1, 2, 3$. In particular,

$$a_{-1}(q) = 2^{-4+2(-1)^q} q^{-2} \prod_{V=1}^{q-1} \exp\left\{2(-1)^V \cos\frac{2V\pi}{q} \log\left(2 - 2\cos\pi\frac{V}{q}\right)\right\}.$$

Combining these results with Theorem B on S_p-series we have

THEOREM 3.4. 1. *$c_q(n)(a_{-1}(q))^{n+1}$ is rational.*

2. *If $j_q(\omega)$ is an S_p-series for some prime p, $a_{-1}(q)$ is algebraic.*

Proof. The first statement of the theorem comes from relating the coefficients of the three different normalizations. To get the second result we write $c_q(n) = r_n(a_{-1}(q))^{-n-1}$ where $r_n \in Q$ and use the fact that $c_q(p^2+p)$ can be expressed recursively over Q in terms of the preceding coefficients. ∎

COROLLARY 3.5. *For all primes p, $j_q(\omega)$ is not an S_p-series if $q = 5, 8, 10, 12$.*

Proof. For $q = 5, 8, 10, 12$, the value of $a_{-1}(q)$ is listed below; in all cases the number is transcendental:

$$a_{-1}(5) = \frac{\sqrt{5}(2+\sqrt{5})^{\sqrt{5}}}{2^6 5^3}, \qquad a_{-1}(8) = \frac{(3+2\sqrt{2})^{\sqrt{2}}}{2^{10}},$$

$$a_{-1}(10) = \frac{\sqrt{5}}{2^3 3^3}\left(\frac{1+\sqrt{5}}{2}\right)^{\sqrt{5}}, \qquad a_{-1}(12) = \frac{1}{2^8 3^3}(7+4\sqrt{3})^{\sqrt{3}}.$$

4. Coefficients of $j_4(\omega)$ and $j_6(\omega)$. We first note that $j_4(\omega)$ and $j_6(\omega)$ fatisfy a uniqueness theorem analogous to Theorem A for all primes p sor which they are S_p-series.

THEOREM A'. *Let p be a prime with $(p, q/2) = 1$, $q = 4, 6$, and let $F_p(X, Y)$ be the polynomial associated with $j_q(\omega)$ when viewed as an S_p-series. Suppose $\varphi(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$ is a formal Laurent series such that $F_p(\varphi(z^p), \varphi(z)) = 0$. Then $\varphi(z)$ converges and defines an analytic function in $\{z\colon 0 < |z| < 1\}$ and $\varphi(z) = \varphi(e^{2\pi i\omega}) = j_q(\omega)$.*

Since $j_q(\omega)$, $q = 4, 6$, is related algebraically to the modular invariant $j(\omega)$, it has been known for quite some time that the coefficients $c_q(n)$ are integers. Now, since $j_4(\omega)$ is an S_3-series, by Theorem B, $c_4(n)$ for $n \geqslant 12$ can be computed in terms of $c_4(k)$, $k = 1, \ldots, 11$. Similarly, since $j_6(\omega)$ is an S_2-series, $c_6(n)$, $n \geqslant 6$, can be computed in terms of $c_6(k)$, $k = 1, \ldots, 5$. By way of example we have calculated the first thirteen coefficients for $j_6(\omega)$. To find the first six coefficients $c_6(0), c_6(1), \dots, c_6(6)$, we use the following identity [6] due to J. Raleigh:

$$j_6(\omega)^3 - 2\cdot 3^2\cdot 7 j_6(\omega)^2 + 2^7\cdot 23\cdot j_6(\omega) = j(3^{1/2}\omega) + j(\omega/3^{1/2}).$$

K. Mahler's coefficient formulae ([4], p. 91) for S_2-series are then used to determine the other coefficients.

$$c_6(0) = 42 = 2\cdot 3\cdot 7,$$
$$c_6(1) = 783 = 3^3\cdot 29,$$
$$c_6(2) = 8{,}672 = 2^5\cdot 271,$$
$$c_6(3) = 65{,}367 = 3^5\cdot 269,$$
$$c_6(4) = 371{,}520 = 2^6\cdot 3^3\cdot 5\cdot 43,$$
$$c_6(5) = 1{,}741{,}655 = 5\cdot 163\cdot 2137,$$
$$c_6(6) = 7{,}161{,}696 = 2^5\cdot 3^6\cdot 307,$$
$$c_6(7) = 26{,}567{,}946 = 2\cdot 3^3\cdot 53\cdot 9283,$$
$$c_6(8) = 90{,}521{,}472 = 2^7\cdot 3\cdot 19^2\cdot 653,$$
$$c_6(9) = 288{,}078{,}201 = 3^7\cdot 157\cdot 839,$$
$$c_6(10) = 864{,}924{,}480 = 2^6\cdot 3^5\cdot 5\cdot 7^2\cdot 227,$$
$$c_6(11) = 2{,}469{,}235{,}686 = 2\cdot 3\cdot 17\cdot 97\cdot 103\cdot 2423,$$
$$c_6(12) = 6{,}748{,}494{,}912 = 2^6\cdot 3^5\cdot 433{,}931.$$

These numerical values suggest the following conjecture.

CONJECTURE. *If $2^a|n$, $a\geqslant 1$, then $2^{a+4}|c_6(n)$.*
If $3^b|n$, $b\geqslant 1$, then $3^{2b+3}|c_6(n)$.

As the first step in verifying this conjecture we have the following result on the divisibility by two of the coefficients of an S_2-series.

THEOREM 4.1. *Let $f(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$ be an S_2-series with integer coefficients. Let 2^a be the largest power of 2 dividing a_2, a_4 and a_8. Then for $n\geqslant 6$*

$$a_n \equiv 0 \pmod{2^a} \quad \textit{whenever} \quad n\equiv 0 \pmod 2. \tag{4.1}$$

Proof. We make extensive use of Mahler's formulae for the coefficients of S_2-series. For $n\geqslant 6$ they are:

$$a_{4k} = a_{2k+1} + \sum_{j=1}^{k-1} a_j a_{2k-j} + (a_k^2 - a_k)/2, \tag{4.2}$$

$$a_{4k+1} = a_{2k+3} + \sum_{j=1}^{k} a_j a_{2k-j+2} - \sum_{j=1}^{2k-1} (-1)^{j-1} a_j a_{4k-j} + \sum_{j=1}^{k-1} a_j a_{4k-4j} - a_2 a_{2k} + (a_{k+1}^2 - a_{k+1})/2 + (a_{2k}^2 - a_{2k})/2, \tag{4.3}$$

$$a_{4k+2} = a_{2k+2} + \sum_{j=1}^{k} a_j a_{2k-j+1}, \tag{4.4}$$

$$a_{4k+3} = a_{2k+4} + \sum_{j=1}^{k+1} a_j a_{2k-j+3} - \sum_{j=1}^{2k} (-1)^{j-1} a_j a_{4k-j+2} + \sum_{j=1}^{k} a_j a_{4k-4j+2} - a_2 a_{2k+1} - (a_{2k+1}^2 - a_{2k+1})/2. \tag{4.5}$$

The proof is by induction. We note that $a_6 = a_4 + a_1 a_2 \equiv 0 \pmod{2^a}$ and $a_{10} = a_6 + a_1 a_4 + a_2 a_3 \equiv 0 \pmod{2^a}$. Now assume that (4.1) holds for all n even, $n < m$, m even. To show that (4.1) holds for m we consider separately the three cases $m\equiv 0 \pmod 8$, $m\equiv 2 \pmod 4$, and $m\equiv 4 \pmod 8$.

We begin with the most difficult case, $m\equiv 0 \pmod 8$, where $m\geqslant 16$. Then by an application of (4.2) followed by an application of (4.3) and the induction hypothesis, we have

$$\begin{aligned} a_m = a_{8k} &= a_{4k+1} + \sum_{j=1}^{2k+1} a_j a_{4k-j} + (a_{2k}^2 - a_{2k})/2 \\ &\equiv a_{2k+3} + \sum_{j=1}^{k} a_j a_{2k-j+2} + (a_{k+1}^2 - a_{k+1})/2 \pmod{2^a}. \end{aligned} \tag{4.6}$$

Applying (4.5) or (4.3) depending on whether or not k is even or odd, we find that when k is even

$$\begin{aligned} a_m &\equiv -\sum_{j=1}^{k} (-1)^{j-1} a_j a_{2k-j+2} - (a_{2k+1}^2 - a_{2k+1})/2 + \\ &\quad + \sum_{j=1}^{k} a_j a_{2k-j+2} + (a_{k+1}^2 - a_{k+1})/2 \pmod{2^a} \\ &\equiv 0 \pmod{2^a} \end{aligned}$$

and when $k = 2k'+1$ is odd

$$(4.7)\quad a_m \equiv a_{2(k'+1)+3} + \sum_{j=1}^{k'+1} a_j a_{2(k'+1)-j+2} - \sum_{j=1}^{2k'+1} (-1)^{j-1} a_j a_{4(k'+1)-j} + $$
$$+ (a_{k'+2}^2 - a_{k'+2})/2 + (a_{2(k'+2)}^2 - a_{2(k'+1)})/2 +$$
$$+ \sum_{j=1}^{k} a_j a_{2k-j+2} + (a_{k+1}^2 - a_{k+1})/2 \pmod{2^\alpha}$$
$$\equiv a_{2(k'+1)+3} + \sum_{j=1}^{k'+1} a_j a_{2(k'+1)-j+2} + (a_{k'+2}^2 - a_{k'+2})/2 \pmod{2^\alpha}.$$

We now note that (4.7) is the same as (4.6) with k replaced by $k'+1$. Therefore, if $k'+1$ is even, that is, if k' is odd,

$$a_m \equiv 0 \pmod{2^\alpha}$$

whereas if $k' = 2k''$ is even,

$$a_m \equiv a_{2(k''+1)+3} + \sum_{j=1}^{k''+1} a_j a_{2(k''+1)-j+2} = (a_{k''+2}^2 - a_{k''+2})/2 \pmod{2^\alpha}.$$

Repeating for $k'', k''', \ldots,$ the argument given above for k', it is clear that eventually we must have $a_m \equiv 0 \pmod{2^\alpha}$.

Next, if $m \equiv 2 \pmod 4$, $m \geq 6$,

$$a_m = a_{4k+2} = a_{2k+2} + \sum_{j=1}^{k} a_j a_{2k-j+1} \equiv 0 \pmod{2^\alpha}$$

by (4.4) and the induction hypothesis.

Finally, if $m \equiv 4 \pmod 8$, $m \geq 12$,

$$a_m = a_{8k+4} = a_{4k+3} + \sum_{j=1}^{2k} a_j a_{4k+2-j} + (a_{2k+1}^2 - a_{2k+1})/2$$
$$\equiv -\sum_{j=1}^{2k} (-1)^{j-1} a_j a_{4k-j+2} - (a_{2k+1}^2 - a_{2k+1})/2 +$$
$$+ \sum_{j=1}^{2k} a_j a_{4k+2-j} + (a_{2k+1}^2 - a_{2k+1})/2$$
$$\equiv 0 \pmod{2^\alpha}$$

by (4.2), (4.5) and the induction hypothesis. ∎

COROLLARY 4.2. *If* $n \equiv 0 \pmod 2$, $c_6(n) \equiv 0 \pmod{2^5}$ *where* $c_6(n)$ *is the* n*-th coefficient of* $j_6(\omega)$. *If* $n \equiv 0 \pmod 2$, $c_3(n) \equiv 0 \pmod{2^{11}}$ *where* $c_3(n)$ *is the* n*-th coefficient of the modular invariant* $j(\omega) = j_3(\omega)$.

Also if $n \equiv 0\ (2)$, $b(n) \equiv 0 \pmod{2^3}$ *where* $b(n)$ *is the* n*-th coefficient of* $j^{1/3}(\omega) = \sum_{n=-1}^{\infty} b(n) e^{2\pi i n\omega/3}$.

Proof. Since $j_6(\omega)$ and $j(\omega)$ are S_2-series, it is just a matter of checking that $\alpha = 5$ for $j_6(\omega)$ and $\alpha = 11$ for $j(\omega)$. The coefficient congruence for $j(\omega)$ is already well known. In [4], K. Mahler verifies that $j^{1/3}(\omega)$ is an S_2-series with $b(2) = 2^3 \cdot 31$, $b(4) = 0$, $b(8) = 2^6 \cdot 3 \cdot 81$. ∎

5. Conclusion. The following interesting questions are as yet unanswered.

5.1. Is there a value for q, other than 3, 4 or 6, for which $j_q(\omega)$ is an S_p-series for some prime p?

5.2. (J. Raleigh [5]) Is $a_{-1}(q)$ transcendental for $q \neq 3, 4, 6$?

5.3. Is the conjecture of § 4 for the coefficients of $j_6(\omega)$ valid?

5.4. What is the analogous conjecture for the coefficients of $j_4(\omega)$ and is it true?

5.5. (C. Pisot in [5]). Except for $q = 3, 4, 6$ is there a value of q for which there is a constant K_q so that $K_q j_q(\omega)$ has integer coefficients? K_q certainly exists if $a_{-1}(q)$ is rational and $j_q(\omega)$ is an S_p-series for a prime p.

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Received on 25.7.1979 (1170)