ACTA ARITHMETICA
XL (1981)

The derivative of p-adic L-functions

by

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The derivatives of a p-adic analogue of $\log \Gamma(x)$ are related to the values of the Kubota–Leopoldt p-adic L-functions at positive integers [5], just as in the case of the classical L-functions. It is natural to ask whether or not there is a similar relation between the antiderivatives of $\log \Gamma(x)$ and the values of the L-functions at negative integers. In this paper, we investigate this question and show that the antiderivatives of $\log \Gamma$ yield the values of the first derivatives of the L-functions at negative integers in both the classical and p-adic cases.

1. The p-adic log gamma functions. Define the Bernoulli numbers by

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

so $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, etc. (note that we have chosen $B_1 = +\frac{1}{2}$ as in [3]). The Bernoulli polynomials are defined by

$$B_n(X) = \sum_{j=0}^{n} \binom{n}{j} B_j X^{n-j}.$$

Let

$$G_p(X) = (X-\tfrac{1}{2})\log_p X - X + \sum_{j=2}^{\infty} \frac{B_j}{j(j-1)} X^{1-j}$$

for $|X|_p > 1$. Here $\log_p X$ is the p-adic logarithm, which may be defined for all p-adic X (see [3]). One readily recognizes $G_p(X)$ as coming from Stirling's asymptotic series for $\log\left(\Gamma(x)/\sqrt{2\pi}\right)$, and hence we may regard it as a p-adic log gamma function. Its derivatives were used in [5] to give the values of p-adic L-functions at positive integers (see below). Subsequently it was defined and studied extensively by Jack Diamond [1]. He showed that it may be continued to $\Omega_p - \boldsymbol{Z}_p$ (where Ω_p = completion of the algebraic closure of $\boldsymbol{Q}_p$) and that it satisfies identities analogous to those for the classical log gamma function. In particular, $G_p(X+1) = G_p(X)+\log_p X$. Also there is the identity $G_p(X)+G_p(1-X) = 0$.

Morita [4] has also defined a p-adic gamma function as follows. If n is a positive integer, let

$$\Gamma_p(n) = (-1)^n \prod_{\substack{j=1 \\ p\nmid j}}^{n-1} j.$$

Then extend Γ_p to all of $\mathbf{Z}_p$ by continuity. It turns out that Γ_p is an analytic function on $\mathbf{Z}_p$.

The following lemma shows the relation between G_p and $\log \Gamma_p$.

LEMMA ([2]). *Extend G_p to all of Ω_p by setting $G_p(X) = 0$ if $X \in \mathbf{Z}_p$. Then*

$$\log_p \Gamma_p(X) = \sum_{a=0}^{p-1} G_p\left(\frac{X+a}{p}\right) \quad \textit{for} \quad X \in \mathbf{Z}_p.$$

Proof. Both sides satisfy the functional equation

$$f(X+1) = f(X)+\delta(X)\log_p X$$

where $\delta(X) = 0$ if $p|X$ and $\delta(X) = 1$ otherwise. Therefore they differ by a constant. But $G_p(y)+G_p(1-y) = 0$, so both sides are equal for $X = 0$, hence equal for all X.

2. p-adic L-functions. Let χ be a Dirichlet character of conductor f and let $L(s,\chi)$ be the associated Dirichlet L-function. Then (see [3])

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}, \quad n \geqslant 1,$$

where $B_{n,\chi}$ is a generalized Bernoulli number.

The p-adic L-function $L_p(s,\chi)$ satisfies a similar property, namely

$$L_p(1-n,\chi) = -\left(1-\chi\omega^{-n}(p)p^{n-1}\right)\frac{B_{n,\chi\omega^{-n}}}{n}, \quad n \geqslant 1.$$

Here ω is the Teichmüller character defined as follows: if $a \in \mathbf{Z}_p^\times$, then $a = \omega(a)\langle a\rangle$ with $\langle a\rangle \equiv 1 \bmod p$ (mod 4 if $p = 2$; references to similar modifications for $p = 2$ will henceforth be omitted) and with $\omega(a)$ a $(p-1)$st root of unity.

In [5] we proved the following: Let F be any multiple of p and f. Then

$$L_p(s,\chi) = \frac{1}{s-1}\,\frac{1}{F}\sum_{\substack{a=1 \\ p\nmid a}}^{F} \chi(a)\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j}\left(\frac{-F}{a}\right)^j B_j.$$

When s is a positive integer with $s \geqslant 2$, this becomes

$$(1_p) \qquad L_p(1+n,\chi) = \frac{(-1)^{n+1}}{n!F^{n+1}}\sum_{\substack{a=1 \\ p\nmid a}}^{F}\chi\omega^n(a)G_p^{(n+1)}\left(\frac{a}{F}\right),$$

where $G_p^{(n+1)}$ denotes the $(n+1)$st derivative. This is the analogue of the classical formula

$$(1_\infty) \qquad L(1+n,\chi) = \frac{(-1)^{n+1}}{n!F^{n+1}}\sum_{a=1}^{F}\chi(a)\frac{d^{n+1}}{dX^{n+1}}\left(\log\left(\Gamma(X)/\sqrt{2\pi}\right)\right)\Big|_{X=a/F}$$

(of course, the $\sqrt{2\pi}$ here is extraneous; but it will be needed later when negative n are considered).

When p does not divide the conductor of $\chi\omega^n$, call it g, we may use the lemma to rewrite (1_p) as

$$(1_p') \qquad L_p(1+n,\chi) = \frac{(-1)^{n+1}}{n!g^{n+1}}\sum_{a=1}^{g}\chi\omega^n(a)\frac{d^{n+1}}{dX^{n+1}}\left(\log\Gamma_p(X)\right)\Big|_{X=a/g}.$$

3. The derivative in the p-adic case. We differentiate the above formula for $L_p(s,\chi)$:

$$\frac{d}{ds}\langle a\rangle^{1-s} = -\langle a\rangle^{1-s}\log_p a.$$

If $1-s \notin \{1,\ldots,j-1\}$, then

$$\frac{d}{ds}\binom{1-s}{j} = -\binom{1-s}{j}\left(\frac{1}{1-s}+\frac{1}{-s}+\ldots+\frac{1}{-s-j+2}\right).$$

If $1-s = n \leqslant j-1$ then

$$\frac{d}{ds}\binom{1-s}{j} = -\frac{n(n-1)\ldots(1)\cdot(-1)(-2)\ldots(n-j+1)}{j!} = \frac{(-1)^{j-n}}{j}\,\frac{1}{\binom{j-1}{n}}.$$

Putting everything together, we obtain

$$\begin{aligned} L_p'(1-n,\chi) = \frac{1}{F}\sum_{\substack{a=1 \\ p\nmid a}}^{F}\chi(a)\Bigg\{&\frac{-\langle a\rangle^n}{n^2}\sum_{j=0}^{\infty}\binom{n}{j}\left(\frac{-F}{a}\right)^j B_j + \\ &+\frac{1}{n}\langle a\rangle^n(\log_p a)\sum_{j=0}^{\infty}\binom{n}{j}\left(\frac{-F}{a}\right)^j B_j + \\ &+\frac{1}{n}\langle a\rangle^n\sum_{j=1}^{n}\binom{n}{j}\left(\frac{1}{n}+\ldots+\frac{1}{n-j+1}\right)\left(\frac{-F}{a}\right)^j B_j - \\ &-\frac{1}{n}\langle a\rangle^n\sum_{j=n+1}^{\infty}(-1)^{j-n}\frac{B_j}{j}\,\frac{1}{\binom{j-1}{n}}\left(\frac{-F}{a}\right)^j\Bigg\}. \end{aligned}$$

This may be rewritten as

$$(2_p)\qquad L_p'(1-n,\chi) = \sum_{a=1}^{F} \chi\omega^{-n}(a)\, H_n^p\!\left(\frac{a}{F}\right),$$

where

$$H_n^p(X) = \frac{(-1)^n F^{n-1}}{n}\Big[(\log_p FX)B_n(-X) - \frac{1}{n}B_n(-X) + \sum_{j=1}^{n}\binom{n}{j}\left(\frac1n+\dots+\frac{1}{n-j+1}\right)(-X)^{n-j}B_j - \sum_{j=n+1}^{\infty}\frac{B_j}{j}\frac{1}{\binom{j-1}{n}}X^{n-j}\Big].$$

Since $B_n'(X)=nB_{n-1}(X)$, we find that

$$H_n^p(X)' = \frac{(-1)^nF^{n-1}}{n}\Big[\frac{1}{X}B_n(-X) - (\log_p FX)nB_{n-1}(-X) + B_{n-1}(-X) - \sum_{j=1}^{n-1}\binom{n}{j}\left(\frac1n+\dots+\frac{1}{n-j+1}\right)(n-j)(-X)^{n-j-1}B_j - \sum_{j=n+1}^{\infty} \frac{B_j}{j}\frac{1}{\binom{j-1}{n}}(n-j)X^{n-j-1}\Big].$$

Since

$$\frac{n-j}{n}\binom{n}{j} = \binom{n-1}{j} \quad\text{and}\quad \frac{n}{n-j}\binom{j-1}{n} = -\binom{j-1}{n-1},$$

it follows that

$$H_n^p(X)' = (n-1)FH_{n-1}^p(X) + (-F)^{n-1}\Big[\frac{-1}{nX}B_n(-X) - \frac1n B_{n-1}(-X) + \frac{1}{n-1}B_{n-1}(-X) + \sum_{j=1}^{n-1}\binom{n-1}{j}\left(\frac1n - \frac{1}{n-j}\right)(-X)^{n-j-1}B_j + \frac{B_n}{nX}\Big].$$

Now

$$\binom{n-1}{j}\left(\frac1n-\frac1{n-j}\right) = \frac1n\left(\binom{n-1}{j}-\binom{n}{j}\right) = -\frac1n\binom{n-1}{j-1}.$$

Therefore the expression in brackets becomes (let $\binom{n-1}{-1}=0$)

$$\sum_{j=0}^{n-1}(-X)^{n-j-1}(B_j)\left\{\frac1n\binom{n}{j} - \frac1n\binom{n-1}{j} - \frac1n\binom{n-1}{j-1} + \frac{1}{n-1}\binom{n-1}{j}\right\}$$
$$= \sum_{j=0}^{n-1}(-X)^{n-j-1}(B_j)\left\{\frac{1}{n-1}\binom{n-1}{j}\right\} = \frac{1}{n-1}B_{n-1}(-X).$$

We now have

$$(3_p)\qquad H_n^p(X)' = (n-1)FH^p_{n-1}(X) + \frac{(-F)^{n-1}}{n-1}B_{n-1}(-X), \qquad n\geqslant 2.$$

Also, from the definition of $H_n^p(X)$,

$$(4_p)\qquad H_1^p(X) = G_p(X) + (X-\tfrac12)\log_p F.$$

Therefore

$$(5_p)\qquad L_p'(1-n,\chi)$$
$$= (n-1)!F^{n-1}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi\omega^{-n}(a)\Big[\int\underset{(n-1)}{\dots}\int G_p(X) + \text{polynomial of degree } n\Big]_{X=a/F}.$$

If p does not divide g = conductor of $\chi\omega^{-n}$, then we may use the lemma to rewrite (5_p) as

$$(5_p')\qquad L_p'(1-n,\chi)$$
$$= (n-1)!\,g^{n-1}\sum_{a=1}^{g}\chi\omega^{-n}(a)\Big[\int\underset{(n-1)}{\dots}\int \log_p\Gamma_p(X) + \text{pseudo-polynomial of degree } n\Big]_{X=a/g};$$

by "pseudo-polynomial" we mean a function of the form

$$\sum_{\substack{k=0\\ X+k\not\equiv 0\,(p)}}^{p-1} P\left(\frac{X+k}{p}\right),$$

where P is a polynomial of degree n.

When $n=1$, this formula is due to Ferrero (see [2]).

4. The derivative in the complex case. We now show that the results of the previous section are analogues of the situation for the classical L-functions. Let

$$H_n(X) = \frac{d}{ds}\left(\frac{1}{F^s}\zeta(s,X)\right)\Big|_{s=1-n},$$

where

$$\zeta(s, X) = \sum_{m=0}^{\infty} \frac{1}{(X+m)^s}, \quad \mathrm{Re}(s) > 1,$$

is the Hurwitz zeta function (which may be continued meromorphically to the whole plane). Then

$$(2_\infty) \qquad L'(1-n, \chi) = \sum_{a=1}^{F} \chi(a) H_n\left(\frac{a}{F}\right).$$

Also

$$H'_n(X) = \frac{d}{ds}\left(\frac{1}{F^s} \sum_{m=0}^{\infty} \frac{-s}{(X+m)^{s+1}}\right)\Bigg|_{s=1-n}$$

$$= (n-1)FH_{n-1}(X) - F^{n-1}\zeta(2-n, X).$$

Since

$$\zeta(2-n, X) = (-1)^n \frac{B_{n-1}(-X)}{n-1}$$

(see [6], p. 267; the formula there may be modified to obtain the present formula), we obtain

$$(3_\infty) \quad H'_n(X) = (n-1)FH_{n-1}(X) + \frac{(-F)^{n-1}}{n-1} B_{n-1}(-X), \qquad n \geqslant 2.$$

In addition ([6], p. 271)

$$(4_\infty) \qquad H_1(X) = \log\left(\Gamma(X)/\sqrt{2\pi}\right) + (X - \tfrac{1}{2})\log F.$$

Therefore

$$(5_\infty) \quad L'(1-n, \chi)$$

$$= (n-1)!\,F^{n-1} \sum_{a=1}^{F} \chi(a) \Big[\int \underset{(n-1)}{\cdots} \int \log\left(\Gamma(x)/\sqrt{2\pi}\right) + \text{polynomial of degree } n \Big]_{X=a/F}.$$

In fact, formulas (3_p) and (3_∞) show that we may choose the antiderivatives so that the polynomials of degree n are identical (except that $\log_p F$ is replaced by $\log F$).

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Received on 21.3.1979 (1147)