[4] P. Deligne, *Formes modulaires et representation de* GL(2), Lecture Notes n° 349, Springer, pp. 55–103.
[5] — *La conjecture de Weil I*, Publ. Math. I. H. E. S., 43 (1974), pp. 273–307.
[6] M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. 67 (1957), pp. 267–298.
[7] — *The basis problem for modular forms and the traces of Hecke operators*, Lecture Notes n° 320, Springer, pp. 75–151.
[8] K. Hatada, *Periods of primitive forms*, Proc. Japan Acad. 53A, (1977), pp. 174–177.
[9] E. Hecke, *Mathematische Werke* (zw. Aufl.), Vandenboeck und Ruprecht, Göttingen 1970.
[10] S. Lang, *Rapport sur la cohomologie de groupes*, Benjamin, New York 1966.
[11] W. Li, *New forms and functional equations*, Math. Ann. 212 (1975), pp. 285–315.
[12] Ju. I. Manin, *Cyclotomic fields and modular curves*, Russian Math. Surveys 26 (1971), pp. 7–78.
[13] — *Parabolic points and zeta functions of modular curves*, Izv. Akad. Nauk SSSR 6 (1972), A.M.S. translation, pp. 19–64.
[14] — *Explicit formulas for the eigenvalues of Hecke operators*, Acta Arith. 24 (1973), pp. 239–249.
[15] — *Periods of parabolic forms and p-adic Hecke series*, Mat. Sbornik 92 (1973), A.M.S. translation, pp. 371–393.
[16] — *The values of p-adic Hecke series at integer points of the critical strip*, Mat. Sbornik 93 (1974), A.M.S. translation, pp. 631–637.
[17] B. Mazur and H. Swinnerton-Dyer, *Arithmetic of Weil curves*, Inventiones Math. 18 (1972), pp. 183–266.
[18] T. Miyake, *On automorphic forms on* GL_2 *and Hecke operators*, Ann. of Math. 94 (1971), pp. 174–189.
[19] M. Razar, *Dirichlet series and Eichler cohomology*, University of Maryland (preprint).
[20] — *Values of Dirichlet series at integers in the critical strip*, Lecture Notes n° 627, Springer, pp. 1–10.
[21] G. Shimura, *Sur les intégrales attachées aux formes automorphes*, J. Math. Soc. of Japan 11 (1959), pp. 291–311.
[22] — *Introduction to the arithmetic theory of automorphic functions*, Iwanami and Princeton University Press, Tokyo and Princeton 1971.
[23] — *The special values of the zeta functions associated with cusp forms*, Comm. Pure appl. Math. XXIX (1976), pp. 783–804.
[24] — *On the periods of modular forms*, Math. Ann. 229 (1977), pp. 211–221.
[25] A. Weil, *Über die Bestimmung Dirichletschen Reihen durch Funktionalgleichungen*, ibid. 168 (1967), pp. 149–156.
[26] — *Remarks on Hecke's lemma and its use*, In: *Algebraic Number Theory*, Japan Society for the Promotion of Science, 1977, pp. 267–274.

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An additive problem in the theory of numbers

by

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1. Introduction. Vinogradov (cf. [5]) proved that every sufficiently large odd integer N can be written as

$$N = p^{(1)}+p^{(2)}+p^{(3)},$$

where $p^{(i)}$'s are odd primes. Here we shall prove

THEOREM. *Let k be an integer $\geqslant 2$. Let $\delta_1, \delta_2, \ldots, \delta_k$ be positive numbers satisfying $\delta_1+\delta_2+\ldots+\delta_k = 1$. Then every sufficiently large odd integer N can be written as*

$$N = n^{(1)}+n^{(2)}+n^{(3)},$$

where $n^{(i)} = p_1^{(i)}p_2^{(i)}\ldots p_k^{(i)}$ with some odd primes $p_j^{(i)}$'s satisfying $p_j^{(i)} \leqslant N^{\delta_j}$ for $j = 1, 2, \ldots, k$ and for $i = 1, 2, 3$.

In fact, we shall prove using Hardy–Littlewood's circle method

$$\sum_{N=n^{(1)}+n^{(2)}+n^{(3)}} \Big(\prod_{i=1}^{3}\prod_{j=1}^{k} \log p_j^{(i)}\Big) = \frac{1}{((k-1)!)^3}\mathfrak{S}(N)\tilde{r}_k(N)+O\big(N^2(\log N)^{-A}\big),$$

where

$$\mathfrak{S}(N) = \prod_{p\nmid N}\Big(1+\frac{1}{(p-1)^3}\Big)\prod_{p|N}\Big(1-\frac{1}{(p-1)^2}\Big),$$

$$\tilde{r}_k(N) = \sum_{N=h_1+h_2+h_3}\Big(\log\frac{N}{h_1}\Big)^{k-1}\Big(\log\frac{N}{h_2}\Big)^{k-1}\Big(\log\frac{N}{h_3}\Big)^{k-1},$$

p runs over primes, h_j's are positive integers and A is a sufficiently large constant. We remark that there are smaller N's which cannot be written as in our theorem.

It might be interesting to ask whether every sufficiently large even integer N can be written as $N = n^{(1)} + n^{(2)}$, where $n^{(1)}$ and $n^{(2)}$ are of the same form as in our theorem.

2. Let N be a sufficiently large odd integer. Suppose that $\delta_1 + \delta_2 + \ldots \ldots + \delta_k = 1$ and $\delta_j > 0$ for $j = 1, 2, \ldots, k$. We put $M_j = N^{\delta_j}$ for $j = 1, 2, \ldots \ldots, k$. We put

$$S_N(\alpha) = \sum_{p_j \leqslant M_j} \log p_1 \ldots \log p_k e(p_1 \ldots p_k \alpha),$$

where $e(x) = \exp(2\pi i x)$ and p_j's run over primes.

We shall estimate the integral

$$r(N) = \int_0^1 S_N^3(\alpha) e(-N\alpha)\, d\alpha.$$

We remark first that

$$r(N) = \sum_{p_j^{(i)} \leqslant M_j} \prod_{i=1}^{3} (\log p_1^{(i)} \log p_2^{(i)} \ldots \log p_k^{(i)}),$$

where $p_j^{(i)}$'s satisfy $\sum_{i=1}^{3} \Big(\prod_{j=1}^{k} p_j^{(i)} \Big) = N$.

We remark second that for any $\alpha \in (0, 1)$, there exist integers q and a such that

$$\left| \alpha - \frac{a}{q} \right| \leqslant \frac{1}{qQ}, \quad 1 \leqslant q \leqslant Q, \quad (a, q) = 1 \quad \text{and} \quad 1 \leqslant a \leqslant q,$$

where we put $Q = N(\log N)^{-B}$ with a sufficiently large constant B. We denote the interval $\{\alpha\colon |\alpha - a/q| \leqslant 1/Q\}$ by J_{aq} and the interval $\{\alpha\colon |\alpha - a/q| \leqslant 1/qQ\}$ by J'_{aq}. We put

$$\sum_{\substack{1 \leqslant q \leqslant (\log N)^B \\ 1 \leqslant a < q, (a,q)=1}} J_{aq} = J_1 \quad \text{and} \quad \left[-\frac{1}{Q}, 1 - \frac{1}{Q} \right) - J_1 = J_2.$$

Then

$$J_2 = \bigcup_{(\log N)^B < q \leqslant Q} \; \bigcup_{(a,q)=1} J'_{aq}.$$

We put

$$r_m(N) = \int_{J_m} S_N^3(\alpha) e(-N\alpha)\, d\alpha \quad \text{for} \quad m = 1, 2.$$

Then $r(N) = r_1(N) + r_2(N)$. We shall estimate $r_2(N)$ in § 3 and § 4. We shall estimate $r_1(N)$ in § 5 and § 6.

3. Let $\alpha = a/q + \beta$, $(a, q) = 1$ and $|\beta| \leqslant 1/Q$. We remark that

$$\begin{aligned} S_N\left(\frac{a}{q} + \beta\right) &= \sum_{p_j \leqslant M_j} \log p_1 \ldots \log p_k e\left(p_1 \ldots p_k \frac{a}{q}\right) e(p_1 \ldots p_k \beta) \\ &= \sum_{h \leqslant N} \big(A(h) - A(h-1)\big) e(h\beta) \\ &= \sum_{h \leqslant N-1} A(h) \Big(e(h\beta) - e\big((h+1)\beta\big)\Big) + A(N) e(N\beta), \end{aligned}$$

where we put

$$A(h) = \sum_{\substack{p_j \leqslant M_j \\ p_1 \ldots p_k \leqslant h}} \log p_1 \ldots \log p_k e\left(p_1 \ldots p_k \frac{a}{q}\right).$$

$$\begin{aligned} A(h) &= \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{ba}{q}\right) \sum_{\substack{p_1 \ldots p_k \leqslant h \\ p_1 \ldots p_k \equiv b \,(\mathrm{mod}\, q) \\ p_j \leqslant M_j}} \log p_1 \ldots \log p_k + O\big(N(\log N)^{-A}\big) \\ &= \frac{1}{\varphi(q)} \sum_{\chi} \Bigg(\sum_{\substack{(b,q)=1 \\ 1 \leqslant b \leqslant q}} e\left(\frac{ba}{q}\right) \bar{\chi}(b) \Bigg) \sum_{\substack{p_1 \ldots p_k \leqslant h \\ p_j \leqslant M_j}} \log p_1 \ldots \log p_k \, \chi(p_1 \ldots p_k) + \\ &\qquad + O\big(N(\log N)^{-A}\big) \\ &= \frac{1}{\varphi(q)} \sum_{\chi} \tau(\bar{\chi}) \chi(a) \vartheta(h, N, \chi) + O\big(N(\log N)^{-A}\big), \end{aligned}$$

where χ runs over all Dirichlet characters mod q, $\tau(\chi)$ is the Gaussian sum, A is a sufficiently large constant and we put

$$\vartheta(h, N, \chi) = \sum_{\substack{p_1 \ldots p_k \leqslant h \\ p_j \leqslant M_j}} \log p_1 \ldots \log p_k \chi(p_1 \ldots p_k).$$

In the following we shall always denote sufficiently large constants by A.

4. We suppose first that $(\log N)^B < q \leqslant Q$ and $|\beta| \leqslant 1/(qQ)$. We shall estimate $A(h)$ for $h \leqslant N$, $S_N(\alpha)$ for $\alpha \in J_2$ and $r_2(N)$. Now

$$\begin{aligned} &\frac{1}{\varphi(q)} \sum_{\chi} \chi(a) \tau(\bar{\chi}) \vartheta(h, N, \chi) \\ &\quad = \frac{1}{\varphi(q)} \sum_{\chi} \chi(a) \tau(\bar{\chi}) \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f_1(s, \chi) \ldots f_k(s, \chi) \frac{h^s}{s}\, ds + \\ &\quad + O\Bigg(\sqrt{q} h^b \Bigg(\sum_{p_1 \leqslant M_1} \frac{\log p_1}{p_1^b} \ldots \sum_{p_k \leqslant M_k} \frac{\log p_k}{p_k^b} \times \\ &\quad \times \min\Big(1/\big(T \left|\log\big(h/(p_1 \ldots p_k)\big)\right|\big), 1\Big)\Bigg)\Bigg) = S_1 + O(S_2), \end{aligned}$$

say, where we put

$$b = 1+\frac{1}{\log N}, \quad T = N^2 \quad \text{and} \quad f_j(s,\chi) = \sum_{p\leqslant M_j}\frac{\chi(p)\log p}{p^s}$$

$$\text{for} \quad j = 1,2,\ldots,k.$$

$$S_1 = \frac{1}{\varphi(q)}\sum_{\chi}\chi(a)\tau(\bar\chi)\frac{1}{2\pi i}\int_{(1/2)-iT}^{(1/2)+iT} f_1(s,\chi)\ldots f_k(s,\chi)\frac{h^s}{s}ds+$$

$$+O\Big(\frac{\sqrt{q}}{T}\int_{1/2}^{b}\prod_{j=1}^{k}\Big(\sum_{p\leqslant M_j}\frac{\log p}{p^\sigma}\Big)h^\sigma d\sigma\Big) = S_3+O(S_4), \text{ say.}$$

$$S_4 \ll N(\log N)^{-A}.$$

$$S_3 \ll \frac{\sqrt{q}}{\varphi(q)}h^{1/2}\int_{-T}^{T}\sum_{\chi}\prod_{j=1}^{k}|f_j(\tfrac12+it,\chi)|\frac{dt}{1+|t|}.$$

$$\sum_{\chi}\prod_{j=1}^{k}|f_j(\tfrac12+it,\chi)| \ll \Big(\sum_{\chi}\Big|\prod_{j=1}^{k-1}f_j(\tfrac12+it,\chi)\Big|^2\Big)^{1/2}\Big(\sum_{\chi}|f_k(\tfrac12+it,\chi)|^2\Big)^{1/2}.$$

Now by Lemma 2 of Gallagher [3],

$$\sum_{\chi}\Big|\prod_{j=1}^{k-1}f_j(\tfrac12+it,\chi)\Big|^2 = \sum_{\chi}\Big|\sum_{n\leqslant N/M_k}\frac{\chi(n)a(n)}{n^{1/2+it}}\Big|^2$$

$$\ll (N/M_k+q)\sum_{n\leqslant N/M_k}\frac{|a(n)|^2}{n},$$

where we put

$$a(n) = \sum_{\substack{n=p^{(1)}\ldots p^{(k-1)}\\ p^{(j)}\leqslant M_j}}\log p^{(1)}\ldots\log p^{(k-1)}.$$

Since

$$\sum_{n\leqslant N/M_k}\frac{|a(n)|^2}{n} \ll \prod_{j=1}^{k-1}\Big(\sum_{p\leqslant M_j}\frac{(\log p)^2}{p}\Big) \ll (\log N)^{2(k-1)},$$

we get

$$\Big(\sum_{\chi}\Big|\prod_{j=1}^{k-1}f_j(\tfrac12+it,\chi)\Big|^2\Big)^{1/2} \ll (\sqrt{N/M_k}+\sqrt{q})(\log N)^{k-1}.$$

Hence we get

$$S_3 \ll \frac{\sqrt{q}}{\varphi(q)}\sqrt{h}\,(\sqrt{N/M_k}+\sqrt{q})(\sqrt{M_k}+\sqrt{q})(\log N)^{k+1} \ll N(\log N)^{-B'},$$

where we put $B' = B/2-k-2$.

Finally,

$$S_2 \ll \sqrt{q}(\log N)^k+\frac{\sqrt{q}h}{T}\sum_{\substack{p_j\leqslant M_j\\ h\neq p_1\ldots p_k}}\frac{\log p_1\ldots\log p_k}{p_1\ldots p_k\Big|\log\dfrac{h}{p_1\ldots p_k}\Big|} \ll N(\log N)^{-A}.$$

Thus we get for $h\leqslant N$,

$$A(h) \ll N(\log N)^{-B'}.$$

Hence for $\alpha = a/q+\beta\in J_2$, we get

$$S_N(\alpha) \ll N(\log N)^{-B'}|\beta|N+N(\log N)^{-B'} \ll N(\log N)^{-B'}.$$

Hence we get

$$r_2(N) \ll \max_{\alpha\in J_2}|S_N(\alpha)|\int_0^1|S_N(\alpha)|^2d\alpha$$

$$\ll N(\log N)^{-B'}\sum_{\substack{p_j^{(i)}\leqslant M_j\\ p_1^{(1)}\ldots p_k^{(1)}=p_1^{(2)}\ldots p_k^{(2)}}}\log p_1^{(1)}\ldots\log p_k^{(1)}\log p_1^{(2)}\ldots\log p_k^{(2)}$$

$$\ll N^2(\log N)^{-((B/2)-2k-2)}.$$

5. Hereafter we suppose that $1\leqslant q\leqslant(\log N)^B$ and $|\beta|\leqslant 1/Q$. By § 3, we have

$$A(h) = \frac{\mu(q)}{\varphi(q)}\sum_{\substack{p_1\ldots p_k\leqslant h\\ p_j\leqslant M_j}}\log p_1\ldots\log p_k+$$

$$+\frac{1}{\varphi(q)}\sum_{\chi}{}'\chi(a)\tau(\bar\chi)\vartheta(h,N,\chi)+O\big(N(\log N)^{-A}\big)$$

$$= S_5+S_6+O\big(N(\log N)^{-A}\big),$$

where the dash indicates that we sum over all non-principal characters mod q. As in § 4,

$$S_6 = \frac{1}{\varphi(q)}\sum_{\chi}{}'\chi(a)\tau(\bar\chi)\frac{1}{2\pi i}\int_{(1/2)-iT}^{(1/2)+iT}f_1(s,\chi)\ldots f_k(s,\chi)\frac{h^s}{s}ds+$$

$$+O\big(N(\log N)^{-A}\big).$$

We remark that for $s = \frac12+it$ and for non principal χ,

$$f_j(s,\chi) \ll M_j^{1/2}(\log N)^{-A}.$$

Hence we get

$$S_6 \ll h^{1/2}q^{1/2}(M_1M_2\dots M_k)^{1/2}(\log N)^{-A}+N(\log N)^{-A}$$
$$\ll N(\log N)^{-A+(B/2)} \ll N(\log N)^{-A}.$$

Next, we shall prove by induction on k that

$$\sum_{\substack{p_1\dots p_k\le h\\ p_j\le M_j}} \log p_1\dots\log p_k = h\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\left(\log\frac{N}{h}\right)^{\nu} + O\big(N(\log N)^{-A}\big),$$

where $h<N=M_1\dots M_k$.

When $k=1$, the above formula is a consequence of the prime number theorem. Suppose that the conclusion is correct for k. We shall prove the above formula for $k+1$. We may suppose that $h\geqslant M_{k+1}$ and $h\geqslant \frac{N}{M_{k+1}}N^{\varepsilon}$, where ε is a sufficiently small positive number.

$$\sum_{\substack{p_1\dots p_{k+1}\le h\\ p_j\le M_j}} \log p_1\dots\log p_k\log p_{k+1} = \sum_{p\le M_{k+1}}\log p \sum_{\substack{p_1\dots p_k\le h/p\\ p_j\le M_j}}\log p_1\dots\log p_k$$
$$= \sum_{p\le (M_{k+1}h)/N}\log p\sum_{p_j\le M_j}\log p_1\dots\log p_k +$$
$$+\sum_{(M_{k+1}h)/N<p\le M_{k+1}}\log p\sum_{\substack{p_1\dots p_k\le h/p\\ p_j\le M_j}}\log p_1\dots\log p_k = S_7+S_8,\ \text{say}.$$

$$S_7 = \frac{N}{M_{k+1}}\Big(1+O\big((\log N)^{-A}\big)\Big)\frac{M_{k+1}h}{N}\Big(1+O\big((\log N)^{-A}\big)\Big)$$
$$= h+O\big(N(\log N)^{-A}\big).$$

$$S_8 = \sum_{(M_{k+1}h)/N<p\le M_{k+1}}\log p\left(\frac{h}{p}\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\left(\log\frac{Np}{M_{k+1}h}\right)^{\nu}+O\left(\frac{N}{M_{k+1}}(\log N)^{-A}\right)\right)$$
$$= h\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\sum_{(M_{k+1}h)/N<p\le M_{k+1}}\frac{\log p}{p}\left(\log\frac{Np}{M_{k+1}h}\right)^{\nu}+O\big(N(\log N)^{-A}\big)$$
$$= h\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\int_{(M_{k+1}h)/N}^{M_{k+1}}\frac{\left(\log\frac{Ny}{M_{k+1}h}\right)^{\nu}}{y}\,d\Big(\sum_{(M_{k+1}h)/N<p\le y}\log p\Big)+O\big(N(\log N)^{-A}\big)$$
$$= h\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\frac{\left(\log\frac{N}{h}\right)^{\nu+1}}{\nu+1}+O\big(N(\log N)^{-A}\big).$$

$$S_7+S_8 = h\sum_{\nu=0}^{k}\frac{\big(\log(N/h)\big)^{\nu}}{\nu!}+O\big(N(\log N)^{-A}\big).$$

This proves our formula described above. Hence we get for $h<N$,

$$A(h) = \frac{\mu(q)}{\varphi(q)}h\sum_{\nu=0}^{k-1}\frac{\big(\log(N/h)\big)^{\nu}}{\nu!}+O\big(N(\log N)^{-A}\big).$$

6. Now for $1\leqslant q\leqslant(\log N)^B$ and $|\beta|\leqslant 1/Q$, we have

$$S_N\left(\frac{a}{q}+\beta\right) = \sum_{h\le N-1}A(h)\Big(e(h\beta)-e\big((h+1)\beta\big)\Big)+A(N)e(N\beta)$$
$$= \frac{\mu(q)}{\varphi(q)}\sum_{h\le N-1}\left(h\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\left(\log\frac{N}{h}\right)^{\nu}\right)\Big(e(h\beta)-e\big((h+1)\beta\big)\Big)+$$
$$+\frac{\mu(q)}{\varphi(q)}Ne(N\beta)+O\big(N^2(\log N)^{-A}|\beta|\big)+O\big(N(\log N)^{-A}\big)$$
$$= \frac{\mu(q)}{\varphi(q)}\sum_{1<h<N}\left(h\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\left(\log\frac{N}{h}\right)^{\nu}-\right.$$
$$\left.-(h-1)\sum_{\nu=0}^{k-1}\frac{1}{\nu!}\left(\log\frac{N}{h-1}\right)^{\nu}\right)e(h\beta)+O\big(N(\log N)^{-A}\big)$$
$$= \frac{\mu(q)}{\varphi(q)}\frac{1}{(k-1)!}\sum_{h\le N}\left(\log\frac{N}{h}\right)^{k-1}e(h\beta)+O\big(N(\log N)^{-A}\big).$$

Now we can estimate $r_1(N)$:

$$r_1(N) = \sum_{1\le q\le(\log N)^B}\sum_{(a,q)=1}e\left(-\frac{aN}{q}\right)\int_{-1/Q}^{1/Q}S_N^3\left(\frac{a}{q}+\beta\right)e(-N\beta)\,d\beta$$
$$= \sum_{1\le q\le(\log N)^B}\frac{\mu^3(q)}{\varphi^3(q)}\sum_{(a,q)=1}e\left(-\frac{aN}{q}\right)\times$$
$$\times\int_{-1/Q}^{1/Q}\left(\frac{1}{(k-1)!}\sum_{h\le N}\left(\log\frac{N}{h}\right)^{k-1}e(h\beta)\right)^{3}e(-N\beta)\,d\beta+$$
$$+O\big(N^2(\log N)^{-A}\big).$$

We remark that

$$\sum_{h\leq N}\left(\log\frac{N}{h}\right)^{k-1} e(h\beta) \ll \frac{(\log N)^{k-1}}{\|\beta\|},$$

where

$$\|\beta\| = \min_{n\in Z}|\beta - n|,$$

because

$$\sum_{h\leq x} e(h\beta) \ll \|\beta\|^{-1}$$

(cf. Lemma 6.7 of [4]).

Hence the last integral is

$$= \int_0^1 \frac{1}{((k-1)!)^3}\left(\sum_{h\leq N}\left(\log\frac{N}{h}\right)^{k-1} e(h\beta)\right)^3 e(-N\beta)\,d\beta + O\left(Q^2(\log N)^{3k-3}\right).$$

Hence we get

$$r_1(N) = \sum_{1\leq q\leq(\log N)^B}\frac{\mu^3(q)}{\varphi^3(q)}\sum_{(a,q)=1} e\left(-\frac{Na}{q}\right)\frac{1}{((k-1)!)^3}\tilde{r}_k(N) + $$
$$+ O\left(N^2(\log N)^{-(2B-3k+3)}\right),$$

where $\tilde{r}_k(N)$ is defined in the introduction.

By Lemma 5.3 of [4], we get

$$r_1(N) = \frac{1}{((k-1)!)^3}\mathfrak{S}(N)\tilde{r}_k(N) + O\left(N^2(\log N)^{-A}\right),$$

where

$$\mathfrak{S}(N) = \prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right)\prod_{p|N}\left(1-\frac{1}{(p-1)^2}\right)$$

and we have taken a sufficiently large constant B. Since $\tilde{r}_k(N) \gg N^2$ and $\mathfrak{S}(N) > 6/\pi^2$, we get our theorem.

References

[1] M. B. Barban, *The large sieve method and its applications in the theory of numbers*, Uspehi Mat. Nauk 21 (1966), pp. 51–102 = Russian Math. Surveys 21 (1966), pp. 49–103.

[2] A. Fujii, *On some analogues of Titchmarsh divisor problem*, Nagoya Math. J. 64 (1976), pp. 149–158.

[3] P. X. Gallagher, *A large sieve density estimate near* $\sigma = 1$, Invent. Math. 11 (1970), pp. 329–339.

[4] K. Prachar, *Primzahlverteilung*, Springer, Berlin 1957.

[5] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Interscience Publishers, London and New York 1954.

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Restricted sums of reciprocal values of additive functions

by

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1. Introduction. Let $\mathscr{F}$ denote the set of all multiplicative arithmetical functions f satisfying

$$\prod_{p|n}(1-p^{-\beta})^{\nu} \leqslant f(n)n^{-a} \leqslant \prod_{p|n}(1-p^{-\beta})^{-\nu} \tag{1.1}$$

for all positive integral n and for some positive reals a, β and ν. We write

$$D_f = \{n|\ f(n) \neq 1\} \quad \text{and} \quad G_f = \{n|\ f(m) > 1 \text{ for all } m \geqslant n\}.$$

Recently de Koninck and Galambos [6] obtained an asymptotic formula for $\sum_{2\leqslant n\leqslant x}\big(\log\sigma_1(n)\big)^{-1}$ (see Remark 3 below) where $\sigma_s(n) = \sum_{d|n} d^s$. Evelyn Scriba [1], generalizing this, established an asymptotic formula for $\sum_{n\leqslant x, n\in G_f}\big(\log f(n)\big)^{-1}$ where f is any member of $\mathscr{F}$ subject to the apparently additional conditions $\nu > a$ and $\beta \leqslant 1$ (see Remark 1 below).

In this paper we establish an asymptotic formula for

$$\sum_{n\leqslant x, n\in D_f\cap S}\big(\log f(n)\big)^{-1}$$

where f is any member of $\mathscr{F}$ and S is a set of positive integers subject to some restrictions (statement in § 2 and proof in § 3). In § 4 we exhibit a succession of particular cases of our theorem (Corollaries 1 through 4) in which Corollary 3, besides covering Scriba's result, affords a refinement of it in certain cases (see Remark 2). § 5 contains a rich class of illustrations which result from an application of our theorem to the set of M-void integers introduced by Rieger ([11]).

2. Notation and statement. A set A of positive integers is said to be *multiplicative* provided, for $(a, b) = 1$, one has $ab \in A$ iff $a \in A$ and $b \in A$ or equivalently when the characteristic function χ_A of A is multiplicative. We write $\mathscr{S}$ to denote the class of all multiplicative sets S for each of which there exist numbers $\delta = \delta_S < 1$, $b = b_S \geqslant 1$ and an arithmetical