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On the mean values and distributions of arithmetic functions

by

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1. Introduction. Let $\{b(n)\}$ be a sequence of integers, for which there exist $T > 0$ and $0 < \alpha < 1$, such that $n^\alpha < Tb(n) \leqslant Tn$ for all integers $n \geqslant 2$. A set of positive integers E is said to *have b-density* $\delta(E)$ if $P_n(E, b) \to \delta(E)$ as $n \to \infty$, where

$$P_n(E, b) = \frac{1}{b(n)} \operatorname{card}\{m \in E\colon\ n < m \leqslant n+b(n)\}.$$

The α-density defined in [5] is a special case of b-density, with $b(n) = [n^\alpha]$. Here $[x]$ denotes the largest integer not exceeding x. As in [5] it can be shown that a set E has a natural density, whenever it has a b-density and in such a case the two densities are the same. Unlike b-density the natural density is not capable of detecting large gaps; see the examples given in [5].

A complex-valued arithmetic function g is said to *have a b-mean value* if

$$\frac{1}{b(n)} \sum_{n<m\leqslant n+b(n)} g(m)$$

tends to a limit as $n \to \infty$. If $|g(m)| \leqslant 1$ for all $m \geqslant 1$ and if g has a b-mean value, then it is clear, as shown in [5], that g has a mean-value in the usual sense.

A complex-valued arithmetic function g is called *multiplicative* if $g(1) = 1$ and $g(mn) = g(m)g(n)$, whenever m and n are prime to each other. Let $\mathscr{M}$ denote the class of all multiplicative functions g satisfying $|g(m)| \leqslant 1$ for all integers $m \geqslant 1$.

A complex-valued arithmetic function f is called *additive* if $f(mn) = f(m)+f(n)$, whenever m and n are prime to each other.

In this paper we obtain conditions for the existence of b-mean values for functions in $\mathscr{M}$. We then use these results to get some conditions for the existence of the distributions of additive functions in the sense of b-density.

In the last section we generalize the Erdös–Kac theorem to the b-density case. We also consider weak convergence of additive functions to infinitely divisible distributions as in [2]. As a special case we obtain the following result. Let $\omega(m)$ denote the number of prime factors of m. For any $\varepsilon > 0$ and $\beta > 0$ we have, except possibly for $o(n^\beta)$ integers $m \in (n, n+n^\beta]$, that

$$|\omega(m) - \log\log m| < (\log\log m)^{1/2+\varepsilon}.$$

We mainly use elementary number theoretic arguments. Probabilistic arguments are used liberally in the last two sections.

Throughout this paper p and q stand for prime numbers, d, k, r, s, n, m for positive integers and j for a non-negative integer; $d|m$ means d divides m and $d\nmid m$ means d does not divide m; $p^j\|m$ means $p^j|m$ and $p^{j+1}\nmid m$. Finally, let

$$\delta_p(m) = \begin{cases} 1 - \dfrac{1}{p} & \text{if } p|m, \\ -\dfrac{1}{p} & \text{otherwise}. \end{cases}$$

2. b-mean values of multiplicative functions. The main object of this section is to prove Theorem 1 stated below. For this purpose, define for a multiplicative function g,

$$g_k(m) = \prod_{p^j\|m,\, p<k} g(p^j).$$

Suppose $u \in \mathcal{M}$ and $\sum_p \frac{1}{p}\big(1-u(p)\big)$ converges. Then the infinite product

$$\prod_p \left(1-\frac{1}{p}\right)\left(1+\sum_{j=1}^{\infty} u(p^j)\right)p^{-j}$$

converges. We denote this product by $\xi(u)$. If $u \in \mathcal{M}$, then obviously $u_k \in \mathcal{M}$ and $\xi(u_k)$ is well defined. Let $\mathcal{M}_b$ denote the class of u in $\mathcal{M}$ satisfying, for each $\varepsilon > 0$,

(1) $$\operatorname{card}\{p^j \leqslant n\colon |1-u(p^j)| > \varepsilon\} = o\big(b(n)\big)$$

as $n \to \infty$.

THEOREM 1. *Suppose $u \in \mathcal{M}_b$ and $\sum_p \frac{1}{p}\big(1-\operatorname{Re} u(p)\big) < \infty$. Then*

(2) $$\frac{1}{b(n)} \sum_{n<m\leqslant n+b(n)} u(m) = \xi(u^n) + o(1),$$

as $n \to \infty$.

Before proving Theorem 1, we consider the following corollary and the remarks.

COROLLARY 1. *Suppose $u \in \mathcal{M}_b$ and $\sum_p \frac{1}{p}\big(1-u(p)\big)$ converges. Then u has b-mean value $\xi(u)$.*

Proof of Corollary 1. If $\sum_p \frac{1}{p}\big(1-u(p)\big)$ converges, then $\xi(u_n) \to \xi(u)$ as $n \to \infty$. So the result follows from Theorem 1.

Remark 1. Without some additional assumption like condition (1), the result is false. An example of a sequence $\{q_n\}$ of primes is constructed in [5] such that $\sum_{n=1}^{\infty} q_n^{-1} < \infty$ and the set E of integers, which are not divisible by any of the primes q_n, does not have α-density. In other words the set E does not have a b-density, if $b(n) = [n^\alpha]$. Let u be the multiplicative function defined by $u(q_n^j) = 0$ for all n and $j \geqslant 1$ and for other primes $u(p^j) = 1$ for all j. Clearly $u \in \mathcal{M}$. But

$$n^{-\alpha} \sum_{n<m\leqslant n+n^\alpha} u(m) = n^{-\alpha}\operatorname{card}\{m \in E\colon n < m \leqslant n+n^\alpha\}$$

does not tend to a limit.

Remark 2. From the results of [3], it follows that $\lim_{n\to\infty} \frac{1}{n}\Big|\sum_{m\leqslant n} g(m)\Big|$ exists for all $g \in \mathcal{M}$. But the $u \in \mathcal{M}$ constructed in Remark 1 shows that this is not the case with b-mean values.

To prove Theorem 1, we require the following lemmas.

LEMMA 1. *Let $k \leqslant s$ and $\{a_p\}$ be a sequence of real numbers. Then*

(3) $$\sum{}' \Big(\sum{}'' a_p \delta_p(m)\Big)^2 \leqslant (2s+8k^2) \sum{}'' \frac{1}{p} a_p^2,$$

where $\sum'$ denotes the sum over all integers $m \in (r, r+s]$ and $\sum''$ denotes the sum over all primes $p < k$.

Proof. The left-hand side of (3) is not more than

(4) $$\sum{}'' a_p^2 \sum{}' \delta_p^2(m) + \sum_{p,q<k,\, p\neq q} \Big|a_p a_q \sum{}' \delta_p(m)\,\delta_q(m)\Big|.$$

We estimate $\sum' \delta_p^2(m)$ and $\big|\sum' \delta_p(m)\,\delta_q(m)\big|$ separately. First note that for any positive integer d, the number of integers $m \in (r, r+s]$ that are divisible by d is $[(r+s)/d]-[r/d]$. This number lies between $s/d-1$ and $s/d+1$. So, for $2 \leqslant p < k$, we have

(5) $$\sum{}' \delta_p^2(m) \leqslant \left(1-\frac{1}{p}\right)^2\left(\frac{s}{p}+1\right) + \frac{1}{p^2}\left(s+1-\frac{s}{p}\right) \leqslant \frac{2s}{p}.$$

Also if $2 \leqslant p, q < k$ and $p \neq q$, then

(6) $$\Big|\sum{}' \delta_p(m)\,\delta_q(m)\Big| \leqslant \Big(\frac{s}{pq}+1\Big)\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big) - \frac{1}{q}\Big(1-\frac{1}{p}\Big)\Big(\frac{s}{p}-\frac{s}{pq}-2\Big) - \frac{1}{p}\Big(1-\frac{1}{q}\Big)\Big(\frac{s}{q}-\frac{s}{pq}-2\Big) + \frac{1}{pq}\Big(s-\frac{s}{p}-\frac{s}{q}+\frac{s}{pq}+3\Big) \leqslant 8.$$

By Cauchy–Schwarz inequality

(7) $$\Big(\sum{}'' |a_p|\Big)^2 \leqslant \Big(\sum{}'' \frac{1}{p} a_p^2\Big)\Big(\sum{}'' p\Big) \leqslant k^2 \sum{}'' \frac{1}{p} a_p^2.$$

The lemma now follows from (4), (5), (6) and (7).

For any complex-valued arithmetic function g and real numbers $x < y$, let

$$M(g, x, y) = \frac{1}{y-x} \sum_{x<m\leqslant y} g(m).$$

LEMMA 2. *Let $s-r \to \infty$ such that $(\log s)^n/(s-r) \to 0$ for some n and let $u, h \in \mathcal{M}$ such that $u(p^j) = 1$ for $p \geqslant n$, $h(p^j) = 1$ for $p < n$ and $h(p^j) = (h(p))^j$ for all $j \geqslant 1$. Suppose $k = k(r, s) \to \infty$ such that*

(8) $$M(h_k, r, s) - \xi(h_k) = o(1).$$

Then

$$M(v_k, r, s) = \xi(u)\,\xi(h_k) + o(1),$$

where v is the multiplicative function defined by $v(m) = u(m)h(m)$ for all $m \geqslant 1$.

Proof. If g is the multiplicative function defined by

$$g(p^j) = v_k(p^j) - h_k(p)\,v_k(p^{j-1})$$

for $j \geqslant 1$, then $g(p^j) = 0$ for $p \geqslant n$, $j \geqslant 1$, $|g(m)| \leqslant 2^{\omega(m)}$ for all m and $g(p) = u(p) - 1$ for all primes. Further, we have

$$\sum_{d=1}^{\infty} \frac{|g(d)|}{d} \leqslant \prod_{p<n}\Big(1+\sum_{j=1}^{\infty} |g(p^j)|p^{-j}\Big) \leqslant \prod_{p<n}\Big(1+\frac{1}{p}|1-u(p)| + O(p^{-2})\Big) < \infty.$$

Hence

$$\sum_{d=1}^{\infty} \frac{g(d)}{d} = \prod_{p<n}\Big(1+\sum_{j=1}^{\infty} g(p^j)p^{-j}\Big) = \xi(u).$$

From the definition of g, we have

$$v_k(m) = \sum_{d|m} g(d)\,h_k(m/d),$$

so that for any $j \geqslant 1$,

(9) $$M(v_k, r, s) = \sum_{d\leqslant s} g(d)\Big(\frac{1}{s-r}\sum_{r<md\leqslant s} h_k(m)\Big) = \sum_{d\leqslant s} \frac{1}{d} g(d) M(h_k, r/d, s/d)$$
$$= \sum_{d\leqslant j} \frac{1}{d} g(d) M(h_k, r/d, s/d) + O\Big(\sum_{d>j} \frac{|g(d)|}{d}\Big) + O\Big(\frac{1}{s-r}\sum_{d\leqslant s} |g(d)|\Big).$$

Since $g(p^j) = 0$ for $p \geqslant n$, $j \geqslant 1$, we have

(10) $$\sum_{d\leqslant s} |g(d)| \leqslant 2^n \operatorname{card}\{d \leqslant s:\ \text{if } p \geqslant n \text{ then } p \nmid d\} \leqslant \big((2\log 2s)/(\log 2)\big)^n.$$

Further, by (8) we have

(11) $$M(h_k, r/d, s/d) - \xi(h_k) = o(1)$$

uniformly for all $1 \leqslant d \leqslant j$. Since $\sum_{d>j} \frac{1}{d}|g(d)| \to 0$ as $j \to \infty$, the lemma follows from (9), (10), (11) and from the hypothesis that $(\log s)^n/(s-r) \to 0$.

LEMMA 3. *Suppose $u \in \mathcal{M}$ is such that $u(p^j) = 1$ for $p \geqslant n$, $j \geqslant 0$. Then we have $M(u, r, s) \to \xi(u)$, whenever $s-r \to \infty$ such that $(\log s)^n/(s-r) \to 0$.*

Proof. The lemma follows from Lemma 2 on taking $h(m) = 1$ for all m.

LEMMA 4. *Let $\{a_p\}$ be a sequence of real numbers such that $\sum_p \frac{1}{p} a_p^2 < \infty$. Let h be the multiplicative function defined by $h(p^j) = e^{ija_p}$. Suppose $s-r \to \infty$ such that $(\log s)^n/(s-r) \to \infty$ for all integers $n \geqslant 2$. Then*

$$M(h_k, r, s) = \xi(h_k) + o(1),$$

where $k \to \infty$ such that $k^2 < s-r$.

Proof. Let $E_n = \sum_{p<n} \frac{1}{p} a_p$ and $D_n = \sum_{p<n} \frac{1}{p} a_p^2$. Since $|e^{iE_n}| = 1$, we have for any $n \leqslant k$,

(12) $$|M(h_k, r, s) - \xi(h_k)| = |M(h_k, r, s)e^{-iE_k} - \xi(h_k)e^{-iE_k}|$$
$$\leqslant |M(h_n, r, s) - \xi(h_n)| + |\xi(h_k)e^{-iE_k} - \xi(h_n)e^{-iE_n}| + \frac{1}{s-r}\Big|\sum_{r<m\leqslant s} \big(h_k(m)e^{-iE_k} - h_n(m)e^{-iE_n}\big)\Big|.$$

Since $\sum_p \frac{1}{p} a_p^2 < \infty$,

$$\xi(h_n)e^{-iE_n} \to \prod_p \left(1-\frac{1}{p}\right)\left(1-\frac{1}{p}e^{ia_p}\right)^{-1} e^{-i\frac{1}{p}a_p} \tag{13}$$

and

$$\sum_{p>n} \frac{1}{p} a_p^2 \to 0 \tag{14}$$

as $n \to \infty$. Further, since $k^2 < s-r$, the number N_1, of integers $m \in (r, s]$ which are divisible by p^2 for some $p \in [n, k)$, does not exceed

$$\sum_{n\leqslant p<k} ([sp^{-2}]-[rp^{-2}]) \leqslant \sum_{n\leqslant p<k} ((s-r)p^{-2}+1).$$

Thus

$$N_1 \leqslant 2(s-r)\sum_{p\geqslant n} p^{-2} \leqslant \frac{4}{n}(s-r). \tag{15}$$

Hence by (13), (14) and (15), for any $\varepsilon > 0$ there exists an integer $n(\varepsilon)$ such that, whenever $k > n \geqslant n(\varepsilon)$

$$N_1 < (s-r)\varepsilon, \qquad D_k - D_n < \varepsilon^3 \tag{16}$$

and

$$|\xi(h_k)e^{-iE_k} - \xi(h_n)e^{-iE_n}| < \varepsilon. \tag{17}$$

If $p^2 \nmid m$ for any $p \in [n, k)$, then

$$|h_k(m)e^{-iE_k} - h_n(m)e^{-iE_n}| = \left|\exp\Big(i\sum_{n\leqslant p<k} a_p\delta_p(m)\Big) - 1\right| \leqslant \Big|\sum_{n\leqslant p<k} a_p\delta_p(m)\Big|.$$

By (16), Lemma 1 and by the Chebyshev's inequality, it follows that the number N_2 of integers $m \in (r, s]$ for which $\Big|\sum_{n\leqslant p<k} a_p\delta_p(m)\Big| > \varepsilon$ is not more than

$$\varepsilon^{-2}\sum_{r<m\leqslant s}\Big|\sum_{n\leqslant p<k} a_p\delta_p(m)\Big|^2 < 10(s-r)\varepsilon^{-2}(D_k-D_n) < 10\varepsilon(s-r). \tag{18}$$

Since $|h_j(m)| = 1$, we have by (12), (16), (17) and (18), that

$$\begin{aligned} |M(h_k, r, s) - \xi(h_k)| &\leqslant |M(h_n, r, s) - \xi(h_n)| + 2\varepsilon + (2N_1+N_2)(s-r)^{-1} \\ &\leqslant |M(h_n, r, s) - \xi(h_n)| + 14\varepsilon, \end{aligned} \tag{19}$$

whenever $s - r > k^2$ and $k > n \geqslant n(\varepsilon)$. The result now follows from (19) and Lemma 3.

LEMMA 5. *Let $u \in \mathcal{M}$ such that $\sum_p \frac{1}{p}(1-\mathrm{Re}\,u(p)) < \infty$. Let $s-r \to \infty$ such that for all integers $n \geqslant 1$, $(\log s)^n/(s-r) \to 0$ and let $k \to \infty$ such that $k^3 < s-r$. Then*

$$M(u_k, r, s) - \xi(u_k) = o(1).$$

Proof. Let h be the multiplicative function defined by

$$h(p) = \big(1-(\mathrm{Im}\,u(p))^2\big)^{1/2} + i\,\mathrm{Im}\,u(p)$$

and $h(p^j) = (h(p))^j$. Clearly $h \in \mathcal{M}$ and $|h(m)| = 1$. So there exists $|a_p| < \pi$ such that $h(p) = e^{ia_p}$. Since $\mathrm{Re}\,u(p) \leqslant \mathrm{Re}\,h(p)$ and $|h(p)-u(p)| \leqslant \mathrm{Re}\,h(p) - \mathrm{Re}\,u(p) \leqslant 1 - \mathrm{Re}\,u(p)$, we have

$$\sum_p \frac{1}{p}(1-\mathrm{Re}\,h(p)) < \infty \tag{20}$$

and

$$\sum_p \frac{1}{p}|h(p)-u(p)| < \infty. \tag{21}$$

Let for any multiplicative function g,

$$g_{nk}(m) = \prod_{p^j\| m,\, n\leqslant p<k} g(p^j).$$

We have clearly,

$$\begin{aligned} |M(u_k, r, s) - \xi(u_k)| \leqslant &\big|M(|u_k - u_n h_{nk}|, r, s)\big| + \\ &+ |M(u_n h_{nk}, r, s) - \xi(u_n)\xi(h_{nk})| + |\xi(u_n)\xi(h_{nk}) - \xi(u_k)|. \end{aligned} \tag{22}$$

We now estimate the terms on the right-hand side of (22) separately. Since $u_n \in \mathcal{M}$,

$$|u_k(m) - u_n(m)h_{nk}(m)| \leqslant |u_{nk}(m) - h_{nk}(m)| \tag{23}$$

for all $m \geqslant 1$. So

$$\begin{aligned} (s-r)M(|u_{nk}-h_{nk}|, r, s) &\leqslant N_1 + \sum_{n\leqslant p<k,\, p|m} |u(p)-h(p)| \\ &\leqslant N_1 + 2(s-r)\sum_{n\leqslant p<k} \frac{1}{p}|u(p)-h(p)|, \end{aligned} \tag{24}$$

where N_1 is defined in the proof of Lemma 4. By (15), (21), (23) and (24), for any $\varepsilon > 0$ there exists a $n(\varepsilon)$ such that for all $k > n \geqslant n(\varepsilon)$,

$$M(|u_k - u_n h_{nk}|, r, s) < \varepsilon. \tag{25}$$

Since $|\xi(u_n)|\leqslant 1$ for all n, we have

$$|\xi(u_n)\xi(h_{nk})-\xi(u_k)|\leqslant|\xi(h_{nk})-\xi(u_{nk})|$$
$$\leqslant\Big|\prod_{n\leqslant p<k}\Big(1+\frac{h(p)-1}{p}+O(p^{-2})\Big)-\prod_{n\leqslant p<k}\Big(1+\frac{u(p)-1}{p}+O(p^{-2})\Big)\Big|$$
$$\leqslant\sum_{n\leqslant p<k}\frac{1}{p}|h(p)-u(p)|+O\Big(\sum_{p\geqslant n}p^{-2}\Big).$$

Thus in view of (21) we have, for any $\varepsilon>0$ there exists an integer $n_1\geqslant n(\varepsilon)$, such that for all $k>n\geqslant n_1$,

$$|\xi(u_n)\xi(h_{nk})-\xi(u_k)|<\varepsilon. \tag{26}$$

Hence, for $k>n\geqslant n_1$ and $s-r>k^3$, we have by (22), (25) and (26) that

$$|M(u_k,r,s)-\xi(u_k)|<2\varepsilon+|M(u_nh_{nk},r,s)-\xi(u_n)\xi(h_{nk})|. \tag{27}$$

Finally, as $1-\operatorname{Re}h(p)=1-\cos a_p=2(\sin\frac{1}{2}a_p)^2\geqslant\theta a_p^2$ for some constant $\theta>0$, we have from (20) that $\sum_p\frac{1}{p}a_p^2<\infty$. So by Lemma 4, we have for each $n<k$, that

$$M(h_{nk},r,s)=\xi(h_{nk})+o(1).$$

Consequently, by Lemma 2, we have for each $n<k$, that

$$M(u_nh_{nk},r,s)=\xi(u_n)\xi(h_{nk})+o(1). \tag{28}$$

The lemma now follows from (27) and (28).

Lemma 6. *Let $u\in\mathcal{M}$ be such that $\sum_p\frac{1}{p}(1-\operatorname{Re}u(p))<\infty$. Then for any $\lambda>1$, $\xi(u_n)-\xi(u_{n^\lambda})\to 0$ as $n\to\infty$.*

Proof. We have by using Cauchy–Schwarz inequality, that

$$|\xi(u_n)-\xi(u_{n^\lambda})|\leqslant\Big|\prod_{n\leqslant p<n^\lambda}\Big(1-\frac{1}{p}\Big)\Big(1+\sum_{j=1}^{\infty}u(p^j)p^{-j}\Big)-1\Big| \tag{29}$$
$$\leqslant\sum_{n\leqslant p<n^\lambda}\frac{1}{p}|1-u(p)|+O\Big(\sum_{p\geqslant n}p^{-2}\Big)$$
$$\leqslant\Big(\sum_{n\leqslant p<n^\lambda}\frac{1}{p}|1-u(p)|^2\Big)^{1/2}\Big(\sum_{n\leqslant p<n^\lambda}\frac{1}{p}\Big)^{1/2}+O(n^{-1}).$$

Since there exists a constant A (see [4]) such that

$$\sum_{p\leqslant x}\frac{1}{p}=\log\log x+A+o(1), \tag{30}$$

as $x\to\infty$ and since $|1-z|^2\leqslant 2(1-\operatorname{Re}z)$ for $|z|\leqslant 1$, the result follows from (29).

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let a be as in the definition of $\{b(n)\}$. Put $k=k(n)=[n^{a/4}]$. We have by Lemmas 5 and 6 that

$$M(u_k,n,n+b(n))=\xi(u_n)+o(1), \tag{31}$$

as $n\to\infty$. To complete the proof, it is enough to show that

$$M(|u-u_k|,n,n+b(n))\to 0$$

as $n\to\infty$. Let for $\varepsilon>0$, $H(n,\varepsilon)$ denote the set of $m\in(n,n+b(n)]$ for which $|u(m)-u_k(m)|>\varepsilon$. Since

$$|u(m)-u_k(m)|\leqslant\sum_{k\leqslant p<2n,\,p^j\|m}|1-u(p^j)|,$$

we have that if $m\in H(n,\varepsilon)$, then there exists a $p\in[k,2n)$ such that $p^j\|m$ and $|1-u(p^j)|>\varepsilon a/8$ for some $j\geqslant 1$. So

$$\operatorname{card}H(n,\varepsilon)\leqslant\sum\nolimits^{*}\operatorname{card}\{m\in(n,n+b(n)]:\ p^j\|m\},$$

where $\sum^*$ denotes the sum over all $p^j\leqslant 2n$ for which $|1-u(p^j)|>\varepsilon a/8$ and $k\leqslant p<2n$. Hence

$$\operatorname{card}H(n,\varepsilon)\leqslant\sum\nolimits^{*}\big(b(n)p^{-j}+1\big)\leqslant b(n)\Big(\sum\nolimits^{**}1/p+\sum_{p>k;\,j\geqslant 2}p^{-j}\Big)+\sum\nolimits^{*}1, \tag{32}$$

where $\sum^{**}$ is the sum over $p\leqslant 2n$ for which $|1-u(p)|>\varepsilon a/8$.

Now

$$\sum\nolimits^{**}\frac{1}{p}\leqslant\frac{64}{\varepsilon^2a^2}\sum_{p>k}\frac{1}{p}|1-u(p)|^2\leqslant\frac{130}{\varepsilon^2a^2}\sum_{p>k}\frac{1}{p}\big(1-\operatorname{Re}u(p)\big)=o(1).$$

It follows now, by (1) and (32), that $\operatorname{card}H(n,\varepsilon)=o(b(n))$. So for any $\varepsilon>0$,

$$M(|u-u_k|,n,n+b(n))\leqslant\varepsilon+\frac{2}{b(n)}\operatorname{card}H(n,\varepsilon)\leqslant\varepsilon+o(1).$$

This completes the proof of the theorem.

3. Distributions of arithmetic functions. Let $\{f_n\}$ be a sequence of real-valued arithmetic functions. The distribution of f_n under $P_n(\cdot,b)$ is given by Q_n, where $Q_n(c)=P_n\big(\{m:\ f_n(m)<c\},b\big)$. If Q_n converges

weakly to F, where F is a distribution function, we write $f_n \overset{b}{\Rightarrow} F$. That is for every continuity point c of F, $P_n\big((m\colon f_n(m) < c), b\big) \to F(c)$ as $n \to \infty$.

A real-valued arithmetic function f is said to have a *b-distribution* if $f_n \overset{b}{\Rightarrow} F$, for the sequence $f_n \equiv f$, where F is some distribution function on the real line.

THEOREM 2. *Let f be a real-valued additive arithmetic function satisfying, for each $\varepsilon > 0$,*

$$\operatorname{card}\{p^j \leqslant n\colon |f(p^j)| > \varepsilon\} = o(b(n)), \tag{33}$$

as $n \to \infty$. Then f has a b-distribution if and only if

$$\sum_p{}' \frac{1}{p} f(p) \quad \text{converges}, \tag{34}$$

$$\sum{}' \frac{1}{p} f^2(p) < \infty \tag{35}$$

and

$$\sum{}'' 1/p < \infty, \tag{36}$$

where $\sum'$ denotes the sum over all primes p for which $|f(p)| < 1$ and $\sum''$ denotes the sum over the remaining primes.

Before proving the theorem we make few remarks.

Remark 3. If $b(n) \leqslant n(\log n)^{-2}$, then (33) implies (36). So in this case condition (36) is redundant.

Remark 4. By the Lévy's continuity theorem (see Theorem 3.6.1 in [9]), a sequence $\{f_n\}$ of real-valued arithmetic functions $\overset{b}{\Rightarrow}$ to a distribution if and only if, for each real number t

$$\int e^{itf_n([x])} P_n(dx, b) = \frac{1}{b(n)} \sum_{n<m\leqslant n+b(n)} e^{itf_n(m)}$$

tends to a limit $\Psi(t)$, which is continuous at $t = 0$.

Proof of Theorem 2. For each real number t, define the multiplicative function $g^{(t)}$ by $g^{(t)}(m) = e^{itf(m)}$. In view of Remark 4, it is enough to show that there exists a function Ψ on the real line which is continuous at zero such that, for each t, $\Psi(t)$ is the b-mean value of $g^{(t)}$. Clearly $g^{(t)} \in \mathscr{M}_b$. Since for any real number x, $|e^{ix} - 1 - ix| \leqslant x^2$, it follows from (34), (35) and (36) that $\sum_p \frac{1}{p}\big(1 - g^{(t)}(p)\big)$ converges. So the infinite product $\xi(g^{(t)})$ converges and by Corollary 1, $\xi(g^{(t)})$ is the b-mean value of $g^{(t)}$. Clearly $\xi(g^{(t)})$ is continuous at zero. Hence f has a b-distribution.

As is mentioned in the introduction if f has a b-distribution, then it has a distribution in the sense of natural density. But in this case, by Erdős–Wintner theorem [8], (34), (35) and (36) hold. This completes the proof of Theorem 2.

Using Cramér–Wold device [1] and Theorem 2, we can easily deduce the following theorem.

THEOREM 3. *Let $f_1, \ldots, f_s$ be real-valued arithmetic functions satisfying for each $\varepsilon > 0$,*

$$\sum_{j=1}^{s} \operatorname{card}\{p^k \leqslant n,\ |f_j(p^k)| > \varepsilon\} = o(b(n)).$$

If the series

$$\sum_{|f_j(p)|<1} \frac{1}{p} f_j(p), \qquad \sum_{|f_j(p)|<1} \frac{1}{p} f_j^2(p) \quad \text{and} \quad \sum_{|f_j(p)|\geqslant 1} \frac{1}{p}$$

converge for $j = 1, \ldots, s$, then there exists a distribution function F on R^s such that at each of its continuity points

$$P_n\big((m\colon f_j(m) < c_j;\ j = 1, \ldots, s), b\big) \to F(c_1, \ldots, c_s)$$

as $n \to \infty$.

Remark 5. Let g be a real-valued multiplicative function. If $g(m) > 0$ for all m, then g has a non-degenerate b-distribution if and only if the additive function f, defined by $f(m) = \log g(m)$, has a non-degenerate b-distribution.

The following two theorems can be proved using the techniques of [6], [7] and the tools developed so far in this paper. We leave the details to the reader.

THEOREM 4. *Let g be a real-valued multiplicative function satisfying*

$$\sum_{g(p)\leqslant 0} 1/p < \infty, \qquad \operatorname{card}\{p^j \leqslant n\colon g(p^j) < 0\} = o(b(n))$$

and for each $\varepsilon > 0$

$$\operatorname{card}\{p^j \leqslant n\colon |g(p^j)| > e^{\varepsilon} \text{ or } |g(p^j)| < e^{-\varepsilon}\} = o(b(n)). \tag{37}$$

Suppose for some $c > 1$, the series

$$\sum{}' \frac{1}{p} \log|g(p)|, \qquad \sum{}' \frac{1}{p} (\log|g(p)|)^2 \quad \text{and} \quad \sum{}'' \frac{1}{p} \tag{38}$$

converge, where $\sum'$ denote the sum over all p such that $c^{-1} < |g(p)| < c$ and $\sum''$ denotes the sum over the remaining primes. Then g has b-distribution.

THEOREM 5. *Let g be a real-valued multiplicative function such that $\sum_{g(p)<0} 1/p = \infty$, $\sum_{g(p)=0} 1/p < \infty$. Let u be the multiplicative function defined by*

$$u(p^j) = \begin{cases} 1 & \text{if} \quad g(p^j) \geqslant 0, \\ -1 & \text{if} \quad g(p^j) < 0. \end{cases}$$

Suppose for each $\varepsilon > 0$, (37) holds, for some $c > 1$, the three series in (38) converge and u has a b-mean value (in which case it will have zero mean value, see [7]). Then g has a symmetric non-degenerate b-distribution.

We do not know under what conditions u defined above has a b-mean value.

4. Convergence to infinitely divisible laws. The object of this section is to prove some of the results of Sections 5 and 6 of [2] for b-density. We shall first state a central limit theorem, which follows easily from a more general case to be proved later in this section. For a real-valued additive function f, let

$$A_n = \sum_{p\leqslant n} \frac{1}{p} f(p) \quad \text{and} \quad B_n^2 = \sum_{p\leqslant n} \frac{1}{p} f^2(p).$$

THEOREM 6. *Let f be a real-valued additive arithmetic function. If $B_n \to \infty$ and $\max_{p\leqslant n} |f(p)|/B_n \to 0$ as $n \to \infty$, then $(f-A_n)/B_n \overset{b}{\Rightarrow} \Phi$, where Φ denotes the normal distribution with mean zero and variance 1.*

As a consequence of this theorem we have that

$$\frac{\omega - \log\log n}{(\log\log n)^{1/2}} \overset{b}{\Rightarrow} \Phi.$$

If $m \in (n, n+b(n)]$, then

$$0 \leqslant \log\log m - \log\log n \leqslant \log\log 2n - \log\log n \to 0$$

as $n \to \infty$. So for every real number x

$$P_n\big((m:\ \omega(m) - \log\log m < x(\log\log m)^{1/2}),\ b\big) \to \Phi(x) \tag{39}$$

as $n \to \infty$. If $\{a(m)\}$ is any sequence of real numbers such that $a(m) \to \infty$ as $m \to \infty$, then from (39) we have

$$P_n\big((m:\ |\omega(m) - \log\log m| > a(m)(\log\log m)^{1/2}),\ b\big) \to 0$$

as $n \to \infty$. In particular for each $\varepsilon > 0$ and $\beta > 0$, except possibly for $o(n^\beta)$ integers $m \in (n, n+n^\beta]$, we have

$$|\omega(m) - \log\log m| < (\log\log m)^{1/2+\varepsilon}.$$

To consider the general case, we start with the class H, introduced by Kubilius [8], consisting of all real-valued additive functions for which $B_n \to \infty$ and for which there exists a sequence $r_n \to \infty$ such that

$$(\log r_n)(\log n)^{-1} \to 0 \quad \text{and} \quad B_n^{-2} \sum_{r_n < p \leqslant n} \frac{1}{p} f^2(p) \to 0$$

as $n \to \infty$. As done in [2], it will be convenient to omit the normalization by B_n; this can be done by considering a new class. Let H_b be the class of arrays $\{f_n\}$ of additive functions for which

$$\sum_{r_n < p \leqslant n} \frac{1}{p} f_n^2(p) \to 0 \tag{40}$$

for some sequence $\{r_n\}$ satisfying $(\log r_n)(\log n)^{-1} \to 0$, for which

$$\sup_n \sum_{p\leqslant n} \frac{1}{p} f_n^2(p) < \infty, \qquad \lim_{n\to\infty} f_n(m) = 0$$

and satisfying for every $\varepsilon > 0$,

$$\operatorname{card}\{p^j \leqslant n:\ |f_n(p^j)| > \varepsilon\} = o\big(b(n)\big). \tag{41}$$

If $f \in H$, $B_n \to \infty$ and for each $\varepsilon > 0$,

$$\operatorname{card}\{p^j \leqslant n:\ |f(p^j)| > \varepsilon B_n\} = o\big(b(n)\big),$$

then $\{f/B_n\}$ belongs to H_b.

Let K_n be the finite measure on the real line R defined, for each interval M, by

$$K_n(M) = \sum_{p\leqslant n, f_n(p)\in M} \frac{1}{p} f_n^2(p).$$

For any finite measure K, define the infinitely divisible charactersitic function

$$\psi_K(u) = \exp\Big(\int_{-\infty}^{+\infty} \big((e^{iux} - 1 - iux)x^{-2}K(dx)\big)\Big),$$

the integrand at $x = 0$ is taken to be $-u^2/2$. If F_K is the distribution function corresponding to ψ_K, F_K has mean zero variance $K(R)$. We write $K_n \underset{v}{\to} K$ to indicate $K_n(I) \to K(I)$ for all finite intervals I, whose boundary has K measure zero. Notice that if $K_n \underset{v}{\to} K$ and $K_n(R) \to K(R)$, then K_n converges weakly to K.

We are now ready to state an analogue of Theorem 5.1 of [2].

THEOREM 7. *If $\{f_n\} \in H_b$, then a necessary and sufficient condition for $f_n - A'_n \overset{b}{\Rightarrow} F$ is that $F = F_K$ and $K_n \underset{v}{\to} K$ for some K, where $A'_n = \sum_{p\leqslant n} \frac{1}{n} f_n(p)$.*

Proof. Let $k = k_n = [n^{\alpha/4}]$, where $n^\alpha/b(n) \leqslant T$ for all n. Put

$$f_{kn}(m) = \sum_{p^j \| m, p>k} f_n(p^j).$$

Since

$$|f_{kn}(m)| \leqslant \sum_{k<p\leqslant 2n, p^j\|m} |f_n(p^j)|,$$

for any $\varepsilon > 0$, $|f_{kn}(m)| > \varepsilon$ implies that for some $p \in (k, 2n]$, $p^j \| m$ and $|f_n(p^j)| > \alpha\varepsilon/10$ for some $j \geqslant 1$. As $\{f_n\} \in H_b$, by (30), (40) and (41) we have

$$b(n)P_n\big((m\colon |f_{kn}(m)| > \varepsilon), b\big) \leqslant \sum{}^{*} \big(b(n)p^{-j}+1\big) = o\big(b(n)\big)$$

as $n \to \infty$, where $\sum^*$ denotes the sum over $p^j \leqslant 2n$ for which $|f_n(p^j)| > \alpha\varepsilon/10$ and $k < p \leqslant 2n$. Also observe that, by Cauchy–Schwarz inequality, (30) and (40),

$$\Big|\sum_{k<p\leqslant n} \frac{1}{p} f_n(p)\Big| \leqslant \Big(\sum_{k<p\leqslant n} \frac{1}{p} f_n^2(p)\Big)^{1/2} \Big(\sum_{k<p\leqslant n} \frac{1}{p}\Big)^{1/2} = o(1)$$

as $n \to \infty$. So it is enough to show that

$$f_n^* - A_n^* \overset{b}{\Rightarrow} F \tag{42}$$

as $n \to \infty$, where f_n^* is the additive function defined by,

$$f_n^*(p^j) = \begin{cases} f_n(p^j) & \text{if } \quad p \leqslant k, \\ 0 & \text{otherwise} \end{cases}$$

and

$$A_n^* = \sum_{p\leqslant k} \frac{1}{p} f_n(p).$$

The proof of (42) is omitted, since it is similar to that of Theorem 5.1 of [2], the only difference being that we use Lemma 1 instead of Lemma 2.1 of [2].

Similarly, all the results of Sections 5 and 6 of [2] can be extended to b-density case with class H' in [2] replaced by H_b.

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