COROLLARY 3. For $k = Q(\zeta)$, $\zeta$ a primitive $l$-th root of 1, the order of $\chi$ in $\hat{A}$ divides $\lambda$, for $\chi$ such that $\chi(J) = -1$ or $\chi = \chi_s$.

We would like to point out that Theorem 4 has been proved independently by R. Gillard [5].

Added in proof: J.-F. Jaulent has recently obtained results similar to some of those in this article.

Théorie d'Iwasawa des tours métabéliennes, Séminaire de théorie des Nombres de Bordeaux, exposé No. 21 (1980–81).

References


On the rationality of periods of primitive forms

by

KAZUYUKI HATADA (Tokyo)

Introduction. In this paper, we give a new proof of the algebraic property of the periods of primitive forms $F$ of Neben type. We also study $p$-adic Hecke series attached to the $F$, which take algebraic values.

Let $\mathcal{I}$ be a finite index subgroup of $\text{SL}(2, \mathbb{Z})$, $w + 2 \geq 2$ be a rational integer, $S_{w+2}(\mathcal{I})$ be the space of cusp forms of weight $w + 2$ with respect to $\mathcal{I}$, $\varphi_w$ be the representation $\text{GL}(2, \mathbb{R}) \to \text{GL}(w + 1, \mathbb{R})$ given by

$$
\varphi_w \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \, d\varphi_w
= \int\frac{(ax + d)^w}{(cx + d)^w} \ldots \, dx,
$$

where $\varphi_w = \{ \varphi_w \}$ is the $C^w$ valued differential form on the upper half plane $H$, $\varphi_{w,l}$ be the restriction of $\varphi_w$ to $\mathcal{I}$, $\text{Ind} \, \varphi_{w,l}$ be the induced representation of $\varphi_{w,l}, P$ be the set consisting of all the parabolic elements in $\text{SL}(2, \mathbb{Z})$ and $H^1_{\text{par}}(\mathcal{I}, \varphi_{w,l}, \mathbb{R})$ (resp. $H^1_{\text{par}}(\text{SL}(2, \mathbb{Z}), \varphi_{w,l}, \mathbb{R})$) be the parabolic cohomology group with the coefficients $\mathbb{R}$.

Let $\pi_{\text{par}}(\mathcal{I}, \varphi_{w,l}, \mathbb{R})$ denote the image of the whole domain: $\text{Image} (\pi_{\text{par}})$ (resp. $\text{Image} (\pi_{\text{par}})$) under the canonical homomorphism

$$
\pi_{\text{par}}: H^1_{\text{par}}(\mathcal{I}, \varphi_{w,l}, \mathbb{Z}) \to H^1_{\text{par}}(\mathcal{I}, \varphi_{w,l}, \mathbb{R})
$$

(resp. $\pi_{\text{par}}: H^1_{\text{par}}(\text{SL}(2, \mathbb{Z}), \varphi_{w,l}, \mathbb{Z}) \to H^1_{\text{par}}(\text{SL}(2, \mathbb{Z}), \varphi_{w,l}, \mathbb{R})$) which is induced by the natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. In § 2, we prove:

Theorem 0.1. (1). For details, see Theorems 2.2–2.4 in § 2.) Let sh be the map of Shapiro:

$$
H^1(\text{SL}(2, \mathbb{Z}), \varphi_{w,l}, \mathbb{Z}) \to H^1(\mathcal{I}, \varphi_{w,l}, \mathbb{Z}).
$$

There exists a canonical \( Z \)-linear map \( \text{sh}_Z \otimes \text{id} : \)

\[
J_1 \left( H^1_{\text{c}}(\text{Ind}_{\text{w}, l, l}(Z)) \right) \to J_0 \left( H^1_{\text{c}}(\text{Ind}_{\text{w}, l, l}(Z)) \right)
\]

induced by the map \( \text{sh} \). This map \( \text{sh}_Z \otimes \text{id} \) is a surjective isomorphism.

We use this theorem for the proof of our main result.

For a positive integer \( N \), let \( I_1(N) \) (resp. \( I_0(N) \)) be the Hecke's congruence subgroup of \( SL(2, Z) \) defined by:

\[
(a \ b
\c \ d) \in I_1(N) \quad \text{(resp. } I_0(N)) \iff (a - 1 \equiv d - 1 \equiv c \equiv 0 \pmod{N})
\]

Every element \( f \) of \( S_{w+2}(I_1(N)) \) has a Fourier expansion \( f(z) = \sum_{n=1}^{\infty} a_n q^n \) \((q = \exp(2\pi i \sqrt{-1}z))\) with complex numbers \( a_n \). We denote by \( Q \) the field generated over the rational number field \( \mathbb{Q} \) by all the coefficients \( a_n \).

For every \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z) \), set \( \sigma g = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \) and

\[
f_{w+2}(g)(z) = f((az + b)(cz + d)) \frac{(cs + d)^w}{(cs + d)^w}.
\]

Let

\[
F(z) = \sum_{n=1}^{\infty} a_n q^n \quad \text{with } a_1 = 1, \; q = \exp(2\pi i \sqrt{-1}z)
\]

be a primitive form in \( S_{w+2}(I_1(N)) \) (in the sense of Atkin–Lehner [1], Li [11], Miyake [18] and others). It is well known that for every automorphism \( \sigma \) of the complex number field \( C \), we can define a primitive form \( F^\sigma(z) \) in \( S_{w+2}(I_1(N)) \) by

\[
F^\sigma(z) = \sum_{n=1}^{\infty} a_n^{\sigma} q^n.
\]

In § 3, we prove the following main result:

**Theorem 0.2.** Let \( F \) be a primitive form in \( S_{w+2}(I_1(N)) \).

(i) There exist complex constants \( c_+^o \) and \( c_-^o \) depending on the \( F^\sigma \) such that

\[
(c_+^o)^{-1} \left( \int_0^{\infty} F_{w+2}(g)(z) \phi(z) dz \right) \left( \int_{\infty}^{\infty} F_{w+2}(g)(z) \phi(z) dz \right) \in Q^p
\]

for all \( l \in Z \) with \( 0 \leq l \leq w \) and all \( g \in SL(2, Z) \).

(ii) For every automorphism \( \sigma \) of the \( C \), we can choose the above \( c_+^o \) as follows.

(a) \( c_+^o = \sigma^e_{+} \) if \( a_{l, q} = \sigma^2_{l, q} \) (\( \sigma \) and \( \sigma' \) are in \( \text{Aut}(C) \)).

(b) \( c_+^o = \pm c_+^o \) for all \( \sigma \in \text{Aut}(C) \) where \( J \) denotes the complex conjugation.

(c) \( \left( \int_{0}^{\infty} F_{w+2}(g)(z) \phi(z) dz \right)^{-1/2} = c_+^o \) for all \( \sigma \in \text{Aut}(C) \) where \( J \) denotes the complex conjugation.

\[
\int_{0}^{\infty} F_{w+2}(g)(z) \phi(z) dz \left( \int_{0}^{\infty} F_{w+2}(g)(z) \phi(z) dz \right)^{-1/2} = c_+^o
\]

for all \( g \in SL(2, Z) \) and all \( l \in Z \) with \( 0 \leq l \leq w \).

This Theorem 0.2 is equivalent to the Theorem 1 (i), (ii) and (iii) in Shimura [24]. We shall give a new proof of Theorem 0.2 along the older lines of Shimura [21] and Manin [15] using the Eichler–Shimura isomorphism and Theorem 0.1. In § 4 we investigate \( F \)-adic measures associated with primitive forms, which take algebraic values, using the following functions \( \mathcal{F}_l^o(z) \) \((z \in Q \cup (i \infty)) \) and \( l \in Z \) with \( 0 \leq l \leq w \):

\[
\mathcal{F}_l^o(z) = \left( \int_{0}^{\infty} F(z + x) \phi(z) dz \right)^{-1/2} \left( \int_{0}^{\infty} F(z - x) \phi(z) dz \right)^{-1/2}.
\]

For details see Theorems 4.3 and 4.4 in § 4. These are generalizations of the original constructions of \( F \)-adic measures in Manin [15], [16] for \( I = SL(2, Z) \) case and Mazur–Swinnerton-Dyer [17] for weight 2 case.

Roughly speaking our method of the proof of Theorem 0.2 (i) (for \( \sigma = 1 \)) is as follows. First we construct a surjective isomorphism \( \Phi \) from \( S_{w+2}(I_1(N)) \) to \( H^1_{\text{c}}(SL(2, Z), \text{Ind}_{\text{w}, l, l}(R)) \) by some integral of cusp forms (Lemma 2.1). Let \( S_{w+2}(I_1(N)) \) be the subspace of \( S_{w+2}(I_1(N)) \) consisting of those forms whose Fourier coefficients at \( z = i \infty \) are all real numbers and \( \langle U^+_1, U^+_2, \ldots, U^+_w \rangle \) (resp. \( \langle U^-_1, U^-_2, \ldots, U^-_w \rangle \)) be a \( Z \)-basis of \( \mathcal{B}_{w+2}(I_1(N)) \) \((\text{resp. } \mathcal{B}_{w+2}(I_1(N)) \cap \mathcal{B}_{w+2}(I_1(N)) \cap \mathcal{B}_{w+2}(I_1(N)) \cap \mathcal{B}_{w+2}(I_1(N))) \) using the fact that \( H^1_{\text{c}}(I_1(N), \text{Ind}_{\text{w}, l, l}(Z)) \) is stable under all the Hecke operators on \( \Phi \), Theorem 0.1 and “Multiplicivity one theorem”; we shall establish there exists a complex number \( c_+^o \) (resp. \( c_-^o \)) and a vector \((a_1, a_2, \ldots, a_j) \) (resp. \((b_1, b_2, \ldots, b_j) \)) with all the coefficients in the \( \Phi \) such that

\[(0.3.1) \quad F = c_+^o \left( a_1 U^+_1 + a_2 U^+_2 + \ldots + a_w U^+_w \right) + c_-^o \left( \beta_1 U^-_1 + \beta_2 U^-_2 + \ldots + \beta_w U^-_w \right)
\]

Then we analyse the action of Hecke operators \( T_{w+2}(p) \) on a certain cocycle \( \in E^2_{\Phi}(SL(2, Z), \text{Ind}_{\text{w}, l, l}(R)) \), expressed as some integral of \( F(z) \) from \( i \infty \) to \( \sigma \left( i \infty \right) \) (\( z \in SL(2, Z) \)), whose cohomology class is equal to the \( \Phi(F) \), by changing the variable of the integral. In this way we deduce, from (0.3.1), Theorem 0.2 (i) using the fact that there exists a prime \( p \) with \( a_p = 1 + p^{w+1} \) and \( p \equiv 1 \pmod{N} \).
The origin of Theorem 0.2 seems due to Shimura [21] where the case of the discriminant function $A(z)$ of weight 12 for $G = SL(2, \mathbb{Z})$ was computed. Manin [15] obtained the theorem for any eigenform of any weight in case of $G = SL(2, \mathbb{Z})$ using the continued fractions of rational numbers. Damerei [3] investigated the values of L functions of imaginary quadratic fields by a different idea. Birch, Manin, Mazur and Swinnerton-Dyer [13], [14], [17]) investigated periods of primitive forms of weight 2 on $\Gamma_0(N)$ in relation to "Modular Symbols" and "Well Parametrization". Recently Shimura, using totally different methods (not using the Eichler-Shimura isomorphism), obtained almost all the results on the rationality in [23], [24]. (Our point in this paper is to do things along the older lines of Manin [15] and Shimura [21]). Razar investigates also the above Theorem 0.2 and obtains partial results in [19] and Theorem 4 [20]. Roughly speaking the distinctions between Razar's and ours are the following.

(i) Razar uses the Eichler-Shimura isomorphism $\phi$ itself instead of $\Phi$.

Let $G = \Gamma_0(N)$ (or $\Gamma_1(N)$).

(ii) He proves that a certain $Q$-subspace $Z$ of the parabolic cocycles $Z_{\Gamma \cap R}(G; \mathbb{Q})$ into whose complexification the space $S_{\eta, \varphi}(G)$ is mapped, is stable under all the Hecke operators on $G$. And he applies "Multiplicite one theorem" to the space $Z \otimes R$. On the other hand we utilize the result that the space

$$j_1([H^1_f|SL(2, \mathbb{Z}), \text{Ind}_{\Gamma \cap SL(2, \mathbb{Z})}]$$

is stable under all the Hecke operators on $G$ through the isomorphism $\Phi$ and apply "Multiplicite one theorem" to its complexification.

(iii) He expresses the coefficients of the Eichler cocycle $(z \in P_{\Gamma \cap R}(G, \mathbb{Q}))$ corresponding to a primitive form $F$ as a pole of some multiple differential. This relates to the theory developed in Weil [26]. We make no use of this technique (iii) in this paper.

(iv) Our technique in deducing Theorem 0.2 (ii) (c) from (0.2.1) is our own.

The author would like to express his gratitude to the referees for their valuable advices on the improvements of this paper and especially for their pointing out the insufficiency of the proof, of Theorem 0.1, given in the earlier version.

The main results of this paper were announced in a Proceedings of Japan Academy Note [8].

1. Notations and preliminary results

$\chi$: a Dirichlet character $(\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathbb{C}^*$.

$S_{\eta, \varphi}(N, \mathbb{Z})$: the subspace of $S_{\eta, \varphi}(\Gamma_0(N))$ consisting of all the cusp forms $f$ with $f((az+b)/(cz+d))(cz+d)^{-w-2} = \chi(z)f(z)$ for every $(a \ b) \in \Gamma_0(N)$. It is well known that $S_{\eta, \varphi}(\Gamma_0(N)) = \mathbb{C} \otimes S_{\eta, \varphi}(N, \mathbb{Z})$, where $\mathbb{C}$ runs over all the Dirichlet characters (mod $N$).

$\Gamma(N)$: the principal congruence subgroup of level $N$.

$\Gamma$: a finite index subgroup of $SL(2, \mathbb{Z})$.

$1_r$: the $r \times r$ identity matrix ($r$: a positive integer).

$SL(2, \mathbb{Z}) = \Gamma(1) = \bigcup_{g \in \mathbb{Z}} \Gamma g$: the left coset decomposition with $g_1 = 1_r$ and $m$: the cardinality of $(\Gamma \setminus \Gamma(1))$.

$R^{(w+1)m}$: the real vector space consisting of the $(w+1)m$ dimensional real column vectors with basis indexed by the pairs $(\{g, g\})$ which are the elements of the product set $(\Gamma \setminus \Gamma(1)) \times [(w, w) \cap \mathbb{Z}]$.

$C^{(w+1)m}$: the lattice of the $R^{(w+1)m}$ with respect to the standard basis.

$x = \left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right),
\tau = \left(\begin{array}{cc}
1 & 1 \\
0 & 1
\end{array}\right),
\eta_1 = \left(\begin{array}{cc}
0 & -1 \\
1 & 1
\end{array}\right),
\eta_2 = \left(\begin{array}{cc}
1 & -1 \\
1 & 1
\end{array}\right)$

$dz$: the $C^{(w+1)}$ valued differential form $\left(\frac{dz}{dz}, \frac{dz}{z}, \frac{dz}{z^2}, \ldots, \frac{dz}{z^w}\right)$ on the upper half plane $H$.

For each cusp form $f \in S_{\eta, \varphi}(\Gamma)$, put

$$D(f)(x) = \left(\begin{array}{c}
\frac{f(\eta_1)(x)}{dz} \frac{dz}{dz} \\
\frac{f(\eta_2)(x)}{dz} \frac{dz}{z} \\
\frac{f(x)}{dz} \frac{dz}{z^2} \\
\vdots
\end{array}\right)$$

and $D(f)(x) = \left(\begin{array}{c}
\frac{\eta_1(x)}{dz} \frac{dz}{dz} \\
\frac{\eta_2(x)}{dz} \frac{dz}{z} \\
\frac{x}{dz} \frac{dz}{z^2} \\
\vdots
\end{array}\right)$

which are $C^{(w+1)m}$ valued differential forms on the $H$. Here $f(x)(dz)g$ denotes the pull back of $f(x)(dz)$ by $g \in SL(2, \mathbb{Z})$.

$\eta_\varphi$: the representation $SL(2, \mathbb{Z}) \rightarrow GL(\mathbb{C}/(w+1)m)$, $g \in SL(2, \mathbb{Z})$, given by

$$\eta_\varphi(g)D(f) = D(f)(g) \quad (f \in S_{\eta, \varphi}(\Gamma), \ g \in SL(2, \mathbb{Z}))$$

which is isomorphic to $Ind_{\Gamma \cap SL(2, \mathbb{Z})}^{\Gamma \cap SL(2, \mathbb{Z})}$. (cf. Eichler [6], Shimura [21], [22]). By an $R$-valued parabolic cocycle of $\eta_\varphi$ (resp. $\eta_\varphi$), we mean a map

$$x: \Gamma(1) \rightarrow R^{(w+1)m} \quad (x: \Gamma \rightarrow R^{(w+1)})$$

satisfying the two conditions:

$$x(gg') = x(g) + \eta_\varphi(g)x(g') \quad (x(hh') = x(h) + \eta_\varphi(h)x(h'))
(g and g' ∈ Γ(1), h and h' ∈ Γ) and
\[ x(\gamma) = (I_{w+1} - \eta_{w}(\gamma))K_{w+1} \text{ for every } \gamma \in P \]
(resp. \( x(\gamma) = (I_{w+1} - \eta_{w}(\gamma))K_{w+1} \) for every \( \gamma \in P \cap \Gamma \)).

A coboundary is a cocycle \( x \) of the form
\[ x(g) = (I_{w+1} - \eta_{w}(g))x_{0} \quad (g \in \Gamma(1)) \]
(resp. \( x(h) = (I_{w+1} - \eta_{w}(h))x_{0} \quad (h \in \Gamma) \))

where \( x_{0} \) is an arbitrarily fixed element of \( R_{w+1} \) (resp. \( R_{w+1} \)). The parabolic cohomology group \( H^{p}_{\text{par}}(\Gamma(1), \eta_{w}, R) \) (resp. \( H^{p}_{\text{par}}(\Gamma, \eta_{w}, R) \)) is the quotient of the group \( Z^{p}_{\text{par}}(\Gamma(1), \eta_{w}, R) \) (resp. \( Z^{p}_{\text{par}}(\Gamma, \eta_{w}, R) \)) by \( R \)-valued parabolic cocycles modulo the subgroup \( B(\Gamma(1), \eta_{w}, R) \) (resp. \( B(\Gamma, \eta_{w}, R) \)) of coboundaries of \( R \). The natural injection \( Z \to R \) induces a canonical homomorphism
\[ f_{1}: H^{p}_{1}(\Gamma(1), \eta_{w}, R) \to H^{p}_{1}(\Gamma(1), \eta_{w}, R) \]
(resp. \( f_{1}: H^{p}_{\text{par}}(\Gamma, \eta_{w}, R) \to H^{p}_{\text{par}}(\Gamma, \eta_{w}, R) \)).

\[ \varphi: S_{w+2}(\Gamma) \to H^{\infty}_{\text{par}}(\Gamma, \eta_{w}, R) \] is the Eichler–Shimura isomorphism for \( S_{w+2} \) (cf. Eichler [6] and Shimura [21], [22]).

Theorem 1.2 (Eichler [6] and Shimura [21], [22]). Let \( f \) be a cusp form in \( S_{w+2}(\Gamma) \) and \( x_{0} \) be a point in the upper half plane \( \mathbb{H} \). Then the map given by
\[ o(f, x_{0}): \mathbb{H} \to \mathbb{R}^{+} \]
\[ f \mapsto f(z) = \int \frac{f(z)}{cz^{2} + b} d\sigma_{w+1} \]

is a cocycle in \( Z_{w+1}(\Gamma, \eta_{w}, R) \). The cohomology class \( \varphi(f) \) of the \( o(f, x_{0}) \) in \( H^{w}_{\text{par}}(\Gamma, \eta_{w}, R) \) is uniquely determined by \( f \) (and independent of the choice of \( x_{0} \)). The map \( f \mapsto \varphi(f) \) is the Eichler–Shimura isomorphism which is an \( R \)-linear surjective isomorphism.

Lemma 1.2 ([8,3,1] and Proposition 8.5 in Shimura [22]). Let \( \varphi \) be the isomorphism given in Lemma 1.2 for \( \Gamma = \Gamma(1) \). Then the group
\[ \varphi^{-1}(H^{\infty}_{\text{par}}(\Gamma(1), \eta_{w}, R)) \]

is stable under all the Hecke operators \( T_{w+2}(n) \) (with \( n, N = 1 \)) acting on \( S_{w+2}(\Gamma(1)) \).

Proof. This is a consequence of Lemmas 1.1 and 1.3. ■

Lemma 1.5 (Proposition 8.6 in Shimura [22]). Regard \( H^{p}_{\text{par}}(\Gamma(1), \eta_{w}, R) \) (resp. \( H^{p}_{\text{par}}(\Gamma, \eta_{w}, R) \)) as a vector space over \( R \). Then we have: The group \( \varphi^{-1}(H^{\infty}_{\text{par}}(\Gamma(1), \eta_{w}, R)) \) (resp. \( \varphi^{-1}(H^{\infty}_{\text{par}}(\Gamma, \eta_{w}, R)) \)) is a lattice (i.e. a discrete subgroup of maximal rank) of the \( H^{p}_{\text{par}}(\Gamma(1), \eta_{w}, R) \) (resp. \( H^{p}_{\text{par}}(\Gamma, \eta_{w}, R) \)).
and \( \text{Ker}(j_2) \) (resp. \( \text{Ker}(j_3) \)) is finite. Hence the map \( j_4 \) (resp. \( j_5 \)) induces the following \( R \)-linear isomorphism:

\[
\begin{align*}
\text{Hom}_R^!(\Gamma(1), \eta_w, Z) \otimes_R R & \cong \text{Hom}_R^!(\Gamma(1), \eta_w, R) \\
(\text{resp. } j_2 \otimes \text{id.}: H^1_p(\Gamma(1), \eta_w, Z) \otimes_R R & \cong H^1_p(\Gamma(1), \eta_w, R))
\end{align*}
\]

**Lemma 1.6.** Let \( \Gamma \) be the group \( \Gamma(N) \) and \( \varphi \) be the isomorphism for \( S_{w+2}(\Gamma) \) given in Theorem 1.2. Then the group

\[
\varphi^{-1}(j_5(H^1_p(\Gamma, \omega_{l1}R, Z)))
\]

is stable under the map \( \varphi \).

**Proof.** Let \( f \) be a cuspidal form in \( \varphi^{-1}(j_5(H^1_p(\Gamma, \omega_{l1}R, Z))) \). Then there exists a real column vector \( \mathbf{A} \) in \( R^{w+1} \) such that

\[
\Re \int_{\infty} f(z) \overline{\varphi(z)} \, dz = \omega_{w}(\mathbf{A}) \mathbf{A} \in \mathbb{Z}^{w+1} \quad \text{for all } g \in \Gamma(N).
\]

By changing the variable of the integration,

\[
\int_{\infty} f'(z) \overline{\varphi(z)} \, dz = \Re \int_{\infty} f(u)(-u)^{-d} \overline{u} \, du = \Re \int_{\infty} (-1)^{d+k} f(z) \overline{\varphi(z)} \, dz \quad (g \in \Gamma).
\]

Namely we have:

\[
\omega_{w}(t)^2 = \omega_{w}(t) \omega_{l1}(t) \quad \text{and } \text{for all } g \in \Gamma(N).
\]

We obtain:

\[
\omega_{w}(t) \int_{\infty} f'(z) \overline{\varphi(z)} \, dz = \omega_{w}(t) \int_{\infty} f(z) \overline{\varphi(z)} \, dz \quad (g \in \Gamma).
\]

Note that \( \omega_{w}(t)^2 = \omega_{w}(t) \omega_{l1}(t) \) for all \( g \in \Gamma(N) \).

We obtain:

\[
\omega_{w}(t) \int_{\infty} f'(z) \overline{\varphi(z)} \, dz = \omega_{w}(t) \int_{\infty} f(z) \overline{\varphi(z)} \, dz \quad (g \in \Gamma).
\]

Namely we have \( f' \in \varphi^{-1}(j_5(H^1_p(\Gamma, \omega_{l1}R, Z))) \).

**Lemma 1.7.** The cohomology group \( H^2(\Gamma(N, Z)^+) \) (resp. \( H^2(N, Z)^- \)) is a lattice in \( S_{w+2}(\Gamma(1)) \) (resp. \( \sqrt{-1} S_{w+2}(\Gamma(1)) \)).

**Proof.** By Lemma 1.5, it is sufficient to show that \( H^2(N, Z)^+ \) (resp. \( H^2(N, Z)^- \)) spans \( S_{w+2}^!(\Gamma(1)) \) (resp. \( \sqrt{-1} S_{w+2}^!(\Gamma(1)) \)) over \( R \). Let \( \Gamma \) be \( \Gamma(N) \) and \( \langle a_0, a_1, a_2, \ldots, a_{2d} \rangle \) be a \( Z \)-basis of \( \varphi^{-1}(j_5(H^1_p(\Gamma, \omega_{l1}R, Z))) \).

From Lemma 1.3, we have:

\[
(1.7.1)^* \quad a_j + a_j' \in H(N, Z)^+ \quad \text{and} \quad a_j - a_j' \in H(N, Z)^-
\]

for all \( j \in \mathbb{Z} \) with \( 1 \leq j \leq 2d \). Let \( f \) be a cuspidal form in \( S_{w+2}(\Gamma) \). Then \( f \) is written as \( f = \sum_{j=1}^{2d} a_j a_j^* \) with some \( a_1, a_2, \ldots, a_{2d} \in R^{w+1} \). We have

\[
\begin{align*}
(1.7.2)^+ & \equiv f + f' = \sum_{j=1}^{2d} (a_j + a_j^*) \quad \text{if} \quad f \in S_{w+2}^!(\Gamma),
(1.7.2)^- & \equiv f - f' = \sum_{j=1}^{2d} (a_j - a_j^*) \quad \text{if} \quad f \in \sqrt{-1} S_{w+2}^!(\Gamma).
\end{align*}
\]

(1.7.1)^* and (1.7.2)^* prove Lemma 1.7.

**Lemma 1.8 (Proposition 3.5.3 in Shimura [22]).** Let \( f \) be an element of \( S_{w+2}(\Gamma(1)) \) which is a common eigenform of all the \( T_{n} \) with \( (n, N) = 1 \). Then the form \( f \) belongs to \( S_{w+2}(\Gamma(1), \psi) \) for a unique character \( \psi \) of \( (Z/NZ)^* \).

**Theorem 1.9 (Multiplicity one theorem. Atkin--Lehner [1], Casselman [2], Deligne [4], Li [11] and Miyake [18]).** Let

\[
F(s) = \sum_{n=1}^{\infty} a_n \exp(2\pi i (-n)^s) \quad (a_1 = 1)
\]

be a primitive form in \( S_{w+2}(\Gamma(1)) \) and \( h(a) \) be an element of \( S_{w+2}(\Gamma(N)) \) satisfying \( h(T_{n}a) = a \cdot h(a) \) for all the positive integer \( n \) with \( (n, N) = 1 \). Then there exists a complex number \( c \) such that \( h = cF \).

**Proof.** By the proof of Proposition 3.5 (namely the above Lemma 1.8) in Shimura [22], \( F \) and \( h \) are contained in the same \( S_{w+2}(\Gamma(N, Z)) \) with some Dirichlet character \( \chi \mod N \). Then this lemma is a direct consequence of Theorem 5 in Li [11].

Now let \( R \) be a commutative ring and \( V_R \) be the \( R \)-module consisting of all the \( R^{w+1} \) (the space of column vectors) valued functions \( f \) on \( \text{SL}(2, Z) \)

\[
(1.10.0) \begin{align*}
& f(g_1x) = f(g_1y) = f(g_2) = f(g_3) = \ldots = f(g_m)
\end{align*}
\]

for all \( f \in V_R \) and \( y \in \text{SL}(2, Z) \).

Hence there exists a surjective \( R[\Gamma(1)] \) isomorphism between the \( R[\Gamma(1)] \) modules: \( V_R \) and the \( R \)-module \( R^{w+1} \) with the action of \( \Gamma(1) \) to the left through the \( \eta_0 \approx \text{Ind}_{\Gamma \cap \Gamma(1)}^{\Gamma(1)} \) \( \omega_{l1} \).

**Lemma 1.10.**}

\[
\begin{align*}
& V_R \to R^{w+1}, \quad f \mapsto \left( \begin{array}{c} f(g_1) \\ f(g_2) \\ \vdots \\ f(g_m) \end{array} \right)
\end{align*}
\]

2 — Acta Arithmetica XL1
Lemma 1.10 (Shapiro's lemma, see e.g. Lang [10]). Let $R$ be a commutative ring. Then the map

$$sh^* R : H^n(\Gamma(1), \eta_w, R) \rightarrow H^n(\Gamma(1), \eta_w, R)$$

induced by the compatible maps $\Gamma \rightarrow SL(2, \mathbb{Z})$ and $\mathbb{A} \rightarrow f \mapsto f(g_1) \in R^{w+1}(g_1 = 1_\mathbb{Z})$, is a surjective isomorphism.

We need only the case of $n = 1$ and set $sh = sh^* Z$.

Lemma 1.11. Let $p$ be a rational prime with $p \equiv 1 \pmod{N}$. Set

$$T(p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| g \text{ is an integral matrix, } ad - bc = p \right\}.$$

Then we have:

$$GT(p) G^{-1} = \{ gG^{-1} | g \in T(p) \} = T(p) \quad \text{for every } G \in SL(2, \mathbb{Z}).$$

Proof. It is easy to see that $GgG^{-1}$ is an integral matrix with the determinant $p$. Note that $g = 1_\mathbb{I}(\text{mod } N)$ and that $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$. Hence $GgG^{-1} = gG^{-1} \equiv 1_\mathbb{I}(\text{mod } N)$. \[\blacksquare\]

Lemma 1.12 (Deligne [5]). Let $\lambda_\mathbb{I}$ be an eigenvalue of $T_{w+2}(p)$ acting on $S_{w+2}(\Gamma(N))$. Then we have:

$$|\lambda_\mathbb{I}| \leq 2p^{(w+1)/2} \quad \text{for every Archimedean absolute value } |\cdot|_\mathbb{R}.$$

For our purpose, it is sufficient to know only

$$|\lambda_\mathbb{I} - 1 - p^{w+1}/2| > 0 \quad \text{for every sufficiently large prime } p.$$

2. Eichler-Shimura isomorphism and Shapiro's lemma. We study Theorem 0.1 in this section. Our main results are Theorems 2.2–4.4 below. Let $\Gamma$ be a finite index subgroup of $\Gamma(1)$ and $\Gamma(1) = \bigcup_{j=1}^{m} \Gamma j_\mathbb{I}$ with $g_1 = 1_\mathbb{I}$ be the left coset decomposition. Let $D(f)$ ($f \in S_{w+2}(\Gamma)$), $\eta_w$ and $\xi_0$ be such as defined in $\S$ 1 for the $\Gamma$.

First we prove:

Lemma 2.1. Let $\mathcal{A}$ be a $(w+1)m$ dimensional real column vector. For each cusp form $f$ in $S_{w+2}(\Gamma)$, the map

$$\Gamma(1) \ni g \mapsto T(g) = \left\{ T_{(f)}(g) = \left( \text{Re} \int_{\xi_0}^{\xi_0} D(f)(\xi) \right) \eta_w(\xi) | \xi \in \mathbb{R}^{w+1}\right\}$$

is a parabolic cocycle in $Z_{\mathcal{A}}(\Gamma(1), \eta_w, \mathbb{R})$. Its cohomology class $\Phi(f)$ in $H_{\mathcal{A}}^n(\Gamma(1), \eta_w, \mathbb{R})$ is determined by $f$ and independent of $\xi_0$ and $\mathcal{A}$.

Proof. The proof goes in a similar way to that of Theorem 8.4 in Shimura [22]. For every $s \in \mathcal{A}$, put

$$(2.1.1) \quad \mathcal{P}(s) = \Re \int_{\xi_0}^{\xi_0} D(f)(\xi) + \mathcal{A}.$$

Since the differential form $D(f)(\xi) + \mathcal{A}$ is holomorphic on the $\mathcal{A}$, $\Phi(s)$ is independent of the choice of the path of the integral. For every elements $g$ and $g'$ of $\Gamma(1)$, we have:

$$(2.1.2) \quad \mathcal{P}(g) = \Re \int_{\xi_0}^{\xi_0} D(f)(\xi) + \mathcal{A}$$

and

$$(2.1.3) \quad \mathcal{P}(g'g) = \Re \int_{\xi_0}^{\xi_0} D(f)(\xi) + \mathcal{A} = \Re \int_{\xi_0}^{\xi_0} D(f)(\xi) + \mathcal{A}.$$
(ii) The composite map $\text{sh}_R \circ \varphi$:

$$S_{w+1}(\Gamma) \to H^1_p(\Gamma(1), \eta_w; R) \to H^1_{\varphi}(\Gamma, \omega_{w+1}; R)$$

is the Eichler-Shimura isomorphism $\varphi$ for $S_{w+1}(\Gamma)$ (given in Theorem 1.2).

(iii) The map $\text{sh}_R \circ \varphi : H^1_p(\Gamma(1), \eta_w; R) \to H^1_{\varphi}(\Gamma, \omega_{w+1}; R)$ is a surjective $R$-linear isomorphism.

**Theorem 2.3.** The map $\varphi$ given in Lemma 2.1 is an $R$-linear isomorphism from $S_{w+1}(\Gamma)$ onto the $H^1_p(\Gamma(1), \eta_w; R)$.

**Theorem 2.4.** (Theorem 0.1 in the Introduction). The image under the map $\text{sh}_R$ of the $j_2[H^1_p(\Gamma(1), \eta_w; \mathbb{Z})]$ coincides with the $j_2[H^1_{\varphi}(\Gamma, \omega_{w+1}; \mathbb{Z})]$.

First, we prove Theorem 2.2.

**Proof of Theorem 2.2 (i).** It is easy to see that

$$\text{sh}_R[H^1_p(\Gamma(1), \eta_w; R)] \subset H^1(\Gamma, \omega_{w+1}; R)$$

by Lemma 1.10. Let $\pi$ be an element of $\Gamma \cap \Gamma$ and $r : \Gamma(1) \to \Gamma(1)/\mathcal{O}$ an $\mathcal{O}$-cocycle in $Z^1(\Gamma(1), \eta_w; R)$. We may assume that for each $g \in \Gamma(1)$, $r(g)$ is an element of $V_Z$ by (1.10.1). Since the cocycle $r$ is parabolic, there exists an element $h_\pi$ in $V_Z$ such that $r(g) = (\eta_w(\pi) - 1_{\omega_{w+1}})h_\pi$. Then we have:

$$\text{sh}_R[r](\pi)(r) = \varphi(\pi)g_\pi(1_\mathcal{O}) = \varphi(\pi)(1_\mathcal{O})$$

$$= (\eta_w(\pi) - 1_{\omega_{w+1}})h_\pi(1_\mathcal{O}) = (\eta_w(\pi) - 1_{\omega_{w+1}})h_\pi(1_\mathcal{O}) = (\omega(\pi) - 1_{\omega_{w+1}})h_\pi(1_\mathcal{O}) = (\omega(\pi) - 1_{\omega_{w+1}})h_\pi(1_\mathcal{O})$$

**Proof of Theorem 2.2 (ii).** Let $f$ be an element of $S_{w+1}(\Gamma)$. By the definition of $\Phi$, $\Phi(f)$ is the cohomology class of the cocycle $\{\Gamma(1) \to R \to \text{Re} \int_{0}^{\mathcal{O}} D(f)\}$. By (1.10.1), $\text{Re} \int_{0}^{\mathcal{O}} D(f)$ corresponds to the function $h_\pi \in V_Z$ such that

$$h_\pi(g_j) = \text{Re} \int_{0}^{\mathcal{O}} (f(w) dw_g) \circ g_j \quad \text{for every } j \in \mathbb{Z} \text{ with } 1 \leq j \leq m.$$  

Then by Lemma 1.10, $\text{sh}_R[\Phi(f)]$ is the cohomology class of the cocycle $\{\Gamma(1) \to h_\pi(g_j)\}$. Since $h_\pi(g_j) = \text{Re} \int_{0}^{\mathcal{O}} f(z) dw_g$, the cohomology class $h_\pi$ is equal to $\varphi(f)$ by Theorem 1.2. Hence $\varphi = (\text{sh}_R \circ \Phi)$.

**Proof of Theorem 2.3.** By Lemma 1.2 and Theorem 2.2 (iii), the map $\varphi = (\text{sh}_R \circ \Phi)$ becomes an $R$-linear surjective isomorphism.

For the proof of Theorem 2.4, we need the following three lemmas.

**Lemma 2.5.** Let $\Gamma(1) = \bigcup_{j=0}^{m} \Gamma_j$ be the left coset decomposition and for each $j \in \mathbb{Z}$ with $1 \leq j \leq m$, $K_j$ be the $(w+1)m \times (w+1)m$ integral matrix:

$$
\begin{pmatrix}
\omega(w^{-1}g_1) & 0 & 0 & 0 \\
\omega(w^{-1}g_2) & c_\omega & 0 & 0 \\
\omega(w^{-1}g_3) & 0 & \ddots & \ddots \\
0 & 0 & \ddots & \omega(w^{-1}g_m)
\end{pmatrix}.
$$

Set $\omega(g) = K_j^{-1} \omega(g) K_j$ for all $g \in \Gamma(1)$.

(i) For each $g \in \Gamma(1)$, the following two representations from $\Gamma(1)$ to $GL((w+1)m, \mathbb{Z})$ are isomorphic to each other:

$$\text{Ind}_{\mathcal{O}^{1}(w+1)}^{\mathcal{O}^{1}(w+1)} (\omega(w^{-1}g) \omega^{-1}).$$

(ii) For each $j \in \mathbb{Z}$ with $1 \leq j \leq m$, let $\text{sh}_R^{(j)}$ be the isomorphism of Shapiro [cf. Lemma 1.10]:

$$H^1(\Gamma(1), \text{Ind}_{\mathcal{O}^{1}(w^{-1}g_{j-1})}^{\mathcal{O}^{1}(w^{-1}g_{j-1})} \omega(w^{-1}g_{j-1}) \omega_{w+1}, \mathbb{Z}).$$

Let $\iota_j$ be the canonical isomorphism induced by the above map in (i):

$$\iota_j : H^1(\Gamma(1), \omega_{w+1}, \mathbb{Z}) \to H^1(\Gamma(1), \text{Ind}_{\mathcal{O}^{1}(w^{-1}g_{j-1})}^{\mathcal{O}^{1}(w^{-1}g_{j-1})} \omega_{w+1}, \mathbb{Z}).$$

Then the composite map $\text{sh}_R^{(j)} \circ \iota_j$ is given by:

The cohomology class of a cocycle $\{\Gamma(1) \to \text{Re} \int_{0}^{\mathcal{O}} D(f) (g) \in \mathcal{O}^{1}(w^{-1}g_{j-1}) \}$ is mapped to the cohomology class of the cocycle $\{\Gamma(1) \to \text{Re} \int_{0}^{\mathcal{O}} D(f) (g) \in \mathcal{O}^{1}(w^{-1}g_{j-1}) \}$.

**Proof of Lemma 2.5 (i).** We may assume $g = g_j$ for some $j \in \mathbb{Z}$ with $1 \leq j \leq m$. Normalize the representation

$$\omega(\pi) = \text{Ind}_{\mathcal{O}^{1}(w^{-1}g_{j-1})}^{\mathcal{O}^{1}(w^{-1}g_{j-1})} (\omega(w^{-1}g_{j-1}))$$

as follows:

$$\eta_{\omega}^{(j)}(g) = \begin{pmatrix}
\omega(w^{-1}g_1) & 0 & 0 & 0 \\
\omega(w^{-1}g_2) & c_\omega & 0 & 0 \\
\omega(w^{-1}g_3) & 0 & \ddots & \ddots \\
0 & 0 & \ddots & \omega(w^{-1}g_m)
\end{pmatrix}.$$

Then we obtain:

$$\eta_{\omega}^{(j)}(g) = K_j^{-1} \eta_{\omega}^{(j)}(g) K_j \quad (g \in \Gamma(1)).$$

Proof of Lemma 2.5 (ii). By the definitions of $\eta^{(1)}_m$, $\eta^{(k)}_m$ and $\text{sh}^{(g)}$, (ii) follows directly (cf. the map (1.10.1)).

**Lemma 2.6.** Let $\mathcal{I}(1) = \bigcup_{n=1}^{m} \mathcal{I}_n$, and $f$ be an integer with $1 \leq j \leq m$.

Let $\theta_j$ be the isomorphism

$$H^1(\mathcal{I}(1), \psi_{\mathfrak{w}}, \mathbb{Z}) \to H^1(\gamma_j^{-1} \mathcal{I}_n, \psi_{\mathfrak{w}} \gamma_j^{-1} \mathcal{I}_n, \mathbb{Z})$$

which is induced by the compatible maps,

$$g_j^{-1} \gamma_j \to \mathcal{I}(1) \ni \gamma \mapsto g_j^{-1} \gamma_j^{-1}$$

and

$$\mathbb{Z}^{w+1} \to \mathbb{Z}^{w+1}: x \mapsto \psi_{\mathfrak{w}}(g_j^{-1} x)$$

Then we have the following commutative diagram:

$$\begin{array}{ccc}
H^1(\mathcal{I}(1), \eta^{(k)}_m, \mathbb{Z}) & \xrightarrow{\text{sh}^{(g)} \circ \theta_j} & H^1(\gamma_j^{-1} \mathcal{I}_n, \psi_{\mathfrak{w}} \gamma_j^{-1} \mathcal{I}_n, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(\gamma_j^{-1} \mathcal{I}_n, \psi_{\mathfrak{w}} \gamma_j^{-1} \mathcal{I}_n, \mathbb{Z}) \end{array}$$

**Remark (2.6).** It is easy to see that the map $\theta_j$ induces the (surjective) isomorphism

$$\theta_j^*: H^1_{\text{pro}}(\mathcal{I}(1), \psi_{\mathfrak{w}}, \mathbb{Z}) \to H^1_{\text{pro}}(\gamma_j^{-1} \mathcal{I}_n, \psi_{\mathfrak{w}} \gamma_j^{-1} \mathcal{I}_n, \mathbb{Z})$$

since "an element $\gamma$ of $\mathcal{I}$ is parabolic" $\Leftrightarrow |\text{Tr}(\gamma)| = 2 \Leftrightarrow |\text{Tr}(g_j^{-1} \gamma_j)| = 2 \Leftrightarrow g_j^{-1} \gamma_j + \gamma_j$ is parabolic.

**Proof of Lemma 2.6.** Let $\mathbb{Z}^{w+1}$ be the lattice of the $\mathbb{R}^{w+1, m}$ with respect to the standard basis, $C: \mathcal{I}(1) \to \mathbb{Z}^{w+1}$ be a cocycle in $\mathfrak{I}(\mathcal{I}(1), \eta^{(k)}_m, \mathbb{Z})$ and $\gamma_j$ denote an element of $g_j^{-1} \gamma_j$. We compute:

$$\theta_j \circ \text{sh}^{(g)} \circ \theta_j(\mathcal{I}(1))$$

the cohomology class of the cocycle $g_j^{-1} \gamma_j \ni \gamma_j \mapsto \psi_{\mathfrak{w}}(g_j^{-1} \gamma_j^{-1} \times \{\text{the first } (w+1) \text{ components of } C(g_j \gamma_j^{-1} \gamma_j)\})$.

Note that

$$C(g_j \gamma_j^{-1} \gamma_j) \equiv (1_{(w+1)m} - \eta^{(k)}_m(g_j \gamma_j^{-1} \gamma_j)) C(g_j) + \eta^{(k)}_m(g_j) C(\gamma_j)$$

since $C$ is a cocycle. Now let $a_i(g_j)$ be the $(w+1)$ components of the $C(g_j)$ and $\bar{a}_i(\gamma_j)$ be the $(w+1)$ components of the $C(\gamma_j)$ from the $(w+1) \times (j-1+1)$-th one to the $(w+1)$-th one. Then we have:

$$\theta_j \circ \text{sh}^{(g)} \circ \theta_j(\mathcal{I}(1))$$

the cohomology class of the cocycle $g_j^{-1} \gamma_j \ni \gamma_j \mapsto \psi_{\mathfrak{w}}(g_j^{-1} \gamma_j^{-1} \times \{\text{the first } (w+1) \text{ components of } C(g_j \gamma_j^{-1} \gamma_j)\})$,

the cohomology class of the cocycle $g_j^{-1} \gamma_j \ni \gamma_j \mapsto \psi_{\mathfrak{w}}(g_j^{-1} \gamma_j^{-1} \times \{\text{the first } (w+1) \text{ components of } C(g_j \gamma_j^{-1} \gamma_j)\})$,

the cohomology class of the cocycle $g_j^{-1} \gamma_j \ni \gamma_j \mapsto \psi_{\mathfrak{w}}(g_j^{-1} \gamma_j^{-1} \times \{\text{the first } (w+1) \text{ components of } C(g_j \gamma_j^{-1} \gamma_j)\})$.

**Lemma 2.7.** Notations being as in Lemmas 2.5 and 2.6, the image under the map $\text{sh}^{(g)}$ of the $H^1_{\text{pro}}(\mathcal{I}(1), \eta^{(k)}_m, \mathbb{Z})$ coincides with the $H^1_{\text{pro}}(\mathcal{I}(1), \eta^{(k)}_m, \mathbb{Z})$.

**Proof.** Let $\psi$ be a cohomology class in $H^1_{\text{pro}}(\mathcal{I}(1), \psi_{\mathfrak{w}}, \mathbb{Z})$. Set

$$u = (\text{sh}^{(g)}(\psi))^{-1}(\psi) \in H^1(\mathcal{I}(1), \psi_{\mathfrak{w}}, \mathbb{Z})$$

By Theorem 2.2 (i), it is sufficient to show $u \in H^1_{\text{pro}}(\mathcal{I}(1), \eta^{(k)}_m, \mathbb{Z})$.

By Theorem 2.2 (ii), there exists a cocycle $C$ in $\mathbb{Z}^1(\mathcal{I}(1), \eta^{(k)}_m, \mathbb{Z})$ whose cohomology class is equal to the class $u$ such that $C(\tau) = (1_{(w+1)m} - \eta^{(k)}_m(\tau)) \mathbb{X}$ with some $(w+1)m$-dimensional real column vector

$$\mathbb{X} = \{x_0^{(i)}, x_1^{(i)}, \ldots, x_{(w+1)}^{(i)}; x_0^{(i)}_{(w+1)}, \ldots, x_{(w+1)}^{(i)}: X_{(w+1)m} \}$$

We shall show that there exists a $(w+1)m$-dimensional column vector

$$\mathbb{X} = \{y_0^{(i)}, y_1^{(i)}, \ldots, y_{(w+1)}^{(i)}; y_0^{(i)}_{(w+1)}, \ldots, y_{(w+1)}^{(i)}: \gamma_{(w+1)m} \}$$

with the coefficients in $\mathbb{Z}$ such that

$$(2.7.1) \quad (1_{(w+1)m} - \eta^{(k)}_m(\tau)) \mathbb{X} = (1_{(w+1)m} - \eta^{(k)}_m(\tau)) \mathbb{X} = C(\tau).$$

Put $C^*(\gamma) = K^{-1} C(g)(\gamma \in \mathcal{I}(1))$, $\mathbb{X} = K^{-1} \mathbb{X}$ and

$$C^*(\mathbb{X}) = \{x_0^{(i)}, x_1^{(i)}, \ldots, x_{(w+1)}^{(i)}; x_0^{(i)}_{(w+1)}, \ldots, x_{(w+1)}^{(i)}: x_{(w+1)m}^{(i)} \}$$

where $K_i \in GL((w+1)m, \mathbb{Z})$ is the matrix defined in Lemma 2.5. Now we fix an arbitrary $\gamma \in \mathbb{Z}$ with $1 \leq j \leq m$. For simplicity, set

$$g_0^{(i)} = g_j, g_1^{(i)} = g_{j+1}, \ldots, g_{(w+1)}^{(i)} = g_{(w+1)m}^{(i)}$$

and

$$\Gamma_\gamma = \Gamma_{\gamma_1} \cdots \Gamma_{\gamma_{(w+1)m}}$$

where $\gamma_j = \gamma_j \cdots \gamma_j$ and $\tau$ be the cohomology class of the cocycle $\mathbb{X}$ in the $H^1(\mathcal{I}(1), \eta^{(k)}_m, \mathbb{Z})$. From Lemma 2.6, we obtain that the cohomology class $\text{sh}^{(g)}(\theta_j \circ \text{sh}^{(g)}(\mu)) = \theta_j \circ \text{sh}^{(g)}(\mu)(\psi) = \theta_j(\psi)$ is a parabolic cohomology class with $\mathbb{Z}$-coefficients. Hence the $(w+1)$ components of the vector $C^*(\tau)$ from the $(w+1)/(b-1)+1$-th one to the $((w+1)b)$-th one are equal to

$$(2.7.2) \quad (1_{w+1} - \eta^{(k)}_m(\tau))^{(w+1)b} \in \mathbb{Z}^{w+1}$$

for some $(w+1)$-dimensional vector $\mathbb{X}^{(w+1)b} = \{x_0^{(w+1)b}, x_1^{(w+1)b}, \ldots, x_{w+1}^{(w+1)b}\}$ with $\mathbb{Z}$ coefficients. For each integer $i$ with $1 \leq i \leq (w+1)b$, set

$$\mathbb{X}^{(w+1)b} = \{x_0^{(w+1)b}, x_1^{(w+1)b}, \ldots, x_{w+1}^{(w+1)b}\}$$

Note that $j(\tau) = b$. Since

$$(2.7.2) \quad C^*(\tau) = K^{-1} C^{(w+1)m} - \eta^{(k)}_m(\tau) \mathbb{X} = (1_{(w+1)m} - \eta^{(k)}_m(\tau)) \mathbb{X}$$

we obtain:

\[(2.7.3) \quad 3^{(0)} - \varphi_0(\tau) 3^{(0)} = \mathcal{B}_1, \ 3^{(0)} - \varphi_1(\tau) 3^{(0)} = \mathcal{B}_2, \ldots \]

\[
\ldots, \quad 3^{(r-1)} - \varphi_{r-1}(\tau) 3^{(0)} = \mathcal{B}_{r-1}, \ 3^{(0)} - \varphi_r(\tau) 3^{(0)} = \mathcal{B}_r,
\]

for some vectors \(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r\) in \(\mathbb{Z}^{n+1}\). Replacing the cocycle \(\{\Gamma(1) \ni g \mapsto \mathcal{C}^*(g)\}\) by \(\{\Gamma(1) \ni g \mapsto \mathcal{C}^*(g) + \{l(\text{det}(g)) - l(\text{det}(g))\}L\}\) for some vector \(L \in \mathbb{Z}^{n+1}\) (namely replacing \(3^{(0)}\) by \(3^{(0)} - \mathcal{B}_1\) for each \(x \in \mathbb{Z}\) with \(1 \leq x \leq r-1\), we may assume that \(\mathcal{B}_1 = \mathcal{B}_2 = \ldots = \mathcal{B}_{r-1} = 0\). Then we obtain from (2.7.3) that \(3^{(0)} - \varphi_r(\tau) 3^{(0)} = \mathcal{B}_r\). Recall \(\varphi^* \in \langle f^{(n)} \rangle\) and note that

\[\mathcal{C}^*(\tau^*) = (\lambda_{(0,1)} - \eta^*(\tau)) + \ldots + (\lambda_{(r-1)} - \eta^*(\tau)) \mathcal{B}_r = (\lambda_{(0,1)} - \eta^*(\tau)) \mathcal{B}_r.\]

By (2.7.2),

\[(2.7.4) \quad (\lambda_{(0,1)} - \eta^*(\tau)) \mathcal{B}_r = (\lambda_{(0,1)} - \eta^*(\tau)) \mathcal{B}_r\]

By (2.7.3),

\[\mathcal{E}^{(0)}(0) = 3^{(0)} - \mathcal{B}_r \quad \text{for all } i \in \mathbb{Z} \quad \text{with } 1 \leq i \leq r-1.\]

Then the \(\mathcal{E}^{(0)}(0)\) satisfies

\[\mathcal{E}^{(0)} - \mathcal{B}_r = (\lambda_{(0,1)} - \eta^*(\tau)) \mathcal{B}_r = (\lambda_{(0,1)} - \eta^*(\tau)) \mathcal{B}_r \quad \text{by} \ (2.7.4).\]

Replace the \(3^{(0)}\) by the \(\mathcal{E}^{(0)}(0)\) for all \(i \in \mathbb{Z}\) with \(1 \leq i \leq r\). Do the above procedure over again for each orbit \(\mathcal{E}_i\) with \(1 \leq i \leq m\). (Here \(\mathcal{E}_i = \bigcup \mathcal{E}_j^*\).) Then (2.7.1) is proved since we can write \(K_{-1}\mathcal{E}\) using the vectors \(\mathcal{E}^*\) obtained by the procedures.

This argument in proving (2.7.1) is effective for any parabolic element \(x \in \Gamma\) in \(\text{SL}(2, \mathbb{Z})\) if we replace \(\tau\) by the \(x\). Hence there is a vector \(\eta_x \in \mathbb{Z}^{n+1}\) for each \(x \in \mathbb{P}\) such that

\[\mathcal{C}(x) = (\lambda_{(0,1)} - \eta^*_x) \mathcal{E}_x.\]

Hence the \(\mathcal{C}\) becomes a parabolic cocycle in the \(\mathbb{C}^b_1(1, \eta, \mathcal{B}_r, \mathcal{Z})\). Namely it is proved that \(u \in \mathbb{C}^b_1(1, \eta, \mathcal{B}_r, \mathcal{Z})\). Lemma 2.7 is proved. \(\blacksquare\)

Proof of Theorem 2.4. By Lemma 2.7,

\[j_2(3^*(\tau^*) 3^{(0)} = j_2^*(3^*(\tau^*) 3^{(0)} = j_2^*(\tau^*) 3^{(0)} = 3^{(0)}.\]

It is easy to see \(j_2 \circ j_2^* = \varphi_2 \circ j_1\). Hence

\[\varphi_2 \circ j_1 \big(3^{(0)} 3^{(0)} = j_2^*(\tau^*) 3^{(0)} = j_2^*(\tau^*) 3^{(0)}.\]

Remark 2.8. It is well known that \(\text{SL}(2, \mathbb{Z})\) is generated by \(s_1\) and \(s_2\). Since \(s_1(0) = s_2(0) = \infty\), the following map is an \(\mathbb{R}\)-linear injection by Theorem 2.3.

\[s_{n+1}(I) \in \mathbb{R}^{n+1} \quad \text{for all } n \in \mathbb{R}.\]

And every cohomology class \(u \in \mathcal{H}_1^1(1, \eta, \mathcal{B}_r, \mathcal{Z})\) has a representative \(\mathcal{E}^* \in \mathcal{H}_1^1(1, \eta, \mathcal{B}_r, \mathcal{Z})\) with \(\mathcal{C}(u) = \mathcal{C}(u)\). Furthermore we can also show that for each cohomology class \(u \in \mathcal{H}_1^1(1, \eta, \mathcal{B}_r, \mathcal{Z})\) there exists a cocycle \(\mathcal{E}^* \in \mathcal{H}_1^1(1, \eta, \mathcal{B}_r, \mathcal{Z})\) with \(\mathcal{C}(u) = \mathcal{C}(u)\) and \(j_1(\mathcal{C}^*) = \mathcal{C}(\eta, \mathcal{B}_r, \mathcal{Z})\).

3. Proof of Theorem 0.2. We use the same notations in \(\S 1\) and \(\S 2\). From Theorems 2.2 and 2.4, we have:

\[\varphi^{-1} \big[ j_2^*(\mathcal{C}^*(\tau^*) 3^{(0)} = \varphi^{-1} \big[ j_2^*(\mathcal{C}^*(\tau^*) 3^{(0)} \big].\]

By (2.7.3),

\[\mathcal{E}^{(0)}(0) = 3^{(0)} - \mathcal{B}_r \quad \text{for all } i \in \mathbb{Z} \quad \text{with } 1 \leq i \leq r-1.\]

Then the \(\mathcal{E}^{(0)}(0)\) satisfies

\[\mathcal{E}^{(0)} - \mathcal{B}_r = (\lambda_{(0,1)} - \eta^*(\tau)) \mathcal{B}_r = (\lambda_{(0,1)} - \eta^*(\tau)) \mathcal{B}_r \quad \text{by} \ (2.7.4).\]

For each positive integer \(n\) with \((n, N) = 1\), there exists a \(d \times d\) matrix \(A^*(n)\) (resp. \(A^*(n)\)) with the coefficients in \(\mathbb{Z}\) such that

\[\{U^0, U^1, U^2, \ldots, U^r\} \quad \text{resp.} \quad \{U^0, U^1, U^2, \ldots, U^r\} \]

\[\{U^0, U^1, U^2, \ldots, U^r\} \quad \text{resp.} \quad \{U^0, U^1, U^2, \ldots, U^r\} \]

\[\text{by Lemma 1.3. Let } \mathbf{P}(x) = \sum \mathbf{B}_x^n \quad \text{be a primeform in } \mathcal{S}_n(1, \mathcal{B}_r, \mathcal{Z}). \text{ By Lemma 1.3, the basis } \mathbf{U}^* \text{ is a primeform in } \mathcal{S}_n(1, \mathcal{B}_r, \mathcal{Z}) \text{ for all positive integers } n \text{ with } (n, N) = 1. \text{ Now consider the following linear equations.}

\[(2.7.1) \quad (a_x, a_y, \ldots, a_z) A^*(n) = A^*(n) \quad \text{for all positive } n \in \mathbb{Z} \text{ with } (n, N) = 1.

\[(2.7.2) \quad (a_x, a_y, \ldots, a_z) A^*(n) = A^*(n) \quad \text{for all positive } n \in \mathbb{Z} \text{ with } (n, N) = 1.

By the Multiplicity one theorem (Lemma 1.10), the space of the solutions \((2.7.1)\) (resp. \((2.7.2)\)) is one dimensional over \(\mathbb{C}\) for every \(x \in \text{Aut}(G)\).
Note that every matrix \((\mathcal{A}^+(n) - a^2_i I_d)\) (resp. \((\mathcal{A}^-(n) - a^2_i I_d)\)) has the coefficients in the field \(Q^r\) \(Q(a^1_1, a^1_2, a^1_3, \ldots)\). Hence the equations (3.1) \((\text{resp. } (3.1)_r)\) have a non-trivial solution
\[
(a^1_1, a^1_2, \ldots, a^1_d) \Rightarrow (a^2_1, a^2_2, \ldots, a^2_d)
\]
with the coefficients in the \(Q^r\). Since the \(\mathcal{A}^\pm(n) (n \in \mathbb{Z})\) has the coefficients in \(\mathbb{Z}\), the vector \((a^1_1, a^1_2, \ldots, a^1_d) = (a^2_1, a^2_2, \ldots, a^2_d)\) (resp. \((a^1_1, a^1_2, \ldots, a^1_d) = (a^2_1, a^2_2, \ldots, a^2_d)\)) becomes a non-trivial solution of the equations (3.1) \((\text{resp. } (3.1)_r)\) for every \(\sigma \in \text{Aut}(C)\). Namely there exists a unique complex number \(c^2_\sigma \) (resp. \(c^2_\sigma\)) such that
\[
(F^t) = c^2_\sigma \times \sum_{r=1}^{d} a^2_r (U_r^t + \sigma (U_r^t)) \quad \text{(resp. } (F^t) = c^2_\sigma \times \sum_{r=1}^{d} a^2_r (U_r^t + \sigma (U_r^t))\).
\]
We shall show that (3.2) \((\text{resp. } 3.2)_r\) assert our Theorem 0.2 (ii) (c). Then it is easy to see, from (3.2) \((\text{resp. } 3.2)_r\), that
\[
(c^2_\sigma)^t = c^2_\sigma \quad \text{if} \quad \sigma |\sigma' = \sigma' |\sigma \quad \text{(and } \sigma' \in \text{Aut}(C))
\]
and
\[
((c^2_\sigma)^t)^t = c^2_\sigma \quad \text{if} \quad (\sigma |\sigma' = \sigma' |\sigma, \sigma' \in \text{Aut}(C)).
\]
for the complex conjugation \(J\) and any \(\sigma \in \text{Aut}(C)\) since \((U_t^t)^t = \pm U_t\) for all \(t \in \mathbb{Z}\) with \(1 \leq t \leq d\). This is a proof of Theorem 0.2 (ii) (a) and (b).

Now we shall prove Theorem 0.2 (ii) (c). For each \(g \in \Gamma(1)\) and \(\sigma \in \text{Aut}(C)\), set
\[
W(F, g, \sigma) = \left(2 c^2_\sigma \right)^{-1} \left( \int_{\text{Max}} D(F^t) - \int_{\text{Min}} D(F^t) \right).
\]
We shall prove below that
\[
\left(2 c^2_\sigma \right)^{-1} \left( \int_{\text{Max}} D(F^t) - \int_{\text{Min}} D(F^t) \right) = 0 \quad \text{for all } g \in \Gamma(1) \text{ and } \sigma \in \text{Aut}(C).
\]
for all \(g \in \Gamma(1)\). Since \(U^t \in \mathbb{R}^{1 \times 1} (\Gamma(1), \mathbb{Z}^N)\) and \(U^t \in \mathbb{R}^{1 \times 1} (\Gamma(1), \mathbb{Z}^N)\), changing the variable of integration from \(z\) to \(-z\), we have:
\[
\int_{\text{Max}} (U^t(z) d\sigma) \Rightarrow (e^2_{\sigma}(t) \int_{\text{Max}} (U^t(z) d\sigma) \Rightarrow (e^2_{\sigma}(t)\}
\]
for all \(g \in \Gamma(1)\). Hence
\[
(3.5) \quad \text{Re} \int_{\text{Max}} D(U^t)^t(z) = \frac{1}{2} \left( \int_{\text{Max}} D(U^t)^t(z) \pm \int_{\text{Min}} D(U^t)^t(z) \right)
\]
for all \(g \in \Gamma(1)\).

We shall prove below that
\[
W(F, g, \sigma) = \left(2 c^2_\sigma \right)^{-1} \left( \int_{\text{Max}} D(F^t) - \int_{\text{Min}} D(F^t) \right) = 0 \quad \text{for all } g \in \Gamma(1) \text{ and } \sigma \in \text{Aut}(C).
\]
for all \(g \in \Gamma(1)\). Since \(U^t \in \mathbb{R}^{1 \times 1} (\Gamma(1), \mathbb{Z}^N)\) and \(U^t \in \mathbb{R}^{1 \times 1} (\Gamma(1), \mathbb{Z}^N)\), changing the variable of integration from \(z\) to \(-z\), we have:
\[
(3.4) \quad \int_{\text{Max}} (U^t(z) d\sigma) \Rightarrow (e^2_{\sigma}(t) \int_{\text{Max}} (U^t(z) d\sigma) \Rightarrow (e^2_{\sigma}(t)\}
\]
for all \(g \in \Gamma(1)\). Hence
\[
(3.5) \quad \text{Re} \int_{\text{Max}} D(U^t)^t(z) = \frac{1}{2} \left( \int_{\text{Max}} D(U^t)^t(z) \pm \int_{\text{Min}} D(U^t)^t(z) \right)
\]
for all \(g \in \Gamma(1)\).
Proof of (3.7)\textsuperscript{\#}.

Case 1 (weight \( w + 2 \geq 3 \)). Let \( k \) be an integer with \( 1 \leq k \leq w \) and \( g \) be an element of \( g_k^1 \Gamma_2(N) g_k \). From (3.8)\textsuperscript{\#}, we obtain:

\[
(3.9)\textsuperscript{\#} \quad \left( (2\omega) \right)^{-1} \left( \int_{G_{\text{Ig}(1)}} F_{|w+2|}[g_k](s) d\sigma_{w} \pm \omega_{w}(t) \int_{G_{\text{Ig}(1)}} F_{|w+1|}[g_k t](s) d\sigma_{w} \right) = \left( (2\omega) \right)^{-1} \left( \int_{G_{\text{Ig}(1)}} F_{|w+2|}[g_k](s) d\sigma_{w} \pm \omega_{w}(t) \int_{G_{\text{Ig}(1)}} F_{|w+1|}[g_k t](s) d\sigma_{w} \right)
\]

for all \( g \in g_k^1 \Gamma_2(N) g_k \). Put the left side of (3.9)\textsuperscript{\#} = \( M_{g}(F, g)\textsuperscript{\#} \). Now set \( g = \left( \begin{array}{cc} 1 & N \\ 0 & 1 \end{array} \right) \) in (3.9)\textsuperscript{\#}. Then \( g(1) = \infty \) and

\[
(3.10)\textsuperscript{\#} \quad \theta = \left( (1-w) \right)^{\left( \begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right)} \left( \lambda_{w,1}^{1}, \lambda_{w,1}^{2}, \ldots \lambda_{w,1}^{\infty} \right)
\]

Using the definition of the representation \( \lambda_{w,1} \), we solve (3.10)\textsuperscript{\#} directly. Then we easily obtain:

\[
(3.10.1)\textsuperscript{\#} \quad \lambda_{w,1}^{1} = \lambda_{w,1}^{2} = \lambda_{w,1}^{\infty} = \ldots = \lambda_{w,1}^{k} = 0.
\]

Only \( \lambda_{w,1}^{0} \) and \( \lambda_{w,1}^{k} \) remain unknown. Now we shall show that \( \lambda_{w,1}^{0} = \lambda_{w,1}^{k} = 0 \).

Let \( p \) be an odd rational prime with \( p = 1 \mod N \) and \( \omega_{w}^{\pi} \neq 1 - p^{-1} \).

Such a prime \( p \) exists by Lemma 1.2. Put

\[
\nu = \left( \begin{array}{cc} 1 & N \\ 0 & p \end{array} \right) \quad \text{for} \quad u \in Z \quad \text{with} \quad 0 \leq u \leq p - 1 \quad \text{and} \quad \nu_{p} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

Note that \( v_{w} \circ \nu_{w}^{-1} \in \Gamma(N) \) for all \( g \in \Gamma(pN) \) and \( u \in Z \) with \( 0 \leq u \leq p \).

Put \( w(g) = v_{w} \circ \nu_{w}^{-1} \) for every \( g \in G \) with \( 0 \leq u \leq p \). We need:

**Lemma 3.1.** Let \( p \) be an odd prime with \( p = 1 \mod N \) and \( g \) be an element of \( \Gamma(pN) \). We obtain:

\[
(3.11)\textsuperscript{\#} \quad M_{g}(F|W_{w+2}(p), g)\textsuperscript{\#} = p^{w+1} \left( \sum_{u=0}^{p} \omega_{w}(v_{w})^{-1} M_{u}(F, w(g))\textsuperscript{\#} \right).
\]

**Proof of Lemma 3.1.** Recall

\[
F|W_{w+2}(p)(s) = p^{w+2} \sum_{u=0}^{p} F_{|w+2|}[v_{w} t_{u}](s) = p^{w+2} \sum_{u=0}^{p} F_{|w+1|}[v_{w} t_{u}](s).
\]

Hence

\[
M_{g}(F|W_{w+2}(p), g)\textsuperscript{\#} = p^{w+2} \sum_{u=0}^{p} \left( (2\omega) \right)^{-1} \left( \int_{G_{\text{Ig}(1)}} F_{|w+2|}[v_{w} t_{u}](s) d\sigma_{w} \pm \omega_{w}(t) \int_{G_{\text{Ig}(1)}} F_{|w+1|}[v_{w} t_{u}](s) d\sigma_{w} \right)
\]

By Lemma 1.1 and the fact that the group \( \Gamma(N) \) is a normal subgroup of \( \Gamma(1) \) with \( \Gamma(N) \subset \Gamma(1) \), for every \( x \in \Gamma(1) \),

\[
\sum_{u=0}^{p} (F_{|w+2|}[v_{w} t_{u}](s) - F_{|w+1|}[v_{w} t_{u}](s)) = 0,
\]

\[
\sum_{u=0}^{p} (F_{|w+2|}[v_{w} t_{u}](s) - F_{|w+1|}[v_{w} t_{u}](s)) = 0,
\]

\[
\sum_{u=0}^{p} (F_{|w+1|}[v_{w} t_{u}](s) - F_{|w+1|}[v_{w} t_{u}](s)) = 0.
\]

Replace the variable \( s \) of the integrals

\[
\int_{G} F_{|w+2|}[v_{w} t_{u}](s) d\sigma_{w} \quad \text{and} \quad \int_{G} F_{|w+1|}[v_{w} t_{u}](s) d\sigma_{w}
\]

(respectively:

\[
\int_{G} F_{|w+2|}[v_{w} t_{u}](s) d\sigma_{w} \quad \text{and} \quad \int_{G} F_{|w+1|}[v_{w} t_{u}](s) d\sigma_{w}
\]

by \( v_{w}^{-1}(s) \) (respectively \( v_{w}^{-1}(s) \)). Note that \( v_{w}^{-1}(s) = v_{w}^{-1}(s) = v_{w}^{-1}(s) = v_{w}^{-1}(s) = v_{w}^{-1}(s) = v_{w}^{-1}(s) \).

and

\[
\nu_{w}(v_{w}) = v_{w}(v_{w}) = v_{w}(v_{w}) = v_{w}(v_{w}) = v_{w}(v_{w}) = v_{w}(v_{w}) \quad \text{(in) (0,\infty)}
\]

Then we obtain:

\[
M_{g}(F|W_{w+2}(p), g)\textsuperscript{\#} = p^{w+1} \sum_{u=0}^{p} \left( (2\omega) \right)^{-1} \left( \int_{G_{\text{Ig}(1)}} F_{|w+2|}[v_{w} t_{u}](s) d\sigma_{w} \pm \omega_{w}(t) \int_{G_{\text{Ig}(1)}} F_{|w+1|}[v_{w} t_{u}](s) d\sigma_{w} \right)
\]

We continue the proof of (3.7)\textsuperscript{\#}. Note that \( F|W_{w+2}(p) = a_{w}^{*} F \) and \( F_{|w+2|}(p) = \alpha_{w}^{*} F \). From (3.9)\textsuperscript{\#} and (3.11)\textsuperscript{\#}, we obtain:

\[
(3.12)\textsuperscript{\#} \quad \left( a_{w}(1 - w) - a_{w}(g) \right) \left( \sum_{u=0}^{p} \omega_{w}(v_{w})^{-1} (1 - w_{u} - u_{w}(g)) \right) \times
\]

\[
\lambda_{w,1}^{*}, \lambda_{w,1}^{2}, \ldots, \lambda_{w,1}^{p} = 0 \quad \text{(in) \( \Gamma(pN) \)).}
\]
Now set $g = \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{pmatrix}$. Then we have:

$$v^{-1}u(g) = \begin{cases} 1/pN & \text{for } u \in \mathbb{Z} \text{ with } 0 \leqslant u \leqslant p-1, \\ 1/N & \text{for } u = p, \\ 0 & \text{for } u = p+1, \end{cases}$$

and, in the same way,

$$\sum_{z=0}^{t_{q(t)}} \left( \int F_{T_z(p)}(z) dz \right) g_j = \sum_{u=0}^{p} \sum_{v=0}^{t_{q(t)}} F_{u}[g_j](z) dz.$$ 

Hence we obtain:

$$W[F(T_z(p), g, c)] = \sum_{u=0}^{p} W(F, g \cdot \gamma(u, g), c).$$

Since $F(T_z(p)) = a_p F$ and $F(T_z(p)) = a_p \gamma(w)$, by (3.8),

$$a_p^2 [I_m - \eta(g)] B(c) = \sum_{u=0}^{p} (I_m - \eta(g \cdot \gamma(u, g))] B(c).$$

Since $\gamma(u, g) \in I(1), N)$ and $\eta([m - \eta(g) = I_m$, Hence by (3.14),

$$(\alpha_p - 1) [I_m - \eta(g)] B(c) = 0 \quad \text{and} \quad [I_m - \eta(g)] B(c) = 0 \quad \text{for all } g \in I(1).$$

In (3.15), set $g = g_j$ for each $j \in \mathbb{Z}$ with $2 \leqslant j \leqslant m$. The first row vector of $[I_m - \eta(g)]$ is $(1, 0, \ldots, 0, 0, 0)$ where $0$ is the $j$th component. Then the first component of the left side of the vector (3.15) is equal to $x_1 - x_j$, which is 0 by (3.15). Hence we obtain:

$$a_p^2 [I_m - \eta(g)] B(c) = \sum_{j=1}^{m} (x_j - x_j) = 0.$$ 

Since $\eta(g) \in I(1))$ is a permutation matrix, we obtain:

$$[I_m - \eta(g)] B(c) = 0 \quad \text{for all } g \in I(1).$$

(3.7) is proved for the case of $w = 2$.

By (3.8), (3.9) assert that $W(F, g, c) = 0$ for all $g \in I(1)$ and all $c \in \text{Aut}(C)$. Set $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $W(F, g, c)$. Then we obtain Theorem 0.2 (ii) (c).

Now consider all the automorphisms $c \in \text{Gal}(C/Q_p)$. Then Theorem 0.2 (i) follows from the Galois theory, Theorem 0.2 (ii) (a) and Theorem 0.2 (ii) (c).

We give two corollaries of Theorem 0.2. For $i \in \mathbb{Z}$ with $0 \leqslant i \leqslant w$, $x \in Q$ and $c \in \text{Aut}(C)$, set

$$P_{i,x}^{c} = \left( \begin{array}{c} c \\ \end{array} \right)^{i-1} \int \int \int F(x + z) s \cdot ds \cdot (-1)^{i-1} \int \int \int F(x + z) s \cdot ds.$$ 

**Corollary 3.16 (Theorem 0.2).** For all $x \in Q, i \in \mathbb{Z}$ with $0 \leqslant i \leqslant w$ and $c \in \text{Aut}(C)$, we have:

$$P_{i,x}^{c} = P_{i,x}^{c} \in Q_p$$ and $$P_{i,x}^{c} = P_{i,x}^{c} \in Q_p.$$ 

For the proof we need:
**Lemma 3.17.** For \( a \in \mathcal{Q}, l \in \mathbb{Z} \) with \( 0 \leq l \leq w \) and \( \sigma \in \text{Aut}(C) \), set
\[
H^l_{\sigma}(a) = \left( c^{-l}_{\sigma} \right)^{-1} \left( \int_{-\infty}^{\infty} F^l(z) z^l dz \pm (-1)^{l+1} \int_{-\infty}^{\infty} F^l(z) z^l dz \right).
\]
We have:
\[
H^l_{\sigma}(a) \in \mathcal{Q}_p^d \quad \text{and} \quad (H^l_{\sigma}(z))^\sigma = H^l_{\sigma}(a).
\]

**Proof.** This is a direct consequence of \( V(F, g, \sigma) \equiv 0 \) (3.7), since the set \( \{ g(i) : g \in G(1) \} \) coincides with \( Q \cup \{ i \infty \} \).

**Proof of Corollary 3.16.**
\[
P^*_\sigma(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n} \right) (-1)^{-l} H^l_{\sigma}(x).
\]

Let \( \psi \) be a primitive Dirichlet character, \( m(\psi) \) be its conductor and \( \Theta(\psi) \) be its Gauss sum \( = \left( \sum_{n=1}^{m(\psi)} \psi(n) \exp(2\pi i n/m(\psi)) \right) \). Set \( F \otimes \psi(z) = \sum_{n=1}^{m(\psi)} \psi(n) q^m, \) where \( q = \exp(2\pi i/1z) \) and \( \psi(n) = 0 \) if \( (n, m(\psi)) \neq 1 \). Let \( \mathcal{Q}(\psi) \) be the field generated over \( Q \) by the values \( \psi \) takes.

**Corollary 3.18 (of Theorem 0.2) (cf. Shimura [23], [24]).** Take the superscript on \( \psi \) as in the formula \( \psi(-1) = \pm(-1)^{l-1} \). We have:
\[
\left( c^l_{\psi} \Theta(\psi) \right)^{-1} \int_{0}^{1} F \otimes \psi(z) z^l dz = \left( c^l_{\psi} \Theta(\psi) \right)^{-1} \int_{0}^{1} F \otimes \psi^\sigma(z) z^l dz \in Q_p^d \cdot \mathcal{Q}(\psi)
\]
for every \( l \in \mathbb{Z} \) with \( 0 \leq l \leq w \) and \( \sigma \in \text{Aut}(C) \).

**Proof.** It is well known (see e.g. Lemma 9.4 in Manin [12] or Weil [23]) that
\[
F \otimes \psi(z) = \frac{\Theta(\psi)}{m(\psi)} \sum_{b \equiv m(\psi) \mod m(\psi)} \psi^{-1}(-b) F \left( z + \frac{b}{m(\psi)} \right).
\]
Hence
\[
(\Theta(\psi)^{-1}) \int_{0}^{1} \left( F \otimes \psi^\sigma(z) \right) z^l dz
\]
\[
= \left( m(\psi)^{-1} \right) \sum_{b \equiv m(\psi) \mod m(\psi)} \left( \psi^{-1}(-b) \right) \int_{0}^{\infty} \left( z + \frac{b}{m(\psi)} \right)^l z^l dz
\]
\[
= \left( m(\psi)^{-1} \right) \sum_{0 < b < m(\psi)/l} \left( \psi^{-1}(-b) \right) \int_{0}^{\infty} \left( z + \frac{b}{m(\psi)} \right)^l z^l dz + (\psi^{-1}(-1)) \int_{0}^{\infty} \left( z - \frac{b}{m(\psi)} \right)^l z^l dz.\]

Now Corollary 3.18 follows from Lemma 3.17.  

**4. On \( p \)-adic measures.** In this section we shall generalize the results on \( p \)-adic measures in Manin [16] to the case of Neben-type primitive forms. Let \( F \) be a primitive form in \( S_{2-k}(\mathbb{Q}, \mathcal{E}) \), \( p \) be a rational prime, \( L \) be a finite extension of \( \mathbb{Q}_p \) and \( A_0 (> 0) \) be a rational integer with \( (A_0, p) = 1 \). Put
\[
\mathcal{F}(z) = \sum_{n=1}^{\infty} a_n q^m, \quad a_n = \exp(2\pi \sqrt{-1} x), \quad \omega_n = \left( \frac{0}{N}, -1, 0 \right),
\]
\( A = A_0 p \) if \( p \geq 3, \ A = A_0 \) if \( p = 2 \) and \( Z_L = \text{proj-lim} [Z/(A_0)] \).

Following (9.4) in Manin [15] we call a finitely additive function \( \mu \) of the open and closed subsets of \( Z_L^* \) with values in the field \( L \) a \( L \)-measure \( \mu \) on \( Z_L^* \). The following proposition is a generalization of Lemma 9.4 in Manin [15]. Let \( \mathcal{Q}_d \) be the set of rational numbers whose denominators divide \( A_0 \) for all \( m \geq 1 \).

**Proposition 4.1.** Let \( \mathcal{R} : \mathcal{Q} \to L \) be a function with the following properties: for some \( A \) and \( B \in L \) and all \( x \in \mathcal{Q}_d \)
\[
\mathcal{R}(a+1) = \mathcal{R}(a) \quad \text{and} \quad \sum_{l=1}^{m-1} \mathcal{R}((x+b)/p) = A \cdot \mathcal{R}(x) + B \cdot \mathcal{R}(ax).
\]

Let \( \psi \) be a Dirichlet character which takes its values in \( L \) and \( q \) denote a root of the equation \( \psi = \psi(p)[A_0 + B_0 \psi(p)], \) with \( q \neq 0 \). Then there exists a \( L(q) \) valued measure \( \mu \) on \( Z_L^* \) such that for all non negative \( m \in Z \) and all \( a \in Z \)
\[
\mu(a + (A_0 p)^m) = e^{-m} \psi(A_0 p) \mathcal{R}(a(A_0 p)^m) + Be^{-m-1} \psi(A_0 p^{m+1}) \mathcal{R}(a(A_0 p^{m-1})).
\]
Here \( (A_0 p)^m \) denotes \( A_0 p^m Z_L^* \).

**Proof.** By (24) in Manin [15], it is sufficient to show
\[
\mu(I_{a,m}) = \sum_{b \equiv m \mod (A_0 p^m)} \mu(I_{a,b,m+1}) \quad (\text{for all } a \in Z \text{ and all } 0 \leq m \leq Z)
\]
where \( I_{a,m} \) denotes \( a + A_0 p^m Z_L^* \). We compute as follows.
\[
\sum_{k=0}^{m-1} \mu(a + A_0 p^m) \frac{1}{(A_0 p^{m+1})} \quad \left( \begin{array}{c}
\sum_{k=0}^{m-1} e^{-m} \psi(A_0 p^{m+1}) \mathcal{R}(a + A_0 p^m) \frac{1}{(A_0 p^{m+1})} + \\
+ Be^{-m-1} \psi(A_0 p^{m+1}) \mathcal{R}(a + A_0 p^m) \frac{1}{(A_0 p^{m+1})}
\end{array} \right)
\]
\[
= e^{-m} \psi(A_0 p^{m+1}) [A \cdot \mathcal{R}(a(A_0 p^m)) + B \cdot \mathcal{R}(a(A_0 p^{m+1})) + \\
+ e^{-m-1} \psi(A_0 p^{m+1}) B \cdot \mathcal{R}(a(A_0 p^m))].
\]

---

2 Acta Arithmetica XL:1

---
We fix an embedding $\iota$ of the algebraic closure $\bar{Q}$ ($= Q_p$) of $Q$ into $\bar{Q}_p$, once and for all and identify the elements of $\bar{Q}$ with those of $(\bar{Q}_p)^*$ by this $\iota$. For each $l \in Z$ with $0 < l \leq w$, we put $\mu^l (\text{resp. } \check{\mu}^l)$ be the $p$-adic measures on $Z^l_1$, constructed by Proposition 4.1, (4.2.4) and (4.2.5), which are defined by:

\begin{equation}
\mu^l (a + (Ap^m)) = \xi^\mu_l \mu^l_1 (a / (Ap^m)) - \tilde{\xi}^\mu_l \mu^l_1 (a / (Ap^{m-1})) \quad (\xi^\mu_l = \tilde{\xi}^\mu_l = \rho^{-1} \sigma_1 - \rho^{-1} \sigma_2)
\end{equation}

\begin{equation}
\check{\mu}^l (a + (Ap^m)) = \check{\xi}^\mu_l \check{\mu}^l_1 (a / (Ap^m)) - \check{\xi}^\mu_l \check{\mu}^l_1 (a / (Ap^{m-1})) \quad (\check{\xi}^\mu_l = \check{\sigma}^\mu_l = \rho^{-1} \sigma_1 - \rho^{-1} \sigma_2)
\end{equation}

In these formulas, we may put $\check{\sigma} = \check{\sigma}_0$, $\check{\xi} = \check{\xi}_0$, $\check{\xi} = p^{-1} \check{\xi}_0$ and $\check{\sigma} = p^{-1} \check{\sigma}_0$ for $l \in Z$ with $0 < l \leq w$. The following theorem is a generalization of Theorem 3 in Manin [16].

**Theorem 4.2.** Let $m$ be a non-negative integer and $\psi$ be a primitive Dirichlet character mod $Ap^m$. Then we obtain:

\begin{equation}
\int_0^\infty \frac{Ap^m}{\psi(s)} \sum_{b \text{ (mod } Ap^m)} \psi^{-1} (-b) \tilde{\mu}^l_1 (b / (Ap^m))
\end{equation}

where the superscripts are taken as in the formula $\psi (1) = \pm (1)^{l+1}$.

\begin{equation}
\int_0^\infty \frac{Ap^{m+1}}{\psi(s)} \sum_{b \text{ (mod } Ap^m)} \psi^{-1} (-b) \tilde{\mu}_l^l (b / (Ap^m))
\end{equation}

where the superscripts are taken as in the formula $\psi (1) = \pm (1)^{l+1}$.

**Proof.** The proof goes in a similar way to that of Theorem 3 in Manin [16]. We prove (4.2.2). (4.2.1) is proved in the same way. By the proof of Corollary 3.18,

the left side of (4.2.2) is $\frac{X(2p^m)}{N^{w+h}} \sum_{b \text{ (mod } Ap^m)} \psi^{-1} (-b) \tilde{\mu}_l^l (b / (Ap^m))$.

On the other hand,
Since \( \hat{F}_x^p(x+1) = \hat{F}_x^p(x) \) and \( \psi \) is primitive,
\[
\int_{\mathbb{Z}_x^*} \psi\,(-a)\,d\hat{\mu}_x^p(a) = \sum_{a \bmod \mathcal{O}_x^*} \psi\,(-a)\,p^{-m} \chi(\mathcal{D}p^m) \hat{F}_x^p(a/(\mathcal{D}p^m))
= \tilde{e}^{-mp\chi(\mathcal{D}p^m)} \sum_{a \bmod \mathcal{O}_x^*} \psi\,(-a)\,\hat{F}_x^p(a/(\mathcal{D}p^m)).
\]

In case of \( p \mid N \), we have \( \bar{a}_x \chi(\mathcal{D}p^m) = a(p) \) (see e.g. Proposition 3.56 in Shimura [23]). Hence we may assume \( \eta = \tilde{\eta} \). The following theorem is a generalization of Theorem 4 and Lemma 8 in Manin [16].

**Theorem 4.3.** Assume \( \rho = \bar{\eta} \) and \( \text{ord}_{\mathcal{O}_x} \rho < 1 \).

(i) \( \hat{d}_{\mu}^p(a) = (-\mathcal{D})^{-1} a d\hat{\mu}_x^p(a) \quad (0 \leq l \leq w) \),

(ii) \( \hat{d}_{\mu}^p(a) = (-\mathcal{D})^{-1} a d\hat{\mu}_x^p(a) \quad (0 \leq l \leq w) \),

(iii) Let \( \mathcal{D} \) be prime to \( N \). Then we have:

\[
\hat{d}_{\mu}^p(-1/(N \mathcal{D})) = (-1)^{w+1} a d\hat{\mu}_x^p(a) \quad \text{on} \quad \mathbb{Z}_x^*.
\]

Moreover let \( \varphi \) be a Dirichlet character: \( \mathbb{Z}_x^* \to \mathbb{C}^* \). Then we have:

\[
\int_{\mathbb{Z}_x^*} \psi\,(-a)\,d\hat{\mu}_x^p(a) = (-1)^{w+1} \varphi(N)N^{-1} \int_{\mathbb{Z}_x^*} \varphi(a)\,a^{-w} d\hat{\mu}_x^p(a).
\]

For the proof of Theorem 4.3, we need:

**Lemma 4.4.** For \( a \in \mathcal{O}_x \), \( l \in \mathbb{Z} \) with \( 0 \leq l \leq w \) and \( \sigma \in \text{Aut}(\mathcal{O}_x) \), set

\[
\hat{H}_{l,x}^p(a) = N^{w+1}(c_x^{\sigma})^{-1} \left( \int_{\mathbb{Z}_x^*} F withdrawn \sum_{\mathbb{Z}_x^*} \psi\,(\mathcal{D}p^m) \right) \int_{\mathbb{Z}_x^*} F withdrawn \int_{\mathbb{Z}_x^*} F withdrawn. \\

Then \( \hat{H}_{l,x}^p(a) \) (resp. \( \hat{H}_{l,x}^p(a) \) in Lemma 3.17) is contained in the finitely generated \( \mathbb{Z}_x^* \)-module

\[
\mathcal{M}_x^p = \sum_{l=0}^w \sum_{a \bmod \mathcal{O}_x^*} \left( (c_x^{\sigma})^{-1} \right) \int_{\mathbb{Z}_x^*} F withdrawn \int_{\mathbb{Z}_x^*} F withdrawn. \\

Proof. By replacing the variable \( s \) of the integrations by \(-N^{-w-1} \), we obtain

\[
\hat{H}_{l,x}^p(a) = (-1)^{w+1} N^{w+1}(c_x^{\sigma})^{-1} \left( \int_{\mathbb{Z}_x^*} F withdrawn \right) \int_{\mathbb{Z}_x^*} F withdrawn \int_{\mathbb{Z}_x^*} F withdrawn.
\]

Let \(-1/(N \mathcal{D}) = b_0/(\mathcal{D}, \mathcal{D})_0, b_0/(\mathcal{D}, \mathcal{D})_0, \ldots, b_0/(\mathcal{D}, \mathcal{D})_0, b_0/(\mathcal{D}, \mathcal{D})_0 = 0/1 \) be the successive convergents obtained by the continued fraction of a rational number \(-1/(N \mathcal{D}) \). We may assume \(-k/(N \mathcal{D}) > 0 \). Set

\[
\mathcal{G}_x = \begin{pmatrix} \eta_x & (-1)^{k-1} \eta_x \\ \mathcal{D} & (-1)^{k-1} \mathcal{D} \end{pmatrix} \quad (1 \leq k \leq w).
\]

It is well known that all the \( \mathcal{G}_x \) are elements of \( \text{SL}(2, \mathbb{Z}) \). Then we have:

\[
\begin{align*}
\hat{F}_x^p(a) &= (-1)^{w-1} a \hat{\mu}_x^p(a) \quad \text{on} \quad \mathbb{Z}_x^*.
\end{align*}
\]

Moreover let \( \varphi \) be a Dirichlet character: \( \mathbb{Z}_x^* \to \mathbb{C}^* \). Then we have:

\[
\int_{\mathbb{Z}_x^*} \psi\,(-a)\,d\hat{\mu}_x^p(a) = (-1)^{w+1} \varphi(N)N^{-1} \int_{\mathbb{Z}_x^*} \varphi(a)\,a^{-w} d\hat{\mu}_x^p(a).
\]

For the proof of Theorem 4.3, we need:

**Lemma 4.4.** For \( a \in \mathcal{O}_x \), \( l \in \mathbb{Z} \) with \( 0 \leq l \leq w \) and \( \sigma \in \text{Aut}(\mathcal{O}_x) \), set

\[
\hat{H}_{l,x}^p(a) = N^{w+1}(c_x^{\sigma})^{-1} \left( \int_{\mathbb{Z}_x^*} F withdrawn \right) \int_{\mathbb{Z}_x^*} F withdrawn \int_{\mathbb{Z}_x^*} F withdrawn. \\

Then \( \hat{H}_{l,x}^p(a) \) (resp. \( \hat{H}_{l,x}^p(a) \) in Lemma 3.17) is contained in the finitely generated \( \mathbb{Z}_x^* \)-module

\[
\mathcal{M}_x^p = \sum_{l=0}^w \sum_{a \bmod \mathcal{O}_x^*} \left( (c_x^{\sigma})^{-1} \right) \int_{\mathbb{Z}_x^*} F withdrawn \int_{\mathbb{Z}_x^*} F withdrawn. \\

Proof. By replacing the variable \( s \) of the integrations by \(-N^{-w-1} \), we obtain

\[
\hat{H}_{l,x}^p(a) = (-1)^{w+1} N^{w+1}(c_x^{\sigma})^{-1} \left( \int_{\mathbb{Z}_x^*} F withdrawn \right) \int_{\mathbb{Z}_x^*} F withdrawn \int_{\mathbb{Z}_x^*} F withdrawn.
\]

We compute as follows:

\[
\hat{F}_x^p(a + (\mathcal{D}p^m)) = e^{-mp\chi(\mathcal{D}p^m)} F_x^p(a/(\mathcal{D}p^m)) - p^{w-\mathcal{D}p^m}(\mathcal{D}p^m)^{-1} \chi(\mathcal{D}p^m+1) F_x^p(a/(\mathcal{D}p^m+1))
= \int_{\mathbb{Z}_x^*} \hat{F}_x^p(a/(\mathcal{D}p^m+1)) d\hat{\mu}_x^p(a/(\mathcal{D}p^m+1)) - \int_{\mathbb{Z}_x^*} \hat{F}_x^p(a/(\mathcal{D}p^m+1)) d\hat{\mu}_x^p(a/(\mathcal{D}p^m+1))
= \int_{\mathbb{Z}_x^*} \hat{F}_x^p(a/(\mathcal{D}p^m+1)) d\hat{\mu}_x^p(a/(\mathcal{D}p^m+1)) - \int_{\mathbb{Z}_x^*} \hat{F}_x^p(a/(\mathcal{D}p^m+1)) d\hat{\mu}_x^p(a/(\mathcal{D}p^m+1)).
\]
Here replace the variable \( z \) of the integrations by \( -N^{-1}z^{-1} \). We have:

\[
(4.3.4) \quad \text{the right side of (4.3.3)}
\]

\[
= (-1)^{w+1}N^{w/2}a^w\chi(Ad^m)\left( \int_{\mathbb{P}^1[N]} \mathcal{F}(1, w)(x)dx \right) \pm (-1)^{w+1}N^{w/2}a^w\chi(Ad^m)\left( \int_{\mathbb{P}^1[N]} \mathcal{F}(1, w)(x)dx \right)
\]

From (4.3.4), we obtain:

\[
(4.3.5) \quad \mu_\mathbb{P}(a + (Ad^m)) = \mu_\mathbb{P}(a' + (Ad^m)) - \mu_\mathbb{P}(a' + (Ad^m))(p)\mathcal{P}(a' + (Ad^m))
\]

\[
= (-1)^{w+1}a^w\chi(Ad^m)\left( \int_{\mathbb{P}^1[N]} \mathcal{F}(1, w)(x)dx \right) \pm (-1)^{w+1}a^w\chi(Ad^m)\left( \int_{\mathbb{P}^1[N]} \mathcal{F}(1, w)(x)dx \right)
\]

(4.3.5) implies (4.3.1). By changing the variable of the integration of the left side of (4.3.2) by \(-1/(Na)\), we obtain from (4.3.1):

\[
\int \psi^{-1}(-Na)^{k}d\mu_\mathbb{P}(a) = \int \psi^{-1}(-Na)^{k}(-N^{-1}a)^{k}d\mu_\mathbb{P}(a)
\]

\[
= (-1)^{w+k+1}N^{-k} \int \psi(a)^{w-k}d\mu_\mathbb{P}(a). \quad \blacksquare
\]

Theorem 4.3 is regarded as a \( p \)-adic analogue for the usual complex functional equation satisfied by the zeta functions associated with the functions \( F \) and \( F_{w+1}(x) \) (cf. Hecke [9], Shimura [22]).

Note added in proof. We explain precisely the method printed in the parentheses at lines 6 and 7, page 24 of this paper: (i) Replace \( \mathfrak{B}^{(2)} \) by \( \mathfrak{B}_{\mathbb{Z}} \) for each \( x \in \mathbb{Z} \) with \( 1 < x < r-1 \). Then \( \mathfrak{B}_{\mathbb{Z}} \) (resp. \( \mathfrak{B}_m \) ) is changed into \( q \mathfrak{B}_{\mathbb{Z}} \) (resp. \( \mathfrak{B}_m \) ) for each \( x \in \mathbb{Z} \) with \( 1 < x < r-1 \). (ii) Carry out (i) inductively \((r-1)\) times. (Hence, for each \( x \in \mathbb{Z} \) with \( 1 < x < r-1 \) the original \( \mathfrak{B}^{(2)} \) (resp. \( \mathfrak{B}_{\mathbb{Q}} \) ) is replaced by \( \mathfrak{B}^{(2)} \) (resp. \( \mathfrak{B}_{\mathbb{Q}} \) ) (resp. \( F_{w+1}(x) \) (resp. \( \mu_\mathbb{P}(a) \)).

References

An additive problem in the theory of numbers

by

AKIO FUJII (Tokyo)

1. Introduction. Vinogradov (cf. [5]) proved that every sufficiently large odd integer $N$ can be written as

$$N = p^{(1)} + p^{(2)} + p^{(3)}$$

where $p^{(i)}$'s are odd primes. Here we shall prove

**Theorem.** Let $k$ be an integer $\geq 2$. Let $b_1, b_2, \ldots, b_k$ be positive numbers satisfying $\delta_1 + \delta_2 + \cdots + \delta_k = 1$. Then every sufficiently large odd integer $N$ can be written as

$$N = n^{(1)} + n^{(2)} + n^{(3)},$$

where $n^{(i)} = p_1^{(i)} p_2^{(i)} \cdots p_j^{(i)}$ with some odd primes $p_j^{(i)}$'s satisfying $p_j^{(i)} \leq N^{\delta_i}$ for $j = 1, 2, \ldots, k$ and for $i = 1, 2, 3$.

In fact, we shall prove using Hardy–Littlewood's circle method

$$\sum_{N = n^{(1)} + n^{(2)} + n^{(3)}} \prod_{i=1}^{k} \log p_i^{(i)}$$


\[
\frac{1}{((k-1)!)^2} \mathcal{S}(N) \mathcal{R}(N) + O(N^2 (\log N)^{-A}),
\]

where

$$\mathcal{S}(N) = \prod_{p \mid N} \left(1 - \frac{1}{p-1}\right) \prod_{p \mid N} \left(1 - \frac{1}{p-1}\right),$$

\[
\mathcal{R}(N) = \sum_{N = n^{(1)} + n^{(2)} + n^{(3)}} \frac{\log N}{h_1} \frac{\log N}{h_2} \frac{\log N}{h_3},
\]

$p$ runs over primes, $h_j$'s are positive integers and $A$ is a sufficiently large constant. We remark that there are smaller $N$'s which cannot be written as in our theorem.