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Restricted sums of reciprocal values of additive functions

by

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1. Introduction. Let $\mathscr{F}$ denote the set of all multiplicative arithmetical functions f satisfying

$$\prod_{p|n}(1-p^{-\beta})^{\nu} \leqslant f(n)n^{-a} \leqslant \prod_{p|n}(1-p^{-\beta})^{-\nu} \tag{1.1}$$

for all positive integral n and for some positive reals a, β and ν. We write

$$D_f = \{n|\ f(n) \neq 1\} \quad \text{and} \quad G_f = \{n|\ f(m) > 1 \text{ for all } m \geqslant n\}.$$

Recently de Koninck and Galambos [6] obtained an asymptotic formula for $\sum_{2\leqslant n\leqslant x}\big(\log\sigma_1(n)\big)^{-1}$ (see Remark 3 below) where $\sigma_s(n) = \sum_{d|n} d^s$. Evelyn Scriba [1], generalizing this, established an asymptotic formula for $\sum_{n\leqslant x, n\in G_f}\big(\log f(n)\big)^{-1}$ where f is any member of $\mathscr{F}$ subject to the apparently additional conditions $\nu > a$ and $\beta \leqslant 1$ (see Remark 1 below).

In this paper we establish an asymptotic formula for

$$\sum_{n\leqslant x, n\in D_f\cap S}\big(\log f(n)\big)^{-1}$$

where f is any member of $\mathscr{F}$ and S is a set of positive integers subject to some restrictions (statement in § 2 and proof in § 3). In § 4 we exhibit a succession of particular cases of our theorem (Corollaries 1 through 4) in which Corollary 3, besides covering Scriba's result, affords a refinement of it in certain cases (see Remark 2). § 5 contains a rich class of illustrations which result from an application of our theorem to the set of M-void integers introduced by Rieger ([11]).

2. Notation and statement. A set A of positive integers is said to be *multiplicative* provided, for $(a, b) = 1$, one has $ab \in A$ iff $a \in A$ and $b \in A$ or equivalently when the characteristic function χ_A of A is multiplicative. We write $\mathscr{S}$ to denote the class of all multiplicative sets S for each of which there exist numbers $\delta = \delta_S < 1$, $b = b_S \geqslant 1$ and an arithmetical function $p = p_S$ such that, as $x \to \infty$,

$$\sum_{\substack{m \leqslant x \\ (m,n)=1}} \chi_S(m) = p(n)x + O(b^{\omega(n)}x^{\delta}) \tag{2.1}$$

uniformly for all positive integral n, where $\omega(n)$ is the number of distinct prime divisors of n. We write $\mu^*(n)$ to denote $(-1)^{\omega(n)}$ and $\gamma(n)$ for the product of the distinct primes dividing n. Writing $d \| n$ to mean that d is a unitary divisor of n, i.e., $d|n$ and $(d, n/d) = 1$, we note that the above μ^* is the unitary analogue of the Möbius function μ in the sense that $\sum_{d\|n} \mu^*(d) = 1$ or 0 according as $n = 1$ or $n > 1$. For $f \in \mathscr{F}$ and real t we write

$$f_t^*(n) = \sum_{d\|n} \left(\frac{f(d)}{d^a}\right)^t \mu^*\left(\frac{n}{d}\right) \quad \text{(see Lemma 1, (i))}.$$

We observe that whenever f satisfies (1.1) with $a = a_0$, $\beta = \beta_0$ and $\nu = \nu_0$ then f satisfies (1.1) also with $a = a_0$, $\beta = \beta_1$ and $\nu = \nu_1$ for all positive $\beta_1 \leqslant \beta_0$ and all $\nu_1 \geqslant \nu_0$. In the sequel, for given $f \in \mathscr{F}$ and $S \in \mathscr{S}$, we assume, as we may in virtue of the above observation, that $\nu > a$, $\beta < 1 - a/\nu$ and $\beta \leqslant 1 - \delta$. For such β we write $c = c_\beta = (1-2^{-\beta})^{-2}$.

Remark 1. In [1] Scriba imposed the restrictions $\beta \leqslant 1$ and $\nu > a$ in the definition of $\mathscr{F}$. These could be dropped in view of the above observation.

The main result of this paper is the following

THEOREM. *Let $f \in \mathscr{F}$ and $S \in \mathscr{S}$. Then as $x \to \infty$*

$$\sum_{n \leqslant x, n \in D_f \cap S} (\log f(n))^{-1} = x \int_{-1/\nu}^{0} F(t) x^{at} dt + O\left\{\frac{x}{\log x}\left(x^{-\beta}\exp\left(\sqrt{b}L(x)\right) + x^{-a/\nu}\right)\right\} \tag{2.2}$$

where

$$F(t) = F_S(t) = \frac{1}{1+at} \sum_{n=1}^{\infty} \frac{\chi_S(n) f_t^*(n) p(n)}{n} \tag{2.3}$$

and

$$L(x) = (\lambda \log x / \log\log x)^{1/2} \tag{2.4}$$

where λ is any number greater than $8c_\beta$. In particular, for each positive integer r, we have as $x \to \infty$,

$$\sum_{\substack{n \leqslant x \\ n \in D_f}} \frac{\chi_S(n)}{\log f(n)} = x \sum_{m=1}^{r} \frac{(-1)^{m-1} F^{(m-1)}(0)}{(a \log x)^m} + O_r\left(\frac{x}{(\log x)^{r+1}}\right). \tag{2.5}$$

3. Lemmas and proof of the theorem.

LEMMA 1. *If $f \in \mathscr{F}$, then*

(i) *$f(n) > 0$ for every n,*

(ii) *there exists an $N > 1$ such that $f(n) \geqslant n^{a/2}$ for all $n \geqslant N$ and*

(iii) *the complement of G_f and hence that of D_f are finite sets.*

Proof. (i) is clear. Since for a prime p and a positive integer m,

$$f(p^m) p^{-ma/2} \geqslant p^{ma/2}(1-2^{-\beta})^{-\nu}$$

from (1.1), we see that $f(n) n^{-a/2} \to \infty$ as $n \to \infty$ (cf. [9], Theorem 316) and we have (ii). (iii) is immediate from (ii).

LEMMA 2. *Let $h > 0$, $k \geqslant 1$ be constants. For $\sigma > 0$, let us define*

$$A_h^{(k)}(\sigma) = \int_{3/2}^{\infty} \log\{1 + kx^{-1}(x^\sigma - 1)^{-1}\} (\log x)^{-h} dx.$$

Then as $\sigma \to 0+$,

$$A_h^{(k)}(\sigma) = kh^{-1}\sigma^{-1}(\log \sigma^{-1})^{-h} + O\{\sigma^{-1}(\log\sigma^{-1})^{-h-1}\log\log\sigma^{-1}\}$$

where the O-constant may depend upon h and k.

The proof of this lemma follows the same lines as that of Lemma 1 in [5] with the choice $x_2 = k\sigma^{-1}$ instead of the choice $x_2 = \sigma^{-1}$ made in line 7 from below, p. 138 in [5] and consequent minor modifications.

LEMMA 3. *If $k \geqslant 1$ then the Dirichlet series $\sum_{n=1}^{\infty} k^{\omega(n)}(\gamma(n))^{-1} n^{-\sigma}$ converges for $\sigma > 0$ and for the sum function $f(\sigma)$, we have as $\sigma \to 0+$,*

$$\log f(\sigma) \sim k\sigma^{-1}(\log\sigma^{-1})^{-1}.$$

Proof. For $\sigma > 0$, the convergence of the Dirichlet series follows, in virtue of Theorem 41 of [7], from the absolute convergence of the product

$$\prod_p (1 + kp^{-1}(p^\sigma - 1)^{-1}) = \prod_p \left(\sum_{m=0}^{\infty} k^{\omega(p^m)}(\gamma(p^m))^{-1} p^{-m\sigma}\right)$$

which then equals $f(\sigma)$ ($\prod_p$ stands for the product taken over all primes p). The proof of the second conclusion follows the same lines as that of Lemma 2 in [5] except that we now use Lemma 2 above in place of Lemma 1 of [5].

LEMMA 4. *Let $A > 0$, $a(n) \geqslant 0$ $(n = 1, 2, \ldots)$ and let $f(\sigma) = \sum_{n=1}^{\infty} a(n) n^{-\sigma}$ converge for $\sigma > 0$. Assume that $\log f(\sigma) \sim A\sigma^{-1}(\log\sigma^{-1})^{-1}$ as $\sigma \to 0+$. Then as $x \to \infty$,*

$$\log\left(\sum_{n \leqslant x} a(n)\right) \sim (8A \log x)^{1/2} (\log\log x)^{-1/2}.$$

This is a special case of a Tauberian theorem given for general Dirichlet series by Hardy and Ramanujan [8] (see also [10]). Combining Lemmas 3 and 4 we obtain

LEMMA 5. *If $k \geqslant 1$ we have, as $x \to \infty$,*

$$\log\Big\{\sum_{n\leqslant x} k^{\omega(n)}\big(\gamma(n)\big)^{-1}\Big\} \sim (8k\log x)^{1/2}(\log\log x)^{-1/2};$$

consequently

$$\sum_{n\leqslant x} k^{\omega(n)}\big(\gamma(n)\big)^{-1} = O\big\{\exp\big((j\log x)^{1/2}(\log\log x)^{-1/2}\big)\big\}$$

for every $j > 8k$.

LEMMA 6. *For $f \in \mathscr{F}$, we have, as $x \to \infty$,*

$$\sum_{n\leqslant x} |f_t^*(n)| = O\big(x^{1-\beta}\exp L(x)\big) \tag{3.1}$$

and

$$\sum_{n> x} |f_t^*(n)| n^{-1} = O\big(x^{-\beta}\exp L(x)\big) \tag{3.2}$$

uniformly for $t \in [-1/\nu, 0]$ where $L(x)$ is given by (2.4).

Proof. We recall the well-known inequalities

$$a^x - 1 \leqslant xa^{x-1}(a-1) \quad \text{if} \quad x \leqslant 0,\ a > 0 \tag{3.3}$$

and

$$a^x - 1 \geqslant xa^{x-1}(a-1) \quad \text{if} \quad 0 \leqslant x \leqslant 1,\ a > 0. \tag{3.4}$$

From the definition of f_t^* we have, for prime p and positive integral m, that

$$\begin{aligned} f_t^*(p^m) &= \big(f(p^m)p^{-m\alpha}\big)^t - 1 \leqslant (1-p^{-\beta})^{\nu t} - 1 \\ &\leqslant -\nu t p^{-\beta}(1-p^{-\beta})^{\nu t-1} \leqslant (1-2^{-\beta})^{-2}p^{-\beta} = c_\beta p^{-\beta} \end{aligned}$$

by (1.1) and (3.3). On the other hand

$$\begin{aligned} f_t^*(p^m) &= \big(f(p^m)p^{-m\alpha}\big)^t - 1 \geqslant (1-p^{-\beta})^{-\nu t} - 1 \\ &\geqslant \nu t p^{-\beta}(1-p^{-\beta})^{-\nu t-1} \geqslant -(1-2^{-\beta})^{-2}p^{-\beta} = -c_\beta p^{-\beta} \end{aligned}$$

by (1.1) and (3.4). Now the multiplicativity of f_t^* yields

$$|f_t^*(n)| \leqslant c^{\omega(n)}\big(\gamma(n)\big)^{-\beta} \tag{3.5}$$

so that we have, by Lemma 5, as $x \to \infty$,

$$\sum_{n\leqslant x} |f_t^*(n)| \leqslant \sum_{n\leqslant x} c^{\omega(n)}\big(\gamma(n)\big)^{-\beta} \leqslant x^{1-\beta}\sum_{n\leqslant x} c^{\omega(n)}\big(\gamma(n)\big)^{-1} = O\big(x^{1-\beta}\exp L(x)\big).$$

Now (3.2) follows from (3.1) by partial summation on noting the fact that $x^{-\beta/2}\exp L(x)$ decreases for large x.

LEMMA 7. *For $f \in \mathscr{F}$ and $S \in \mathscr{S}$ we have, as $x \to \infty$,*

$$\sum_{n\leqslant x} \chi_S(n)\big(f(n)\big)^t = F(t)x^{1+\alpha t} + O\big(x^{1+\alpha t-\beta}\exp\big(\sqrt{b}L(x)\big)\big)$$

uniformly for $t \in [-1/\nu, 0]$ where $F(t)$ is given by (2.3).

Proof. By (2.1), (3.2) and (3.5) we have, as $x \to \infty$,

$$\begin{aligned} \Sigma_x &\equiv \sum_{n\leqslant x} \chi_S(n)\big(f(n)n^{-\alpha}\big)^t = \sum_{n\leqslant x}\chi_S(n)\sum_{\substack{rs=n\\(r,s)=1}} f_t^*(r) \\ &= \sum_{r\leqslant x}\chi_S(r)f_t^*(r)\Big\{\sum_{\substack{s\leqslant x/r\\(s,r)=1}}\chi_S(s)\Big\} = \sum_{r\leqslant x}\chi_S(r)f_t^*(r)\Big\{\frac{x}{r}p(r) + O\Big(b^{\omega(r)}\Big(\frac{x}{r}\Big)^{\delta}\Big)\Big\} \\ &= x\sum_{r=1}^{\infty}\chi_S(r)f_t^*(r)p(r)r^{-1} + O\Big(x\sum_{r>x}|f_t^*(r)|r^{-1}\Big) + O\Big(x^{\delta}\sum_{r\leqslant x}|f_t^*(r)|b^{\omega(r)}r^{-\delta}\Big) \\ &= (1+\alpha t)xF(t) + O\big(x^{1-\beta}\exp L(x)\big) + O\Big(x^{\delta}\sum_{r\leqslant x}(bc)^{\omega(r)}\big(\gamma(r)\big)^{-\beta}r^{-\delta}\Big) \end{aligned}$$

on noting that $0 \leqslant p(n) \leqslant 1$ for all positive integer n. The second O-term above is $O\Big(x^{\delta}\sum_{r\leqslant x}(bc)^{\omega(r)}\big(\gamma(r)\big)^{-\beta-\delta}\Big)$ or $O\Big(\sum_{r\leqslant x}(bc)^{\omega(r)}\big(\gamma(r)\big)^{-\beta}\Big)$ according as $\delta > 0$ or not. In either case this reduces to $O\Big(x^{1-\beta}\sum_{r\leqslant x}(bc)^{\omega(r)}\big(\gamma(r)\big)^{-1}\Big)$ and hence to $O\big(x^{1-\beta}\exp\big(\sqrt{b}L(x)\big)\big)$. Thus as $x \to \infty$,

$$\Sigma_x = (1+\alpha t)xF(t) + O\big(x^{1-\beta}\exp\big(\sqrt{b}L(x)\big)\big)$$

uniformly for $t \in [-1/\nu, 0]$. Now by partial summation,

$$\begin{aligned} \sum_{n\leqslant x}\chi_S(n)\big(f(n)\big)^t &= x^{\alpha t}\Sigma_x - \alpha t\int_1^x \Sigma_u u^{\alpha t-1}du \\ &= (1+\alpha t)x^{1+\alpha t}F(t) + O\big(x^{1+\alpha t-\beta}\exp\big(\sqrt{b}L(x)\big)\big) - \\ &\quad -\alpha t(1+\alpha t)F(t)\int_1^x u^{\alpha t}du + O\Big(\exp\big(\sqrt{b}L(x)\big)\int_3^x u^{\alpha t-\beta}du\Big) \\ &= x^{1+\alpha t}F(t) + O\big(x^{1+\alpha t-\beta}\exp\big(\sqrt{b}L(x)\big)\big) \end{aligned}$$

uniformly for $t \in [-1/\nu, 0]$ since $1+\alpha t-\beta \geqslant 1-\alpha/\nu-\beta > 0$.

Proof of the theorem. With N as in (ii) of Lemma 1, we have

$$\int_{-1/\nu}^{0}\Big(\sum_{\substack{n\leqslant x\\ n\in D_f}}\chi_S(n)\big(f(n)\big)^t\Big)dt = \sum_{\substack{n\leqslant x\\ n\in D_f}}\chi_S(n)\int_{-1/\nu}^{0}\big(f(n)\big)^t dt = \sum_{\substack{n\leqslant x\\ n\in D_f}}\chi_S(n)\big(\log f(n)\big)^{-1} - \Sigma$$

where

$$\Sigma = \Big(\sum_{\substack{n<N\\ n\in D_f}} + \sum_{\substack{N\leqslant n\leqslant x\\ n\in D_f}}\Big)\chi_S(n)(f(n))^{-1/\nu}(\log f(n))^{-1}$$
$$= O\Big(1+\sum_{N\leqslant n\leqslant x}\chi_S(n)(f(n))^{-1/\nu}(\log f(n))^{-1}\Big) = O\big(x^{1-a/\nu}(\log x)^{-1}\big)$$

as $x\to\infty$ in virtue of Lemma 7. Now by (iii) of Lemma 1 and Lemma 7 we have, as $x\to\infty$,

$$\sum_{n\leqslant x, n\in D_f\cap S}(\log f(n))^{-1}$$
$$= \int_{-1/\nu}^{0}\Big(\sum_{n\leqslant x, n\in D_f}\chi_S(n)(f(n))^{t}\Big)dt + O\big(x^{1-a/\nu}(\log x)^{-1}\big)$$
$$= \int_{-1/\nu}^{0}\big\{F(t)x^{1+at}+O\big(x^{1+at-\beta}\exp(\sqrt{b}L(x))\big)\big\}dt + O(1)+O\big(x^{1-a/\nu}(\log x)^{-1}\big)$$
$$= x\int_{-1/\nu}^{0}F(t)x^{at}dt + O\Big(\frac{x^{1-\beta}}{\log x}\exp(\sqrt{b}L(x))\Big) + O\big(x^{1-a/\nu}(\log x)^{-1}\big),$$

thus completing the proof of the theorem.

4. Sums over certain semigroups. In this section we specialise our theorem to the case where S is a suitable semigroup of positive integers. At the outset we introduce a Möbius-type function relative to a set of primes P by writing

$$\mu_P(n) = \begin{cases}0 & \text{if either } n \text{ is not square free or } n \text{ has a prime divisor outside } P;\\ (-1)^{\omega(n)} & \text{otherwise.}\end{cases}$$

By using Dirichlet series or otherwise one verifies that

$$\sum_{d|n}\mu_P(d) = s_P(n) \tag{4.1}$$

where s_P is the characteristic function of the multiplicative semigroup $S(P)$ generated by P.

Let C be a set of primes for which there exists a $\delta<1$ such that $\sum_{p\in C}p^{-\delta}<\infty$. For each positive integer n we write $C_n = \{p|\ p\in C$ or p is a prime divisor of $n\}$. We shall apply our theorem to the semigroup $S(C')$ where C' is the set of all primes outside C by first proving

LEMMA 8. *$S(C')\in\mathcal{S}$. More precisely, as $x\to\infty$*

$$\sum_{\substack{m\leqslant x\\ (m,n)=1}}\chi_{S(C')}(m) = x\prod_{p\in C_n}\Big(1-\frac{1}{p}\Big)+O(2^{\omega(n)}x^{\delta})$$

uniformly for all positive integers n.

Proof. That $S(C')$ is multiplicative is clear. Observing that $\prod_{p\in C_n}(1+p^{-\delta})$ converges, we obtain, by Theorem 41 of [7], that $\sum_{m=1}^{\infty}|\mu_{C_n}(m)|m^{-\delta}$ converges so that we have as $x\to\infty$,

$$\sum_{m\leqslant x}|\mu_{C_n}(m)|m^{-\delta}\leqslant\sum_{m=1}^{\infty}|\mu_{C_n}(m)|m^{-\delta} = \prod_{p\in C_n}(1+p^{-\delta})$$
$$= O\Big(\prod_{p|n}(1+p^{-\delta})\Big) = O(2^{\omega(n)})$$

uniformly for all positive integral n. Now by partial summation,

$$\sum_{m\leqslant x}|\mu_{C_n}(m)| = O(2^{\omega(n)}x^{\delta}) \tag{4.2}$$

and

$$\sum_{m>x}|\mu_{C_n}(m)|m^{-1} = O(2^{\omega(n)}x^{-1+\delta}) \tag{4.3}$$

uniformly in n. Hence by (4.1), (4.2) and (4.3),

$$\sum_{\substack{m\leqslant x\\ (m,n)=1}}\chi_{S(C')}(m) = \sum_{m\leqslant x}\chi_{S(C'_n)}(m) = \sum_{d\delta\leqslant x}\mu_{C_n}(d) = \sum_{d\leqslant x}\mu_{C_n}(d)\Big[\frac{x}{d}\Big]$$
$$= x\sum_{d=1}^{\infty}\mu_{C_n}(d)d^{-1}+O\Big(x\sum_{d>x}|\mu_{C_n}(d)|d^{-1}\Big)+O\Big(\sum_{d\leqslant x}|\mu_{C_n}(d)|\Big)$$
$$= x\prod_{p\in C_n}\Big(1-\frac{1}{p}\Big)+O(2^{\omega(n)}x^{\delta}).$$

This completes the proof of Lemma 8. Now we can prove

COROLLARY 1. *With $S = S(C')$ as above, we have, for $f\in\mathcal{F}$, as $x\to\infty$,*

$$\sum_{n\leqslant x, n\in D_f\cap S}(\log f(n))^{-1} = x\int_{-1/\nu}^{0}F_S(t)x^{at}dt + O\Big\{\frac{x}{\log x}\big(x^{-\beta}\exp(\sqrt{2}L(x))+x^{-a/\nu}\big)\Big\}$$

where

$$F_S(t) = (1+at)^{-1}\prod_{p\in C}(1-p^{-1})\prod_{p\in C'}\Big\{(1-p^{-1})\sum_{l=0}^{\infty}(f(p^l))^t p^{-l(1+at)}\Big\}$$

and $L(x)$ is given by (2.4).

Proof. By (2.3) and Lemma 8 we have, with $S = S(C')$,

$$(1+at)F_S(t) = \sum_{n=1}^{\infty} \chi_{S(C')}(n) f_t^*(n) n^{-1} \prod_{p\in C} (1-p^{-1}) \prod_{p|n} (1-p^{-1})$$

$$= \prod_{p\in C} (1-p^{-1}) \prod_{p\in C'} \left\{1+(1-p^{-1}) \sum_{l=1}^{\infty} \frac{(f(p^l)p^{-al})^t-1}{p^l}\right\}$$

$$= \prod_{p\in C} (1-p^{-1}) \prod_{p\in C'} \left\{(1-p^{-1}) \sum_{l=0}^{\infty} (f(p^l))^t p^{-l(1+at)}\right\}.$$

This, together with Lemma 8 and our theorem, yields Corollary 1.

Fixing a positive integer k and taking C to be the set of all prime divisors of k in Corollary 1, we obtain

COROLLARY 2. *For positive integer k and $f \in \mathscr{F}$, we have as $x \to \infty$,*

$$\sum_{\substack{n\leqslant x, n\in D_f \\ (n,k)=1}} (\log f(n))^{-1} = x \int_{-1/\nu}^{0} F(t; k) x^{at} dt + O\left\{\frac{x}{\log x}\left(x^{-\beta}\exp\left(\sqrt{2}\,L(x)\right)+x^{-a/\nu}\right)\right\}$$

where

$$F(t; k) = (1+at)^{-1}\varphi(k)k^{-1} \prod_{p\nmid k} \left\{(1-p^{-1}) \sum_{l=0}^{\infty} (f(p^l))^t p^{-l(1+at)}\right\}$$

and β is subject to the only restriction $\beta < 1-a/\nu$ (φ being the Euler totient function).

COROLLARY 3 (Corollary 2, $k = 1$). *For $f \in \mathscr{F}$, we have as $x \to \infty$,*

$$(4.4)\qquad \sum_{n\leqslant x, n\in D_f} (\log f(n))^{-1}$$
$$= x \int_{-1/\nu}^{0} F(t) x^{at} dt + O\left\{\frac{x}{\log x}\left(x^{-\beta}\exp\left(\sqrt{2}\,L(x)\right)+x^{-a/\nu}\right)\right\}$$

with

$$F(t) = F(t; 1) = (1+at)^{-1} \prod_{p} \left\{(1-p^{-1}) \sum_{l=0}^{\infty} (f(p^l))^t p^{-l(1+at)}\right\}.$$

Remark 2. While comparing (4.4) above with (1.4) of [1], it should be noted that the β above is subject to the restriction $\beta < 1-a/\nu$. However one easily verifies that

(a) in case $f \in \mathscr{F}$ but (1.1) fails for $\beta \geqslant \min\{a/\nu, 1-a/\nu\}$, (4.4) above yields an improvement over (1.4) of [1] and

(b) otherwise (4.4) above is equivalent to (1.4) of [1].

To illustrate the case (a) above we may take $f(n) = n^{a+\beta\nu}(J_\beta(n))^{-\nu}$ with $a \geqslant 1$, $\nu = 2a$ and $\beta < 1/2$ where J_s is the Jordan totient function

of order s defined by $J_s(n) = n^s \prod_{p|n}(1-p^{-s})$. In this case the O-terms in (4.4) above and (1.4) of [1] turn out respectively to be $O\left(x^{1-\beta}\exp L(x)\right)$ and $O(x^{1-\beta+\varepsilon})$ with arbitrary positive ε.

Applying Corollary 3 to the function σ_a we obtain

COROLLARY 4. *If $a > 0$ then as $x \to \infty$,*

$$\sum_{2\leqslant n\leqslant x} (\log \sigma_a(n))^{-1} = x \int_{-1/\nu}^{0} F(t) x^{at} dt + \Delta_a(x)$$

where

$$F(t) = \prod_{p} \left\{(1-p^{-1}) \sum_{m=0}^{\infty} p^{-m}(1+p^{-a}+ \ldots +p^{-am})^t\right\}$$

and

(a) $\nu = 2a$ and $\Delta_a(x) = O(x^{1/2+\varepsilon})$ for each $\varepsilon > 0$ if $a \geqslant 1/2$ and
(b) $\nu = 1$ and $\Delta_a(x) = O\left(x^{1-a}(\log x)^{-1}\right)$ if $a < 1/2$.

In particular for each positive integer r, we have, as $x \to \infty$,

$$(4.5)\qquad \sum_{2\leqslant n\leqslant x} (\log \sigma_a(n))^{-1} = x \sum_{m=1}^{r} \frac{(-1)^{m-1}F^{(m-1)}(0)}{(a\log x)^m} + O_r\left(\frac{x}{(\log x)^{r+1}}\right).$$

Remark 3. The special case $a = 1$ of (4.5) was obtained by de Koninck and Galambos ([6], Theorem on p. 161). In the course of their proof (line 5 from below, p. 162) it was made out that $\sum_{n\leqslant d}\left(\prod_{p|n} p^{-1}\right) < \sum_{n\leqslant d} n^{-1}$ which is not true. However, observing that $\sum_{n\leqslant d}\left(\prod_{p|n} p^{-1}\right) = O_\varepsilon(d^\varepsilon)$ for each $\varepsilon > 0$ (which incidentally follows also from Lemma 3 above) and making consequent modifications in their proof, one can arrive at their final result.

5. Some more illustrations. Let M be a set of integers with $\min M = r \geqslant 2$. Following Rieger ([11]) we say that an integer is *M-void* if it is positive and in its canonical factorisation $\prod_p p^{l_p}$, no l_p belongs to M. Denoting the set of all M-void integers by Q_M and its characteristic function by q_M we now apply our theorem to the case $S = Q_M$. Clearly S is a multiplicative set and it follows, in virtue of Satz 1 of [11], that (2.1) holds for this set with $b = 2$, $\delta = 1/r$ and

$$p(n) = \theta_M \prod_{p|n} \left(\frac{p-1}{p\left(1-(p-1)\sum_{m\in M} p^{-m-1}\right)}\right)$$

where

$$\theta_M = \prod_{p} \left(1-(p-1) \sum_{m\in M} p^{-m-1}\right).$$

Further with this $p(n)$ we have

$$F(t) = F_{Q_M}(t) = (1+at)^{-1}\sum_{n=1}^{\infty} q_M(n)f_t^*(n)p(n)n^{-1}$$

$$= \frac{\theta_M}{1+at}\prod_p \left\{1+\sum_{\substack{m=1\\ m\notin M}}^{\infty}\frac{\left(f(p^m)p^{-ma}\right)^t-1}{p^m}\,\frac{p-1}{p\left(1-(p-1)\sum_{m\in M}p^{-m-1}\right)}\right\}$$

$$= \frac{1}{1+at}\prod_p \left\{\left(1-\frac{1}{p}\right)\sum_{\substack{m=0\\ m\notin M}}^{\infty}\left(f(p^m)\right)^t p^{-m(1+at)}\right\}$$

by a straightforward calculation. Thus we obtain

Corollary 5. *Let M be a set of integers with* $\min M = r \geqslant 2$. *For each $f \in \mathscr{F}$, we have, as $x \to \infty$,*

$$\sum_{n\leqslant x,\, n\in D_f\cap Q_M}\left(\log f(n)\right)^{-1}$$

$$= x\int_{-1/\nu}^{0} F_{Q_M}(t)x^{at}dt + O\left\{\frac{x}{\log x}\left(x^{-\beta}\exp\left(\sqrt{2}\,L(x)\right)+x^{-a/\nu}\right)\right\}$$

where $\nu > a$, $\beta \leqslant 1-1/r$ and $\beta < 1-a/\nu$.

Remark 4. Corollary 5 affords us with a rich class of illustrations of our theorem. To this end, let t, k, r be integers such that $t \geqslant 1$ and $k > r \geqslant 2$. We write

$M_1 = M_1(r) = \{n \mid n \text{ is integral, } n \geqslant r\}$,

$M_2 = M_2(k, r) = \{n \mid n \text{ is congruent to one of } r, r+1, \ldots, k-1 \pmod k\}$,

$M_3 = M_3(t, r) = \{jr \mid j = 1, 2, \ldots, t\}$,

$M_4 = M_4(r) = \{jr \mid j = 1, 2, \ldots\}$ and

$M_5 = M_5(r) = \{r\}$.

The elements of the sets Q_{M_1} through Q_{M_5} (usually denoted respectively by Q_r, $Q_{k,r}$, $Q^*_{t,r}$, Q^*_r and Q_r^{s*}) are known as r-free integers, (k, r)-integers ([12], [3]), unitarily (t, r)-integers, unitarily r-free integers ([2], [4]) and semi r-free integers ([13]) respectively. Specializing Corollary 5 to these sets of integers one can obtain a number of illustrations of our theorem.

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