COROLLARY 3. *For $k = \boldsymbol{Q}(\zeta)$, ζ a primitive l-th root of 1, the order of χ in $\hat{\Delta}$ divides λ_χ for χ such that $\chi(J) = -1$ or $\chi = \chi_0$.*

We would like to point out that Theorem 4 has been proved independently by R. Gillard [5].

Added in proof: J.-F. Jaulent has recently obtained results similar to some of those in this article.

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Received on 19.1.1976
and in revised form on 20.4.1979 (806)

ACTA ARITHMETICA
XL (1981)

On the rationality of periods of primitive forms

by

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Introduction. In this paper, we give a new proof of the algebraic property of the periods of primitive forms F of Neben type. We also study p-adic Hecke series attached to the F, which take algebraic values.

Let Γ be a finite index subgroup of $\mathrm{SL}(2, \boldsymbol{Z})$, $w+2 \geqslant 2$ be a rational integer, $S_{w+2}(\Gamma)$ be the space of cusp forms of weight $w+2$ with respect to Γ, ϱ_w be the representation $\mathrm{GL}(2, \boldsymbol{R}) \to \mathrm{GL}(w+1, \boldsymbol{R})$ given by

$$\varrho_w\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) dz_w = {}^t\big((cz+d)^w,\ (cz+d)^{w-1}(az+b),\ (cz+d)^{w-2}(az+b)^2,\ \ldots,\ (az+b)^w\big)dz$$

($dz_w = {}^t(dz, zdz, z^2dz, \ldots, z^wdz)$: the $\boldsymbol{C}^{w+1}$ valued differential form on the upper half plane H), $\varrho_w|_\Gamma$ be the restriction of ϱ_w to Γ, $\operatorname*{Ind}_{\Gamma\uparrow \mathrm{SL}(2,\boldsymbol{Z})} \varrho_w|_\Gamma$ be the induced representation of $\varrho_w|_\Gamma$, P be the set consisting of all the parabolic elements in $\mathrm{SL}(2, \boldsymbol{Z})$ and $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, R)$ (resp. $H^1_P\big(\mathrm{SL}(2, \boldsymbol{Z}), \operatorname*{Ind}_{\Gamma\uparrow \mathrm{SL}(2,\boldsymbol{Z})} \varrho_w|_\Gamma, R\big)$ be the parabolic cohomology group with the coefficients in a commutative ring R. Now let $j_2\big(H^1_{P\cap\Gamma}(\varrho_w|_\Gamma, \boldsymbol{Z})\big)$ (resp. $j_1\big(H^1_P(\operatorname{Ind} \varrho_w|_\Gamma, \boldsymbol{Z})\big)$) denote the image of the whole domain: Image (j_2) (resp. Image (j_1)) under the canonical homomorphism

$$j_2\colon H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{Z}) \to H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{R})$$

(resp. $j_1\colon H^1_P\big(\mathrm{SL}(2, \boldsymbol{Z}), \operatorname*{Ind}_{\Gamma\uparrow \mathrm{SL}(2,\boldsymbol{Z})} \varrho_w|_\Gamma, \boldsymbol{Z}\big) \to H^1_P\big(\mathrm{SL}(2, \boldsymbol{Z}), \operatorname*{Ind}_{\Gamma\uparrow \mathrm{SL}(2,\boldsymbol{Z})} \varrho_w|_\Gamma, \boldsymbol{R}\big)$) which is induced by the natural inclusion $\boldsymbol{Z} \hookrightarrow \boldsymbol{R}$. In § 2, we prove:

THEOREM 0.1 (1). (For details, see Theorems 2.2–2.4 in § 2.) *Let sh be the map of Shapiro:*

$$H^1\big(\mathrm{SL}(2, \boldsymbol{Z}), \operatorname*{Ind}_{\Gamma\uparrow \mathrm{SL}(2,\boldsymbol{Z})} \varrho_w|_\Gamma, \boldsymbol{Z}\big) \to H^1(\Gamma, \varrho_w|_\Gamma, \boldsymbol{Z}).$$

(1) This theorem has some applications to congruence properties of eigenvalues of Hecke operators. (Cf. K. Hatada: *On the divisibility by 2 of the eigenvalues of Hecke operators*, Proc. Japan Acad. 53A, (1977), pp. 37–40, and K. Hatada: *Congruences of the eigenvalues of Hecke operators*, Proc. Japan Acad. 53A, (1977), pp. 125–128. Also cf. K. Hatada: *Eigenvalues of Hecke operators on* $\mathrm{SL}(2, \boldsymbol{Z})$, to appear in Math. Ann.)

There exists a canonical $\boldsymbol{Z}$*-linear map* $\mathrm{sh}_P\boldsymbol{Z}\otimes\mathrm{id}$:

$$j_1\big(H^1_P(\mathrm{Ind}\,\varrho_w|_\Gamma,\boldsymbol{Z})\big)\to j_2\big(H^1_{P\cap\Gamma}(\varrho_w|_\Gamma,\boldsymbol{Z})\big)$$

induced by the map sh. *This map* $\mathrm{sh}_P\boldsymbol{Z}\otimes\mathrm{id}$ *is a surjective isomorphism.*

We use this theorem for the proof of our main results.

For a positive integer N, let $\Gamma_1(N)$ (resp. $\Gamma_0(N)$) be the Hecke's congruence subgroup of $\mathrm{SL}(2,\boldsymbol{Z})$ defined by:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1(N)\quad(\text{resp. }\Gamma_0(N))\Leftrightarrow\begin{cases}a-1\equiv d-1\equiv c\equiv 0\ (\mathrm{mod}\ N)\\ (\text{resp. } c\equiv 0\ (\mathrm{mod}\ N)).\end{cases}$$

Every element f of $S_{w+2}(\Gamma_1(N))$ has a Fourier expansion $f(z)=\sum_{n=1}^{\infty}u_nq^n$ ($q=\exp(2\pi\sqrt{-1}z)$) with complex numbers u_n. We denote by Q_f the field generated over the rational number field $\boldsymbol{Q}$ by all the coefficients u_n. For every $\mathbf{g}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\boldsymbol{Z})$, set $\mathbf{tgt}=\begin{pmatrix}a&-b\\-c&d\end{pmatrix}$ and

$$f|_{w+2}[\mathbf{g}](z)=f\big((az+b)/(cz+d)\big)(cz+d)^{-w-2}.$$

Let

$$F(z)=\sum_{n=1}^{\infty}a_nq^n\qquad(a_1=1,\ q=\exp(2\pi\sqrt{-1}z))$$

be a primitive form in $S_{w+2}(\Gamma_1(N))$ (in the sense of Atkin–Lehner [1], Li [11], Miyake [18] and others). It is well known that for every automorphism σ of the complex number field $\boldsymbol{C}$, we can define a primitive form $F^\sigma(z)$ in $S_{w+2}(\Gamma_1(N))$ by

$$F^\sigma(z)=\sum_{n=1}^{\infty}a_n^\sigma q^n.$$

In § 3, we prove the following main results.

THEOREM 0.2. *Let* F *be a primitive form in* $S_{w+2}(\Gamma_1(N))$.

(i) *There exist complex constants* c_σ^+ *and* c_σ^- *depending on the* F^σ *such that*

$$(c_\sigma^\pm)^{-1}\Big(\int_0^{i\infty}F^\sigma|_{w+2}[\mathbf{g}](z)z^ldz\pm(-1)^{l+1}\int_0^{i\infty}F^\sigma|_{w+2}[\mathbf{tgt}](z)z^ldz\Big)\in Q_F^\sigma$$

for all $l\in\boldsymbol{Z}$ *with* $0\leqslant l\leqslant w$ *and all* $\mathbf{g}\in\mathrm{SL}(2,\boldsymbol{Z})$.

(ii) *For every automorphism* σ *of the* $\boldsymbol{C}$, *we can choose the above* $c_\sigma^\pm$ *as follows.*

(a) $c_\sigma^\pm=c_{\sigma'}^\pm$ *if* $\sigma|_{Q_F}=\sigma'|_{Q_F}$ (σ *and* $\sigma'\in\mathrm{Aut}(\boldsymbol{C})$).

(b) $(c_\sigma^\pm)^J=\pm c_{\sigma J}^\pm$ *for all* $\sigma\in\mathrm{Aut}(\boldsymbol{C})$ *where* J *denotes the complex conjugation.*

(c) $$\Big((c_1^\pm)^{-1}\Big(\int_0^{i\infty}F|_{w+2}[\mathbf{g}](z)z^ldz\pm(-1)^{l+1}\int_0^{i\infty}F|_{w+2}[\mathbf{tgt}](z)z^ldz\Big)\Big)^\sigma$$
$$=(c_\sigma^\pm)^{-1}\Big(\int_0^{i\infty}F^\sigma|_{w+2}[\mathbf{g}](z)z^ldz\pm(-1)^{l+1}\int_0^{i\infty}F^\sigma|_{w+2}[\mathbf{tgt}](z)z^ldz\Big)$$

for all $\mathbf{g}\in\mathrm{SL}(2,\boldsymbol{Z})$ *and all* $l\in\boldsymbol{Z}$ *with* $0\leqslant l\leqslant w$.

This Theorem 0.2 is equivalent to the Theorem 1 (i), (ii) and (iii) in Shimura [24]. We shall give a new proof of Theorem 0.2 along the older lines of Shimura [21] and Manin [15] using the Eichler–Shimura isomorphism and Theorem 0.1. In § 4 we investigate p-adic measures associated with primitive forms, which take algebraic values, using the following functions $\mathscr{P}_l^\pm(x)$ ($x\in\boldsymbol{Q}\cup\{i\infty\}$ and $l\in\boldsymbol{Z}$ with $0\leqslant l\leqslant w$):

$$\mathscr{P}_l^\pm(x)=(c_1^\pm)^{-1}\Big(\int_0^{i\infty}F(z+x)z^ldz\pm(-1)^{l+1}\int_0^{i\infty}F(z-x)z^ldz\Big).$$

For details see Theorems 4.3 and 4.4 in § 4. These are generalizations of the original constructions of p-adic measures in Manin [15], [16] for $\Gamma=\mathrm{SL}(2,\boldsymbol{Z})$ case and Mazur–Swinnerton-Dyer [17] for weight 2 case.

Roughly speaking our method of the proof of Theorem 0.2 (i) (for $\sigma=1$) is as follows. First we construct a surjective isomorphism Φ from $S_{w+2}(\Gamma_1(N))$ to $H^1_P(\mathrm{SL}(2,\boldsymbol{Z}),\mathrm{Ind}\,\varrho_w|_\Gamma,\boldsymbol{R})$ by some integral of cusp forms (Lemma 2.1). Let $S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))$ be the subspace of $S_{w+2}(\Gamma_1(N))$ consisting of those forms whose Fourier coefficients at $z=i\infty$ are all real numbers and $\langle U_1^+,U_2^+,\ldots,U_d^+\rangle$ (resp. $\langle U_1^-,U_2^-,\ldots,U_d^-\rangle$) be a $\boldsymbol{Z}$-basis of $S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))\cap\Phi^{-1}\big(j_1\big(H^1_P(\mathrm{Ind}\,\varrho_w|_\Gamma,\boldsymbol{Z})\big)\big)$ (resp. $\sqrt{-1}S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))\cap\cap\Phi^{-1}\big(j_1\big(H^1_P(\mathrm{Ind}\,\varrho_w|_\Gamma,\boldsymbol{Z})\big)\big)$). Using the fact that $H^1_{P\cap\Gamma}(\Gamma,\varrho_w|_\Gamma,\boldsymbol{Z})$ is stable under all the Hecke operators on Γ, Theorem 0.1 and "Multiplicity one theorem", we show that there exists a complex number c_1^+ (resp. c_1^-) and a vector $(\alpha_1,\alpha_2,\ldots,\alpha_d)$ (resp. $(\beta_1,\beta_2,\ldots,\beta_d)$) with all the coefficients in the Q_F such that

$$\text{(0.2.1)}\qquad F=c_1^+(\alpha_1U_1^++\alpha_2U_2^++\ldots+\alpha_dU_d^+)$$
$$(\text{resp. } F=c_1^-(\beta_1U_1^-+\beta_2U_2^-+\ldots+\beta_dU_d^-)).$$

Then we analyse the action of Hecke operators $T_{w+2}(p)$ on a certain cocycle $\in Z^1_P(\mathrm{SL}(2,\boldsymbol{Z}),\mathrm{Ind}\,\varrho_w|_\Gamma,\boldsymbol{R})$, expressed as some integral of $F(z)$ from $i\infty$ to $\sigma(i\infty)$ ($\sigma\in\mathrm{SL}(2,\boldsymbol{Z})$), whose cohomology class is equal to the $\Phi(F)$, by changing the variable of the integral. In this way we deduce, from (0.2.1), Theorem 0.2 (i) using the fact that there exists a prime p with $a_p\neq 1+p^{w+1}$ and $p\equiv 1\ (\mathrm{mod}\ N)$.

The origin of Theorem 0.2 seems due to Shimura [21] where the case of the discriminant function $\Delta(z)$ of weight 12 for $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ was computed. Manin [15] obtained the theorem for any eigenform of any weight in case of $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ using the continued fractions of rational numbers. Damerell [3] investigated the values of L functions of imaginary quadratic fields by a different idea. Birch, Manin, Mazur and Swinnerton-Dyer ([13], [14], [17]) investigated periods of primitive forms of weight 2 on $\Gamma_0(N)$ in relation to "Modular Symbols" and "Weil Parametrization". Recently Shimura, using totally different methods (not using the Eichler–Shimura isomorphism), obtained almost all the results on the rationality in [23], [24]. (Our point in this paper is to do things along the older lines of Manin [15] and Shimura [21].) Razar investigates also the above Theorem 0.2 and obtains partial results in [19] and Theorem 4 [20]. Roughly speaking the distinctions between Razar's and ours are the following.

(i) Razar uses the Eichler–Shimura isomorphism φ itself instead of the Φ.

Let Γ be $\Gamma_0(N)$ (or $\Gamma_1(N)$).

(ii) He proves that a certain $\mathbf{Q}$-subspace $\tilde{Z}$ of the parabolic cocycles $Z^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{Q})$ into whose complexification the space $S_{w+2}(\Gamma)$ is mapped, is stable under all the Hecke operators on Γ. And he applies "Multiplicity one theorem" to the space $\tilde{Z}\otimes_{\mathbf{Q}}\mathbf{R}$. On the other hand we utilize the result that the space

$$j_1\big(H^1_P(\mathrm{SL}(2, \mathbf{Z}), \operatorname*{Ind}_{\Gamma\uparrow \mathrm{SL}(2,\mathbf{Z})} \varrho_w|_\Gamma, \mathbf{Z})\big)$$

is stable under all the Hecke operators on Γ through the isomorphism Φ and apply "Multiplicity one theorem" to its complexification.

(iii) He expresses the coefficients of the Eichler cocycle ($\in Z^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{R})$) corresponding to a primitive form F as a pole of some multiple differential. This relates to the theory developed in Weil [26]. We make no use of this technique (iii) in this paper.

(iv) Our technique in deducing Theorem 0.2 (ii) (c) from (0.2.1) is our own.

The author would like to express his gratitude to the referees for their valuable advices on the improvements of this paper and especially for their pointing out the insufficiency of the proof, of Theorem 0.1, given in the earlier version.

The main results of this paper were announced in a Proceedings of Japan Academy Note [8].

1. Notations and preliminary results

χ: a Dirichlet character $(\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$.

$S_{w+2}(N, \chi)$: the subspace of $S_{w+2}(\Gamma_1(N))$ consisting of all the cusp forms f with $f((az+b)/(cz+d))(cz+d)^{-w-2} = \chi(d)f(z)$ for every $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(N)$. It is well known that $S_{w+2}(\Gamma_1(N)) = \oplus_\chi S_{w+2}(N, \chi)$, where χ runs over all the Dirichlet characters (mod N).

$\Gamma(N)$: the principal congruence subgroup of level N.

Γ: a finite index subgroup of $\mathrm{SL}(2, \mathbf{Z})$.

$\mathbf{1}_r$: the $r\times r$ identity matrix (r: a positive integer).

$\mathrm{SL}(2, \mathbf{Z}) = \Gamma(1) = \bigcup_{j=1}^{m} \Gamma g_j$: the left coset decomposition with $g_1 = \mathbf{1}_2$ and m = the cardinality of $(\Gamma\backslash\Gamma(1))$.

$\mathbf{R}^{(w+1)m}$: the real vector space consisting of the $(w+1)m$ dimensional real column vectors with basis indexed by the pairs $\{(\Gamma g_j, u)\}$ which are the elements of the product set $(\Gamma\backslash\Gamma(1))\times([0, w]\cap\mathbf{Z})$.

$\mathbf{Z}^{(w+1)m}$: the lattice of the $\mathbf{R}^{(w+1)m}$ with respect to the standard basis.

$$\mathrm{t} = \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & -1\\ 1 & 0\end{pmatrix}.$$

dz_w: the $\mathbf{C}^{w+1}$ valued differential form ${}^t(dz, zdz, z^2dz, \ldots, z^wdz)$ on the upper half plane $\mathbf{H}$.

For each cusp form $f\in S_{w+2}(\Gamma)$, put

$$\mathbf{D}(f)(z) = \begin{pmatrix} (f(z)\,dz_w)\circ g_1\\ (f(z)\,dz_w)\circ g_2\\ (f(z)\,dz_w)\circ g_3\\ \vdots\\ (f(z)\,dz_w)\circ g_m\end{pmatrix} \quad\text{and}\quad \mathbf{D}(f)^E(z) = \begin{pmatrix} \varrho_w(\mathrm{t})(f(z)\,dz_w)\circ(\mathrm{t}g_1\mathrm{t})\\ \varrho_w(\mathrm{t})(f(z)\,dz_w)\circ(\mathrm{t}g_2\mathrm{t})\\ \varrho_w(\mathrm{t})(f(z)\,dz_w)\circ(\mathrm{t}g_3\mathrm{t})\\ \vdots\\ \varrho_w(\mathrm{t})(f(z)\,dz_w)\circ(\mathrm{t}g_m\mathrm{t})\end{pmatrix}$$

which are $\mathbf{C}^{(w+1)m}$ valued differential forms on the $\mathbf{H}$. Here $(f(z)\,dz_w)\circ g$ denotes the pull back of $(f(z)\,dz_w)$ by $g\in\mathrm{SL}(2, \mathbf{Z})$.

η_w: the representation $\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{GL}((w+1)m, \mathbf{Z})$ given by

$$\eta_w(\mathbf{g})\mathbf{D}(f) = \mathbf{D}(f)\circ\mathbf{g} \quad (f\in S_{w+2}(\Gamma),\ \mathbf{g}\in\mathrm{SL}(2, \mathbf{Z})),$$

which is isomorphic to $\operatorname*{Ind}_{\Gamma\uparrow\mathrm{SL}(2,\mathbf{Z})}(\varrho_w|_\Gamma)$.

R: a commutative ring with a unit element 1.

Now we give the definitions of the Eichler cohomology group $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, R)$ and the parabolic cohomology group $H^1_P(\Gamma(1), \eta_w, R)$ (cf. Eichler [6], Shimura [21], [22]). By an R-valued parabolic cocycle of η_w (resp. $\varrho_w|_\Gamma$), we mean a map

$$\mathfrak{x}\colon \Gamma(1)\to R^{(w+1)m} \quad (\text{resp. } \mathfrak{x}\colon \Gamma\to R^{w+1})$$

satisfying the two conditions:

$$\mathfrak{x}(\mathbf{g}\mathbf{g}') = \mathfrak{x}(\mathbf{g}) + \eta_w(\mathbf{g})\mathfrak{x}(\mathbf{g}') \quad (\text{resp. } \mathfrak{x}(\mathbf{h}\mathbf{h}') = \mathfrak{x}(\mathbf{h}) + \varrho_w(\mathbf{h})\mathfrak{x}(\mathbf{h}'))$$

($\mathbf{g}$ and $\mathbf{g}' \in \Gamma(1)$, $\mathbf{h}$ and $\mathbf{h}' \in \Gamma$) and

$$\mathfrak{x}(\gamma) \in \big(\mathbf{1}_{(w+1)m} - \eta_w(\gamma)\big) R^{(w+1)m} \quad \text{for every } \gamma \in P$$

$$(\text{resp. } \mathfrak{x}(\gamma) \in \big(\mathbf{1}_{w+1} - \varrho_w(\gamma)\big) R^{w+1} \quad \text{for every } \gamma \in P \cap \Gamma).$$

A coboundary is a cocycle $\mathfrak{x}$ of the form

$$\mathfrak{x}(\mathbf{g}) = \big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big)\mathfrak{x}_0 \quad (\mathbf{g} \in \Gamma(1))$$

$$(\text{resp. } \mathfrak{x}(\mathbf{h}) = \big(\mathbf{1}_{w+1} - \varrho_w(\mathbf{h})\big)\mathfrak{x}_0 \quad (\mathbf{h} \in \Gamma))$$

where $\mathfrak{x}_0$ is an arbitrarily fixed element of $R^{(w+1)m}$ (resp. R^{w+1}). The parabolic cohomology group $H^1_P(\Gamma(1), \eta_w, R)$ (resp. $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, R)$) is the quotient of the group $Z^1_P(\Gamma(1), \eta_w, R)$ (resp. $Z^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, R)$) of all the R-valued parabolic cocycles modulo the subgroup $B^1(\Gamma(1), \eta_w, R)$ (resp. $B^1(\Gamma, \varrho_w|_\Gamma, R)$) of coboundaries. The natural injection $\mathbf{Z} \to \mathbf{R}$ induces a canonical homomorphism

$$j_1\colon H^1_P(\Gamma(1), \eta_w, \mathbf{Z}) \to H^1_P(\Gamma(1), \eta_w, \mathbf{R})$$

$$(\text{resp. } j_2\colon H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{Z}) \to H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{R})).$$

$\varphi\colon S_{w+2}(\Gamma) \to H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{R})$: the Eichler–Shimura isomorphism for $S_{w+2}(\Gamma)$ (cf. Eichler [6] and Shimura [21], [22]).

$$H(N, \mathbf{Z})^+ = \varphi^{-1}\Big(j_2\big(H^1_{P\cap\Gamma_1(N)}(\Gamma_1(N), \varrho_w|_{\Gamma_1(N)}, \mathbf{Z})\big)\Big) \cap S^{\mathbf{R}}_{w+2}(\Gamma_1(N)).$$

$$H(N, \mathbf{Z})^- = \varphi^{-1}\Big(j_2\big(H^1_{P\cap\Gamma_1(N)}(\Gamma_1(N), \varrho_w|_{\Gamma_1(N)}, \mathbf{Z})\big)\Big) \cap \sqrt{-1}\, S^{\mathbf{R}}_{w+2}(\Gamma_1(N)).$$

$$\varepsilon\colon S_{w+2}(\Gamma_1(N)) \to S_{w+2}(\Gamma_1(N)), \quad f \mapsto f^\varepsilon(z) = \overline{f(-\bar{z})}.$$

$\mathrm{Aut}(\mathbf{C})$: the group of the automorphisms of the complex number field.

For every $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{GL}(2, \mathbf{R})$ and every $f \in S_{w+2}(\Gamma_1(N))$, put

$$f|_{w+2}\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right] = (ad-bc)^{(w+2)/2} f\big((az+b)/(cz+d)\big)(cz+d)^{-w-2}.$$

For every $q \in \mathbf{Z}$ with $(q, N) = 1$, we choose an element $\sigma_q \in \mathrm{SL}(2, \mathbf{Z})$ such that $\sigma_q \equiv \begin{pmatrix} q^{-1} & 0 \\ 0 & q\end{pmatrix} \pmod N$. We recall the actions of the Hecke operators $T_{w+2}(n)$ with $(n, N) = 1$ on $S_{w+2}(\Gamma_1(N))$ (cf. Hecke [9]):

$$f|T_{w+2}(n)(z) = n^{w/2} \sum_{\substack{a\geq 1\\ ad=n\\ 0\leq b<d}} f|_{w+2}\left[\sigma_a \begin{pmatrix} a & b \\ c & d\end{pmatrix}\right](z) \quad (f \in S_{w+2}(\Gamma_1(N))).$$

In particular

$$f|T_{w+2}(n)(z) = n^{w/2} \sum_{\substack{a\geq 1\\ ad=n\\ 0\leq b<d}} \chi(a) f|_{w+2}\left[\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right](z) \quad (f \in S_{w+2}(N, \chi)).$$

For the proofs of our theorems, we need the following lemmas.

Lemma 1.1 (cf. Theorem 3.52 in Shimura [22]).

(i) *If $w+2 \geq 2$, $S_{w+2}(\Gamma_1(N))$ has a basis consisting of cusp forms of which the Fourier coefficients at $z = i\infty$ are rational integers.*

(ii) *Let $f(z) = \sum_{n=1}^{\infty} u_n q^n$ ($q = \exp(2\pi\sqrt{-1}z)$) be a cusp form in $S_{w+2}(\Gamma_1(N))$ with all $u_n \in \mathbf{Q}$. Then for each positive $n \in \mathbf{Z}$ with $(n, N) = 1$, $f|T_{w+2}(n)$ has also its Fourier coefficients at $z = i\infty$ all in $\mathbf{Q}$. Hence*

$$h^\varepsilon|T_{w+2}(n) = \big(h|T_{w+2}(n)\big)^\varepsilon \quad \text{for all } h \in S_{w+2}(\Gamma_1(N))$$

and

$$T_{w+2}(n)\big(S^{\mathbf{R}}_{w+2}(\Gamma_1(N))\big) \subset S^{\mathbf{R}}_{w+2}(\Gamma_1(N)).$$

Theorem 1.2 (Eichler [6] and Shimura [21], [22]). *Let f be a cusp form in $S_{w+2}(\Gamma)$ and z_0 be a point in the upper half plane $\mathbf{H}$. Then the map given by*

$$c(f, z_0)\colon \Gamma \ni \mathbf{g} \mapsto \mathrm{Re}\int_{z_0}^{\mathbf{g}z_0} f(z)\, dz_w \in \mathbf{R}^{w+1}$$

is a cocycle in $Z^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{R})$. The cohomology class $\varphi(f)$ of the $c(f, z_0)$ in $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{R})$ is uniquely determined by f (and independent of the choice of z_0). The map $f \mapsto \varphi(f)$ is the Eichler–Shimura isomorphism which is an $\mathbf{R}$-linear surjective isomorphism.

Lemma 1.3 ((8.3.1) and Proposition 8.5 in Shimura [22]). *Let φ be the isomorphism given in Lemma 1.2 for $\Gamma = \Gamma_1(N)$. Then the group*

$$\varphi^{-1}\Big(j_2\big(H^1_{P\cap\Gamma_1(N)}(\Gamma_1(N), \varrho_w|_{\Gamma_1(N)}, \mathbf{Z})\big)\Big)$$

is stable under all the Hecke operators $T_{w+2}(n)$ (with $(n, N) = 1$) acting on $S_{w+2}(\Gamma_1(N))$.

Lemma 1.4. *The cohomology group $H(N, \mathbf{Z})^-$ (resp. $H(N, \mathbf{Z})^+$) is stable under all the Hecke operators $T_{w+2}(n)$ (with $(n, N) = 1$) on $S_{w+2}(\Gamma_1(N))$.*

Proof. This is a consequence of Lemmas 1.1 and 1.3. ∎

Lemma 1.5 (Proposition 8.6 in Shimura [22]). *Regard $H^1_P(\Gamma(1), \eta_w, \mathbf{R})$ (resp. $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{R})$) as a vector space over $\mathbf{R}$. Then we have: The group $j_1\big(H^1_P(\Gamma(1), \eta_w, \mathbf{Z})\big)$ (resp. $j_2(H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{Z}))$) is a lattice (i.e. a discrete subgroup of maximal rank) of the $H^1_P(\Gamma(1), \eta_w, \mathbf{R})$ (resp. $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{R})$)*

and $\mathrm{Ker}(j_1)$ (resp. $\mathrm{Ker}(j_2)$) *is finite. Hence the map* j_1 (*resp.* j_2) *induces the following* $\boldsymbol{R}$*-linear isomorphism*:

$$j_1 \otimes \mathrm{id.}\colon H^1_P(\Gamma(1), \eta_w, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \boldsymbol{R} \simeq H^1_P(\Gamma(1), \eta_w, \boldsymbol{R})$$

(resp. $j_2 \otimes \mathrm{id.}\colon H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \boldsymbol{R} \simeq H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{R})$).

LEMMA 1.6. *Let* Γ *be the group* $\Gamma_1(N)$ *and* φ *be the isomorphism for* $S_{w+2}(\Gamma)$ *given in Theorem 1.2. Then the group*

$$\varphi^{-1}\big(j_2\big(H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{Z})\big)\big)$$

is stable under the map ε.

Proof. Let f be a cusp form in $\varphi^{-1}\big(j_2\big(H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{Z})\big)\big)$. Then there exists a real column vector $\mathfrak{A}$ in $\boldsymbol{R}^{w+1}$ such that

$$\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} f(z)\,dz_w + \big(\mathbf{1}_{w+1} - \varrho_w(\mathbf{g})\big)\mathfrak{A} \in \boldsymbol{Z}^{w+1} \quad \text{for all } \mathbf{g} \text{ in } \Gamma_1(N).$$

By changing the variable of the integration,

$$\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} f^{\varepsilon}(z)z^k dz = \mathrm{Re}\int_{u=i\infty}^{-\overline{\mathbf{g}(i\infty)}} \overline{f(u)}\,(-\bar{u})^k d(-\bar{u})$$

$$= \mathrm{Re}\int_{u=i\infty}^{\mathbf{tgt}(i\infty)} (-1)^{k+1}\overline{f(u)\,u^k du} = \mathrm{Re}\int_{i\infty}^{\mathbf{tgt}(i\infty)} (-1)^{k+1} f(z)z^k dz \qquad (\mathbf{g} \in \Gamma).$$

Namely we have:

$$\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} f^{\varepsilon}(z)\,dz_w = \varrho_w(\mathbf{t})\,\mathrm{Re}\int_{i\infty}^{\mathbf{tgt}(i\infty)} f(z)\,dz_w \qquad (\mathbf{g} \in \Gamma).$$

Note that $\varrho_w(\mathbf{t})^2 = \varrho_w(\mathbf{t}^2) = \mathbf{1}_{w+1}$ and $\mathbf{tgt} \in \Gamma_1(N)$ for all $\mathbf{g} \in \Gamma_1(N)$. We obtain:

$$\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} f^{\varepsilon}(z)\,dz_w + \big(\mathbf{1}_{w+1} - \varrho_w(\mathbf{g})\big)\varrho_w(\mathbf{t})\mathfrak{A}$$

$$= \varrho_w(\mathbf{t})\Big(\int_{i\infty}^{\mathbf{tgt}(i\infty)} f(z)\,dz_w + \big(\mathbf{1}_{w+1} - \varrho_w(\mathbf{tgt})\big)\mathfrak{A}\Big) \in \boldsymbol{Z}^{w+1} \quad \text{for all } \mathbf{g} \in \Gamma_1(N).$$

Namely we have $f^{\varepsilon} \in \varphi^{-1}\big(j_2\big(H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{Z})\big)\big)$. ∎

LEMMA 1.7. *The cohomology group* $H(N, \boldsymbol{Z})^+$ (*resp.* $H(N, \boldsymbol{Z})^-$) *is a lattice in* $S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))$ (*resp.* $\sqrt{-1}S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))$).

Proof. By Lemma 1.5, it is sufficient to show that $H(N, \boldsymbol{Z})^+$ (resp. $H(N, \boldsymbol{Z})^-$) spans $S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))$ (resp. $\sqrt{-1}S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))$) over $\boldsymbol{R}$. Let Γ be $\Gamma_1(N)$ and $\langle a_1, a_2, a_3, \ldots, a_{2d}\rangle$ be a $\boldsymbol{Z}$-basis of $\varphi^{-1}\big(j_2\big(H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \boldsymbol{Z})\big)\big)$. From Lemma 1.6, we have:

$$(1.7.1)^{\pm} \qquad a_j + a_j^{\varepsilon} \in H(N, \boldsymbol{Z})^+ \quad \text{and} \quad a_j - a_j^{\varepsilon} \in H(N, \boldsymbol{Z})^-$$

for all $j \in \boldsymbol{Z}$ with $1 \leqslant j \leqslant 2d$. Let f be a cusp form in $S_{w+2}(\Gamma)$. Then f is

written as $f = \sum_{j=1}^{2d} \alpha_j a_j$ with some $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{2d}) \in \boldsymbol{R}^{2d}$. We have

$$(1.7.2)^{+} \qquad \tfrac{1}{2}(f+f^{\varepsilon}) = f = \sum_{j=1}^{2d} \tfrac{1}{2}\alpha_j(a_j + a_j^{\varepsilon}) \quad \text{if} \quad f \in S^{\boldsymbol{R}}_{w+2}(\Gamma),$$

$$(1.7.2)^{-} \qquad \tfrac{1}{2}(f-f^{\varepsilon}) = f = \sum_{j=1}^{2d} \tfrac{1}{2}\alpha_j(a_j - a_j^{\varepsilon}) \quad \text{if} \quad f \in \sqrt{-1}S^{\boldsymbol{R}}_{w+2}(\Gamma).$$

$(1.7.1)^{\pm}$ and $(1.7.2)^{\pm}$ prove Lemma 1.7. ∎

LEMMA 1.8 (Proposition 3.53 in Shimura [22]). *Let* f *be an element of* $S_{w+2}(\Gamma_1(N))$ *which is a common eigenfunction of all the* $T_{w+2}(n)$ *with* $(n, N) = 1$. *Then the form* f *belongs to* $S_{w+2}(N, \psi)$ *with a unique character* ψ *of* $(\boldsymbol{Z}/N\boldsymbol{Z})^*$.

THEOREM 1.9 (Multiplicity one theorem. Atkin–Lehner [1], Casselman [2], Deligne [4], Li [11] and Miyake [18]). *Let*

$$F(z) = \sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz) \qquad (a_1 = 1)$$

be a primitive form in $S_{w+2}(\Gamma_1(N))$ *and* $h(z)$ *be an element of* $S_{w+2}(\Gamma_1(N))$ *satisfying* $h|T_{w+2}(n) = a_n h$ *for all the positive integers* n *with* $(n, N) = 1$. *Then there exists a complex number* c *such that* $h = cF$.

Proof. By the proof of Proposition 3.53 (namely the above Lemma 1.8) in Shimura [22], F and h are contained in the same $S_{w+2}(N, \chi)$ with some Dirichlet character $\chi \bmod N$. Then this lemma is a direct consequence of Theorem 5 in Li [11]. ∎

Now let R be a commutative ring and V_R be the R-module consisting of all the R^{w+1} (the space of column vectors) valued functions f on $\mathrm{SL}(2, \boldsymbol{Z})$ such that $f(\mathbf{xy}) = \varrho_w(\mathbf{x})f(\mathbf{y})$ for all $\mathbf{x} \in \Gamma$ and $\mathbf{y} \in \mathrm{SL}(2, \boldsymbol{Z})$. (Of course we set $(rf)(\mathbf{y}) = r(f(\mathbf{y}))$ for $r \in R$ and $\mathbf{y} \in \mathrm{SL}(2, \boldsymbol{Z})$.) From the definition of η_w, we obtain easily

$$(1.10.0) \qquad \begin{pmatrix} f(g_1\mathbf{y}) \\ f(g_2\mathbf{y}) \\ f(g_3\mathbf{y}) \\ \vdots \\ f(g_m\mathbf{y}) \end{pmatrix} = \eta_w(\mathbf{y}) \begin{pmatrix} f(g_1) \\ f(g_2) \\ f(g_3) \\ \vdots \\ f(g_m) \end{pmatrix} \quad \text{for all } f \in V_R \text{ and } \mathbf{y} \in \mathrm{SL}(2, \boldsymbol{Z}).$$

Hence there exists a surjective $R[\Gamma(1)]$ isomorphism between the $R[\Gamma(1)]$ modules: V_R and the R-module $R^{(w+1)m}$ with the action of $\Gamma(1)$ to the left through the η_w $(\simeq \operatorname{Ind}_{\Gamma\uparrow\Gamma(1)}(\varrho_w|_\Gamma))$. Namely

$$(1.10.1) \qquad V_R \to R^{(w+1)m}, \quad f \mapsto \begin{pmatrix} f(g_1) \\ f(g_2) \\ \vdots \\ f(g_m) \end{pmatrix}.$$

LEMMA 1.10 (Shapiro's lemma, see e.g. Lang [10]). *Let R be a commutative ring. Then the map*

$$\mathrm{sh}^n R\colon H^n(\Gamma(1), \eta_w, R) \to H^n(\Gamma, \varrho_w|_\Gamma, R)$$

induced by the compatible maps $\Gamma \hookrightarrow \mathrm{SL}(2, \boldsymbol{Z})$ and $V_R \ni f \mapsto f(g_1) \in R^{w+1}$ $(g_1 = \mathbf{1}_2)$, is a surjective isomorphism.

We need only the case of $n = 1$ and set $\mathrm{sh} = \mathrm{sh}^1\boldsymbol{Z}$.

LEMMA 1.11. *Let p be a rational prime with $p \equiv 1 \pmod N$. Set*

$$T(p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, g \text{ is an integral matrix, } ad-bc = p,\ b \equiv c \equiv a-1 \equiv 0 \pmod N \right\}.$$

Then we have:

$$GT(p)G^{-1} = \{GgG^{-1} \mid g \in T(p)\} = T(p) \quad \text{for every } G \in \mathrm{SL}(2, \boldsymbol{Z}).$$

Proof. It is easy to see that GgG^{-1} is an integral matrix with the determinant p. Note that $g \equiv \mathbf{1}_2 \pmod N$ and that $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$. Hence $GgG^{-1} \equiv GG^{-1} \equiv \mathbf{1}_2 \pmod N$. ∎

LEMMA 1.12 (Deligne [5]). *Let λ_p be an eigenvalue of $T_{w+2}(p)$ acting on $S_{w+2}(\Gamma_1(N))$. Then we have:*

$$|\lambda_p| \leqslant 2p^{(w+1)/2} \quad \text{for every Archimedean absolute value } |\cdot|.$$

For our purpose, it is sufficient to know only

$$|\lambda_p - 1 - p^{w+1}| \gtrneqq 0 \quad \text{for every sufficiently large prime } p.$$

2. Eichler-Shimura isomorphism and Shapiro's lemma. We study Theorem 0.1 in this section. Our main results are Theorems 2.2–2.4 below. Let Γ be a finite index subgroup of $\Gamma(1)$ and $\Gamma(1) = \bigcup_{j=1}^{m} \Gamma g_j$ with $g_1 = \mathbf{1}_2$ be the left coset decomposition. Let $\mathbf{D}(f)$ $(f \in S_{w+2}(\Gamma))$, η_w and z_0 be such as defined in § 1 for the Γ.

First we prove:

LEMMA 2.1. *Let $\mathfrak{A}$ be a $(w+1)m$ dimensional real column vector. For each cusp form f in $S_{w+2}(\Gamma)$, the map*

$$\Gamma(1) \ni \mathbf{g} \mapsto T(\mathbf{g}) = \Big(\mathrm{Re}\int_{z_0}^{\mathbf{g}z_0} \mathbf{D}(f)(z) + (\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g}))\mathfrak{A}\Big) \in \boldsymbol{R}^{(w+1)m}$$

is a parabolic cocycle in $Z_P^1(\Gamma(1), \eta_w, \boldsymbol{R})$. Its cohomology class $\Phi(f)$ in $H_P^1(\Gamma(1), \eta_w, \boldsymbol{R})$ is determined by f and independent of z_0 and $\mathfrak{A}$.

Proof. The proof goes in a similar way to that of Theorem 8.4 in Shimura [22]. For every $z \in H$, put

$$(2.1.1)\qquad \Psi(z) = \Big(\mathrm{Re}\int_{z_0}^{z} \mathbf{D}(f)(u)\Big) + \mathfrak{A}.$$

Since the differential form $\mathbf{D}(f)(u)$ is holomorphic on the H, $\Phi(z)$ is independent of the choice of the path of the integral. For every elements $\mathbf{g}$ and $\mathbf{g}'$ of $\Gamma(1)$, we have:

$$\begin{aligned}
(2.1.2)\qquad \Psi(\mathbf{g}z) &= \mathrm{Re}\int_{z_0}^{\mathbf{g}z} \mathbf{D}(f)(u) + \mathfrak{A} \\
&= \mathrm{Re}\int_{\mathbf{g}z_0}^{\mathbf{g}z} \mathbf{D}(f)(u) + \mathrm{Re}\int_{z_0}^{\mathbf{g}z_0} \mathbf{D}(f)(u) + \mathfrak{A} \\
&= \mathrm{Re}\int_{u=z_0}^{u=z} \mathbf{D}(f)\circ\mathbf{g}(u) + \mathrm{Re}\int_{z_0}^{\mathbf{g}z_0} \mathbf{D}(f)(u) + \mathfrak{A} \\
&= \eta_w(\mathbf{g})\Psi(z) + T(\mathbf{g}),
\end{aligned}$$

and

$$\begin{aligned}
\Psi(\mathbf{g}'\mathbf{g}z) &= \eta_w(\mathbf{g}'\mathbf{g})\Psi(z) + T(\mathbf{g}'\mathbf{g}) \\
&= \eta_w(\mathbf{g}')\Psi(\mathbf{g}z) + T(\mathbf{g}') = \eta_w(\mathbf{g}')\big(\eta_w(\mathbf{g})\Psi(z) + T(\mathbf{g})\big) + T(\mathbf{g}').
\end{aligned}$$

Hence $T(\mathbf{g}'\mathbf{g}) = T(\mathbf{g}') + \eta_w(\mathbf{g}')T(\mathbf{g})$ for every $\mathbf{g}$ and $\mathbf{g}'$ in $\Gamma(1)$. Namely T is a cocycle in $Z^1(\Gamma(1), \eta_w, \boldsymbol{R})$. From (2.1.1) and (2.1.2), we observe that the change of $\mathfrak{A}$ (and hence the change of z_0) affects T only by an addition of an element of the coboundaries $B^1(\Gamma(1), \eta_w, \boldsymbol{R})$. Let s be a cusp of Γ (viz. $s \in \boldsymbol{Q} \cup \{i\infty\}$). It is well known that for each $\mathbf{g} \in \mathrm{SL}(2, \boldsymbol{Z})$, $\mathbf{g}(s)$ is also a cusp of Γ. Hence $\Psi(s) = \lim_{z\to s} \Psi(z)$ exists when the limit is taken along a geodesic line (cf. p. 233 in Shimura [22]). Let $\pi \in \Gamma(1)$ be a stabilizer of s. Then

$$\Psi(s) = \Psi(\pi(s)) = \lim_{z\to s} \Psi(\pi(z)) = \eta_w(\pi)\Psi(s) + T(\pi).$$

Hence T becomes a parabolic cocycle in $Z_P^1(\Gamma(1), \eta_w, \boldsymbol{R})$. ∎

Hence Φ defines an $\boldsymbol{R}$-linear map: $S_{w+2}(\Gamma) \to H_P^1(\Gamma(1), \eta_w, \boldsymbol{R})$.

We use the following three theorems for the proof of Theorem 0.2.

THEOREM 2.2. *Let R be the field $\boldsymbol{R}$ of the real numbers or the ring $\boldsymbol{Z}$ of the rational integers.*

(i) *The image, under the map $\mathrm{sh}^1 R$ given in Lemma 1.10, of the cohomology group $H_P^1(\Gamma(1), \eta_w, R)$ is contained in the cohomology group $H_{P\cap\Gamma}^1(\Gamma, \varrho_w|_\Gamma, R)$.*

Let $\mathrm{sh}_P R$ be the restriction of the map $\mathrm{sh}^1 R$ given in Lemma 1.10 to the $H_P^1(\Gamma(1), \eta_w, R)$.

(ii) *The composite map* $\mathrm{sh}_P\boldsymbol{R}\circ\Phi$:

$$S_{w+2}(\Gamma)\to H^1_P(\Gamma(1),\eta_w,\boldsymbol{R})\to H^1_{P\cap\Gamma}(\Gamma,\varrho_w|_\Gamma,\boldsymbol{R})$$

is the Eichler–Shimura isomorphism φ *for* $S_{w+2}(\Gamma)$ *(given in Theorem* 1.2*).*

(iii) *The map* $\mathrm{sh}_P\boldsymbol{R}\colon H^1_P(\Gamma(1),\eta_w,\boldsymbol{R})\to H^1_{P\cap\Gamma}(\Gamma,\varrho_w|_\Gamma,\boldsymbol{R})$ *is a surjective* $\boldsymbol{R}$*-linear isomorphism.*

THEOREM 2.3. *The map* Φ *given in Lemma* 2.1 *is an* $\boldsymbol{R}$*-linear isomorphism from* $S_{w+2}(\Gamma)$ *onto the* $H^1_P(\Gamma(1),\eta_w,\boldsymbol{R})$.

THEOREM 2.4 (Theorem 0.1 in the Introduction). *The image under the map* $\mathrm{sh}_P\boldsymbol{R}$ *of the* $j_1(H^1_P(\Gamma(1),\eta_w,\boldsymbol{Z}))$ *coincides with the* $j_2(H^1_{P\cap\Gamma}(\Gamma,\varrho_w|_\Gamma,\boldsymbol{Z}))$.

First we prove Theorem 2.2.

Proof of Theorem 2.2 (i). It is easy to see that

$$\mathrm{sh}^1R\big(H^1_P(\Gamma(1),\eta_w,R)\big)\subset H^1(\Gamma,\varrho_w|_\Gamma,R)$$

by Lemma 1.10. Let π be an element of $P\cap\Gamma$ and $\boldsymbol{r}\colon\Gamma(1)\ni\mathbf{g}\mapsto\boldsymbol{r}(\mathbf{g})\in R^{(w+1)m}$ be a parabolic cocycle in $Z^1_P(\Gamma(1),\eta_w,R)$. We may assume that for each $\mathbf{g}\in\Gamma(1)$, $r(\mathbf{g})$ is an element of V_R by (1.10.1). Since the cocycle $\boldsymbol{r}$ is parabolic, there exists an element h_π in V_R such that $\boldsymbol{r}(\pi)=(\eta_w(\pi)-\mathbf{1}_{(w+1)m})h_\pi$. Then we have:

$$\begin{aligned}(\mathrm{sh}^1R(r))(\pi)&=(r(\pi))(g_1)=(r(\pi))(\mathbf{1}_2)\\&=\big((\eta_w(\pi)-\mathbf{1}_{(w+1)m})h_\pi\big)(\mathbf{1}_2)=(\eta_w(\pi)h_\pi-h_\pi)(\mathbf{1}_2)=h_\pi(\pi)-h_\pi(\mathbf{1}_2)\\&=(\varrho_w(\pi)-\mathbf{1}_{w+1})(h_\pi(\mathbf{1}_2))\in(\varrho_w(\pi)-\mathbf{1}_{w+1})R^{w+1}.\ \blacksquare\end{aligned}$$

Proof of Theorem 2.2 (ii). Let f be an element of $S_{w+2}(\Gamma)$. By the definition of Φ, $\Phi(f)$ is the cohomology class of the cocycle $\{\Gamma(1)\ni\mathbf{g}\mapsto\mathrm{Re}\int_{z_0}^{gz_0}\mathbf{D}(f)\}$. By (1.10.1), $\mathrm{Re}\int_{z_0}^{gz_0}\mathbf{D}(f)$ corresponds to the function $h_{\mathbf{g}}\in V_R$ such that

$$h_{\mathbf{g}}(g_j)=\mathrm{Re}\int_{u=z_0}^{u=gz_0}(f(u)\,du_w)\circ g_j\quad\text{ for every } j\in\boldsymbol{Z}\text{ with } 1\leqslant j\leqslant m.$$

Then by Lemma 1.10, $\mathrm{sh}_P\boldsymbol{R}(\Phi(f))$ is the cohomology class $(=X_f)$ of the cocycle $\{\Gamma\ni\mathbf{g}\mapsto h_{\mathbf{g}}(g_1)\}$. Since $h_{\mathbf{g}}(g_1)=\mathrm{Re}\int_{z_0}^{gz_0}f(z)\,dz_w$, the cohomology class X_f is equal to $\varphi(f)$ by Theorem 1.2. Hence $\varphi=(\mathrm{sh}_P\boldsymbol{R})\circ\Phi$. ■

Proof of Theorem 2.2 (iii). The map $\varphi=(\mathrm{sh}_P\boldsymbol{R})\circ\Phi$ is surjective by Theorem 1.2. Hence the map $\mathrm{sh}_P\boldsymbol{R}$ is also surjective. On the other hand, the map $\mathrm{sh}_P\boldsymbol{R}$ is injective by Lemma 1.10. ■

Proof of Theorem 2.3. By Lemma 1.2 and Theorem 2.2 (iii), the map $\Phi=(\mathrm{sh}_P\boldsymbol{R})^{-1}\circ\varphi$ becomes an $\boldsymbol{R}$-linear surjective isomorphism. ■

For the proof of Theorem 2.4, we need the following three lemmas.

LEMMA 2.5. *Let* $\Gamma(1)=\bigcup_{j=1}^{m}\Gamma g_j$ *be the left coset decomposition and for each* $j\in\boldsymbol{Z}$ *with* $1\leqslant j\leqslant m$, K_j *be the* $(w+1)m\times(w+1)m$ *integral matrix:*

$$\begin{pmatrix}\varrho_w(g_j^{-1}g_1)&0&0&0\\0&\varrho_w(g_j^{-1}g_2)&0&0\\0&0&\ddots&0\\0&0&0&\varrho_w(g_j^{-1}g_m)\end{pmatrix}.$$

Set $\eta^*_w(\mathbf{g})=K_1^{-1}\eta_w(\mathbf{g})K_1$ *for all* $\mathbf{g}\in\Gamma(1)$.

(i) *For each* $g\in\Gamma(1)$, *the following two representations from* $\Gamma(1)$ *to* $\mathrm{GL}((w+1)m,\boldsymbol{Z})$ *are isomorphic to each other.*

$$\operatorname*{Ind}_{(g^{-1}\Gamma g)\uparrow\Gamma(1)}(\varrho_w|_{g^{-1}\Gamma g})\simeq\eta^*_w.$$

(ii) *For each* $j\in\boldsymbol{Z}$ *with* $1\leqslant j\leqslant m$, *let* $\mathrm{sh}^{(j)}$ *be the isomorphism of Shapiro (cf. Lemma* 1.10*):*

$$H^1\big(\Gamma(1),\operatorname*{Ind}_{(g_j^{-1}\Gamma g_j)\uparrow\Gamma(1)}(\varrho_w|_{g_j^{-1}\Gamma g_j}),\boldsymbol{Z}\big)\to H^1(g_j^{-1}\Gamma g_j,\varrho_w|_{g_j^{-1}\Gamma g_j},\boldsymbol{Z}).$$

Let ι_j *be the canonical isomorphism induced by the above map in* (i):

$$\iota_j\colon H^1(\Gamma(1),\eta^*_w,\boldsymbol{Z})\to H^1\big(\Gamma(1),\operatorname*{Ind}_{(g_j^{-1}\Gamma g_j)\uparrow\Gamma(1)}(\varrho_w|_{g_j^{-1}\Gamma g_j}),\boldsymbol{Z}\big).$$

Then the composite map $\mathrm{sh}^{(j)}\circ\iota_j$ *is given by:*

The cohomology class of a cocycle $\{\Gamma(1)\ni\mathbf{g}\mapsto T^*(\mathbf{g})\in\boldsymbol{Z}^{(w+1)m}\}$
$\xmapsto{\mathrm{sh}^{(j)}\circ\iota_j}$ *The cohomology class of the cocycle* $\{g_j^{-1}\Gamma g_j\ni\mathbf{g}\mapsto$ *the* $(w+1)$ *components of the* $T^*(\mathbf{g})$ *from the* $((w+1)(j-1)+1)$*-st one to the* $((w+1)j)$*-th one*$\}$.

Proof of Lemma 2.5 (i). We may assume $g=g_j$ for some $j\in\boldsymbol{Z}$ with $1\leqslant j\leqslant m$. Normalize the representation

$$\eta_w^{(j)}\overset{\text{def.}}{=}\operatorname*{Ind}_{(g_j^{-1}\Gamma g_j)\uparrow\Gamma(1)}(\varrho_w|_{g_j^{-1}\Gamma g_j})$$

as follows:

$$\eta_w^{(j)}(\mathbf{g})\begin{pmatrix}(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_1)\\(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_2)\\(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_3)\\\vdots\\(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_m)\end{pmatrix}=\begin{pmatrix}(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_1\mathbf{g})\\(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_2\mathbf{g})\\(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_3\mathbf{g})\\\vdots\\(f|_{w+2}[g_j](z)\,dz_w)\circ(g_j^{-1}g_m\mathbf{g})\end{pmatrix}$$

$(\mathbf{g}\in\Gamma(1)$ and $f\in S_{w+2}(\Gamma))$. Then we obtain:

$$\eta^*_w(\mathbf{g})=K_j^{-1}\eta^{(j)}_w(\mathbf{g})K_j\quad(\mathbf{g}\in\Gamma(1)).\ \blacksquare$$

Proof of Lemma 2.5 (ii). By the definitions of $\eta_w^{(j)}$, η_w^* and $\mathrm{sh}^{(j)}$, (ii) follows directly (cf. the map (1.10.1)). ∎

Lemma 2.6. *Let* $\Gamma(1) = \bigcup_{x=1}^{m} \Gamma g_x$ *and* j *be an integer with* $1 \leqslant j \leqslant m$. *Let* θ_j *be the isomorphism*

$$H^1(\Gamma, \varrho_w|_\Gamma, \mathbf{Z}) \to H^1(g_j^{-1}\Gamma g_j, \varrho_w|_{g_j^{-1}\Gamma g_j}, \mathbf{Z})$$

which is induced by the compatible maps,

$$g_j^{-1}\Gamma g_j \to \Gamma\colon \gamma \mapsto g_j \gamma g_j^{-1}$$

and

$$\mathbf{Z}^{w+1} \to \mathbf{Z}^{w+1}\colon \mathfrak{x} \mapsto \varrho_w(g_j)^{-1}\mathfrak{x}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\Gamma(1), \eta_w^*, \mathbf{Z}) & \xrightarrow{\mathrm{sh}^{(1)}\circ \iota_1} & H^1(\Gamma, \varrho_w|_\Gamma, \mathbf{Z}) \\ & \searrow^{\mathrm{sh}^{(j)}\circ \iota_j} \quad \circlearrowleft & \downarrow \theta_j \\ & & H^1(g_j^{-1}\Gamma g_j, \varrho_w|_{g_j^{-1}\Gamma g_j}, \mathbf{Z}) \end{array}$$

Remark (2.6)′. It is easy to see that the map θ_j induces the (surjective) isomorphism

$$\theta_j^*\colon H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{Z}) \to H^1_{P\cap g_j^{-1}\Gamma g_j}(g_j^{-1}\Gamma g_j, \varrho_w|_{g_j^{-1}\Gamma g_j}, \mathbf{Z})$$

since "an element γ of Γ is parabolic" $\Leftrightarrow |\mathrm{Tr}(\gamma) = 2| \Leftrightarrow |\mathrm{Tr}(g_j^{-1}\gamma g_j)| = 2 \Leftrightarrow$ "$g_j^{-1}\gamma g_j$ is parabolic".

Proof of Lemma 2.6. Let $\mathbf{Z}^{(w+1)m}$ be the lattice of the $\mathbf{R}^{(w+1)m}$ with respect to the standard basis, $C\colon \Gamma(1) \to \mathbf{Z}^{(w+1)m}$ be a cocycle in $Z^1(\Gamma(1), \eta_w^*, \mathbf{Z})$ and γ_j denote an element of $g_j^{-1}\Gamma g_j$. We compute:

$\theta_j \circ \mathrm{sh}^{(1)} \circ \iota_1(C)$

$=$ the cohomology class of the cocycle $\{g_j^{-1}\Gamma g_j \ni \gamma_j \mapsto \varrho_w(g_j)^{-1} \times$ (the first $(w+1)$ components of the $C(g_j\gamma_j g_j^{-1})$)$\}$.

Note that

$$C(g_j\gamma_j g_j^{-1}) = \big(\mathbf{1}_{(w+1)m} - \eta_w^*(g_j\gamma_j g_j^{-1})\big)C(g_j) + \eta_w^*(g_j)C(\gamma_j)$$

since C is a cocycle. Now let $c_1(g_j)$ be the first $(w+1)$ components of the $C(g_j)$ and $c_j(\gamma_j)$ be the $(w+1)$ components of the $C(\gamma_j)$ from the $((w+1)\times(j-1)+1)$-st one to the $((w+1)j)$-th one. Then we have:

$\theta_j \circ \mathrm{sh}^{(1)} \circ \iota_1(C)$

$=$ the cohomology class of the cocycle $\{g_j^{-1}\Gamma g_j \ni \gamma_j \mapsto \varrho_w(g_j)^{-1}\big(\big(\mathbf{1}_{w+1} - \varrho_w(g_j\gamma_j g_j^{-1})\big)c_1(g_j) + \varrho_w(g_j)c_j(\gamma_j)\big)\}$

$=$ the cohomology class of the cocycle $\{g_j^{-1}\Gamma g_j \ni \gamma_j \mapsto \big(\big(\mathbf{1}_{w+1} - \varrho_w(\gamma_j)\big)\varrho_w(g_j)^{-1}c_1(g_j) + c_j(\gamma_j)\big)\}$

$=$ the cohomology class of the cocycle $\{g_j^{-1}\Gamma g_j \ni \gamma_j \mapsto c_j(\gamma_j)\}$

$= \mathrm{sh}^{(j)} \circ \iota_j(C)$. ∎

Lemma 2.7. *Notations being as in Lemmas 2.5 and 2.6, the image under the map* $\mathrm{sh}^{(1)}$ *of the* $H^1_P(\Gamma(1), \eta_w, \mathbf{Z})$ *coincides with the* $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{Z})$. *Namely*

$$\mathrm{sh}_P\mathbf{Z}\big(H^1_P(\Gamma(1), \eta_w, \mathbf{Z})\big) = H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{Z}).$$

Proof. Let v be a cohomology class in $H^1_{P\cap\Gamma}(\Gamma, \varrho_w|_\Gamma, \mathbf{Z})$. Set

$$u = (\mathrm{sh}^{(1)})^{-1}(v) \in H^1(\Gamma(1), \eta_w, \mathbf{Z}).$$

By Theorem 2.2 (i), it is sufficient to show $u \in H^1_P(\Gamma(1), \eta_w, \mathbf{Z})$. By Theorem 2.2 (iii), there exists a cocycle C in the $Z^1(\Gamma(1), \eta_w, \mathbf{Z})$ whose cohomology class is equal to the class u such that $C(\tau) = \big(\mathbf{1}_{(w+1)m} - \eta_w(\tau)\big)\mathfrak{X}$ with some $(w+1)m$ dimensional real column vector

$$\mathfrak{X} = {}^t(x_0^{(1)}, x_1^{(1)}, \ldots, x_w^{(1)}, x_0^{(2)}, \ldots, x_w^{(2)}, \ldots, x_0^{(m)}, \ldots, x_w^{(m)}).$$

We shall show that there exists a $(w+1)m$ dimensional column vector

$$\mathfrak{Y} = {}^t(y_0^{(1)}, y_1^{(1)}, \ldots, y_w^{(1)}, y_0^{(2)}, \ldots, y_w^{(2)}, \ldots, y_0^{(m)}, \ldots, y_w^{(m)})$$

with the coefficients in $\mathbf{Z}$ such that

$$(2.7.1)\qquad \big(\mathbf{1}_{(w+1)m} - \eta_w(\tau)\big)\mathfrak{Y} = \big(\mathbf{1}_{(w+1)m} - \eta_w(\tau)\big)\mathfrak{X} = C(\tau).$$

Put $C^*(g) = K_1^{-1}C(g)$ $(g \in \Gamma(1))$, $\mathfrak{Z} = K_1^{-1}\mathfrak{X}$ and

$$\mathfrak{Z} = {}^t(z_0^{(1)}, \ldots, z_w^{(1)}, z_0^{(2)}, \ldots, z_w^{(2)}, \ldots, z_0^{(m)}, \ldots, z_w^{(m)})$$

where $K_1 \in \mathrm{GL}((w+1)m, \mathbf{Z})$ is the matrix defined in Lemma 2.5. Now we fix an arbitrary $j \in \mathbf{Z}$ with $1 \leqslant j \leqslant m$. For simplicity, set

$$g^{(1)} = g_j, \quad g^{(2)} = g_j\tau, \quad \ldots, \quad g^{(r)} = g_j\tau^{r-1}$$

where $\Gamma g_j = \Gamma g_j \tau^r$ and $\Gamma g^{(i)} \neq \Gamma g^{(j)}$ if $i \neq j$. Then we have $\tau^r \in g_j^{-1}\Gamma g_j$ and $\tau^r \in (g^{(r)})^{-1}\Gamma g^{(r)}$. Let b be the integer with $1 \leqslant b \leqslant m$ such that $\Gamma g^{(r)} = \Gamma g_b$ and u^* be the cohomology class of the cocycle C^* in the $H^1(\Gamma(1), \eta_w^*, \mathbf{Z})$. From Lemma 2.6, we obtain that the cohomology class

$$\mathrm{sh}^{(b)} \circ \iota_b(u^*) = \theta_b \circ \mathrm{sh}^{(1)} \circ \iota_1(u^*) = \theta_b \circ \mathrm{sh}^{(1)}(u) = \theta_b(v)$$

is a parabolic cohomology class with $\mathbf{Z}$-coefficients. Hence the $(w+1)$ components of the vector $C^*(\tau^r)$ from the $((w+1)(b-1)+1)$-st one to the $((w+1)b)$-th one are equal to

$$(2.7.2)\qquad \big(\mathbf{1}_{w+1} - \varrho_w(\tau^r)\big){}^t(\xi_0^{(b)}, \xi_1^{(b)}, \ldots, \xi_w^{(b)})$$

for some $(w+1)$ dimensional vector $\Xi^{(b)} = {}^t(\xi_0^{(b)}, \xi_1^{(b)}, \ldots, \xi_w^{(b)})$ with $\mathbf{Z}$ coefficients. For each integer i with $1 \leqslant i \leqslant r$, let $j(i)$ be the integer with $1 \leqslant j(i) \leqslant m$ such that $\Gamma g_{j(i)} = \Gamma g^{(i)}$, and set

$$\mathfrak{Z}^{(i)} = {}^t(z_0^{(j(i))}, z_1^{(j(i))}, \ldots, z_w^{(j(i))}).$$

Note that $j(r) = b$. Since

$$C^*(\tau) = K_1^{-1}C(\tau) = K_1^{-1}\big(\mathbf{1}_{(w+1)m} - \eta_w(\tau)\big)\mathfrak{X} = \big(\mathbf{1}_{(w+1)m} - \eta_w^*(\tau)\big)\mathfrak{Z} \in \mathbf{Z}^{(w+1)m},$$

we obtain:

$$(2.7.3)\qquad \mathfrak{Z}^{(1)}-\varrho_w(\tau)\mathfrak{Z}^{(2)}=\mathfrak{V}_1,\ \mathfrak{Z}^{(2)}-\varrho_w(\tau)\mathfrak{Z}^{(3)}=\mathfrak{V}_2,\ \ldots,\ \mathfrak{Z}^{(r-1)}-\varrho_w(\tau)\mathfrak{Z}^{(r)}=\mathfrak{V}_{r-1},\ \mathfrak{Z}^{(r)}-\varrho_w(\tau)\mathfrak{Z}^{(1)}=\mathfrak{V}_r$$

for some vectors $\mathfrak{V}_1, \mathfrak{V}_2, \ldots, \mathfrak{V}_r$ in $\boldsymbol{Z}^{w+1}$. Replacing the cocycle $\{\Gamma(1)\ni \mathbf{g}\mapsto C^*(\mathbf{g})\}$ by $\{\Gamma(1)\ni \mathbf{g}\mapsto C^*(\mathbf{g})+(\mathbf{1}_{(w+1)m}-\eta_w^*(\mathbf{g}))L\}$ for some vector $L\in\boldsymbol{Z}^{(w+1)m}$ (namely replacing $\mathfrak{Z}^{(x)}$ by $\mathfrak{Z}^{(x)}-\mathfrak{V}_x$ for each $x\in\boldsymbol{Z}$ with $1\leqslant x\leqslant r-1$), we may assume that $\mathfrak{V}_1=\mathfrak{V}_2=\ldots=\mathfrak{V}_{r-1}=0$. Then we obtain from (2.7.3) that $\mathfrak{Z}^{(r)}-\varrho_w(\tau)^r\mathfrak{Z}^{(r)}=\mathfrak{V}_r$. Recall $\tau^r\in(g^{(r)})^{-1}\Gamma g^{(r)}$ and note that

$$\begin{aligned}C^*(\tau^r)&=(\mathbf{1}_{(w+1)m}+\eta_w^*(\tau)+\ldots+\eta_w^*(\tau)^{r-1})(\mathbf{1}_{(w+1)m}-\eta_w^*(\tau))\mathfrak{Z}\\&=(\mathbf{1}_{(w+1)m}-\eta_w^*(\tau^r))\mathfrak{Z}.\end{aligned}$$

By (2.7.2),

$$(2.7.4)\qquad (\mathbf{1}_{w+1}-\varrho_w(\tau^r))\mathfrak{Z}^{(r)}=(\mathbf{1}_{w+1}-\varrho_w(\tau^r))\Xi^{(b)}.$$

By (2.7.3), set

$$\Xi^{(j(i))}=\varrho_w(\tau)^{r-i}\,\Xi^{(b)}\qquad(\in\boldsymbol{Z}^{w+1})$$

for all $i\in\boldsymbol{Z}$ with $1\leqslant i\leqslant r-1$. Then the $\Xi^{(j(1))}$ satisfies

$$\begin{aligned}\Xi^{(b)}-\varrho_w(\tau)\Xi^{(j(1))}&=(\mathbf{1}_{w+1}-\varrho_w(\tau)^r)\,\Xi^{(b)}\\&=(\mathbf{1}_{w+1}-\varrho_w(\tau^r))\,\mathfrak{Z}^{(r)}=\mathfrak{V}_r\qquad\text{by (2.7.4)}.\end{aligned}$$

Replace the $\mathfrak{Z}^{(i)}$ by the $\Xi^{(j(i))}$ for all $i\in\boldsymbol{Z}$ with $1\leqslant i\leqslant r$. Do the above procedure over again for each orbit $\Gamma g_j\langle\tau\rangle$ with $1\leqslant j\leqslant m$. (Here $\Gamma(1)=\bigcup_{j=1}^{m}\Gamma g_j$.) Then (2.7.1) is proved since we can write $K_1^{-1}\mathfrak{Y}$ using the vectors $\{\Xi^*\}$ obtained by the procedures.

This argument in proving (2.7.1) is effective for any parabolic element π $(\in P)$ in $\mathrm{SL}(2,\boldsymbol{Z})$ if we replace τ by the π. Hence there exists a vector $\mathfrak{Y}_\pi\in\boldsymbol{Z}^{(w+1)m}$ for each $\pi\in P$ such that

$$C(\pi)=(\mathbf{1}_{(w+1)m}-\eta_w(\pi))\mathfrak{Y}_\pi.$$

Hence the C becomes a parabolic cocycle in the $Z_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})$. Namely it is proved that $u\in H_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})$. Lemma 2.7 is proved. ∎

Proof of Theorem 2.4. By Lemma 2.7,

$$j_2\Big(\mathrm{sh}_P\boldsymbol{Z}\big(H_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})\big)\Big)=j_2\big(H^1_{P\cap\Gamma}(\Gamma,\varrho_w|_\Gamma,\boldsymbol{Z})\big).$$

It is easy to see $j_2\circ\mathrm{sh}_P\boldsymbol{Z}=\mathrm{sh}_P\boldsymbol{R}\circ j_1$. Hence

$$\mathrm{sh}_P\boldsymbol{R}\Big(j_1\big(H_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})\big)\Big)=j_2\big(H^1_{P\cap\Gamma}(\Gamma,\varrho_w|_\Gamma,\boldsymbol{Z})\big).\ ∎$$

Remark 2.8. It is well known that $\mathrm{SL}(2,\boldsymbol{Z})$ is generated by s_1 and s_2. Since $s_1(0)=s_2(0)=i\infty$, the following map is an $\boldsymbol{R}$-linear injection by Theorem 2.3.

$$S_{w+2}(\Gamma)\to\boldsymbol{R}^{(w+1)m};\qquad f\mapsto \mathrm{Re}\int_0^{i\infty}\mathbf{D}(f).$$

And every cohomology class $\mathfrak{u}$ in the $H_P^1(\Gamma(1),\eta_w,\boldsymbol{R})$ has a representative $\mathfrak{C}$ in the $Z_P^1(\Gamma(1),\eta_w,\boldsymbol{R})$ with $\mathfrak{C}(s_1)=\mathfrak{C}(s_2)$. Furthermore we can also show that for each cohomology class $\mathfrak{u}$ in the $H_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})$, there exists a cocycle $\mathfrak{C}$, in the $Z_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})$, with $\mathfrak{C}(s_1)=\mathfrak{C}(s_2)$ and j_1 (the cohomology class of the $\mathfrak{C})=j_1(\mathfrak{u})$.

3. Proof of Theorem 0.2. We use the same notations in § 1 and § 2. From Theorems 2.2 and 2.4, we have:

$$\Phi^{-1}\Big(j_1\big(H_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})\big)\Big)=\varphi^{-1}\Big(j_2\big(H^1_{P\cap\Gamma}(\Gamma,\varrho_w|_\Gamma,\boldsymbol{Z})\big)\Big)$$

and

$$\Phi^{-1}\Big(j_1\big(H_P^1(\Gamma(1),\eta_w,\boldsymbol{Z})\big)\Big)\cap(-1)^{(1\pm(-1))/4}S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))=H(N,\boldsymbol{Z})^{\pm}.$$

Let $U^+=\langle U_1^+,U_2^+,\ldots,U_d^+\rangle$ (resp. $U^-=\langle U_1^-,U_2^-,\ldots,U_d^-\rangle$) be a $\boldsymbol{Z}$-basis of $H(N,\boldsymbol{Z})^+$ (resp. $H(N,\boldsymbol{Z})^-$). By Lemmas 1.4 and 1.7 the basis U^- (resp. U^+) becomes a $\boldsymbol{C}$-basis of $S_{w+2}(\Gamma_1(N))$ since we have

$$S_{w+2}(\Gamma_1(N))=S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N))\oplus_{\boldsymbol{R}}\sqrt{-1}S^{\boldsymbol{R}}_{w+2}(\Gamma_1(N)).$$

For each positive integer n with $(n,N)=1$, there exists a $d\times d$ matrix $A^+(n)$ (resp. $A^-(n)$) with the coefficients in $\boldsymbol{Z}$ such that:

$${}^t\big(U_1^+|T_{w+2}(n),\,U_2^+|T_{w+2}(n),\,\ldots,\,U_d^+|T_{w+2}(n)\big)=A^+(n)\,{}^t(U_1^+,U_2^+,\ldots,U_d^+)$$

(resp.

$${}^t\big(U_1^-|T_{w+2}(n),\,U_2^-|T_{w+2}(n),\,\ldots,\,U_d^-|T_{w+2}(n)\big)=A^-(n)\,{}^t(U_1^-,U_2^-,\ldots,U_d^-))$$

by Lemma 1.3. Let $F(z)=\sum_{n=1}^{\infty}a_nq^n$ $(a_1=1,\ q=\exp(2\pi\sqrt{-1}z))$ be a primitive form in $S_{w+2}(\Gamma_1(N))$ and σ be an element of $\mathrm{Aut}(\boldsymbol{C})$. It is well known that $F^\sigma|T_{w+2}(n)=a_n^\sigma F^\sigma$ for all positive integers n with $(n,N)=1$. Now consider the following linear equations.

$$(3.1)^+_\sigma:\qquad (x_1^+,x_2^+,\ldots,x_d^+)A^+(n)=(a_n^\sigma x_1^+,a_n^\sigma x_2^+,\ldots,a_n^\sigma x_d^+)$$

for all positive $n\in\boldsymbol{Z}$ with $(n,N)=1$.

$$(3.1)^-_\sigma:\qquad (x_1^-,x_2^-,\ldots,x_d^-)A^-(n)=(a_n^\sigma x_1^-,a_n^\sigma x_2^-,\ldots,a_n^\sigma x_d^-)$$

for all positive $n\in\boldsymbol{Z}$ with $(n,N)=1$.

By the Multiplicity one theorem (Lemma 1.10), the space of the solutions $(3.1)^+_\sigma$ (resp. $(3.1)^-_\sigma$) is one dimensional over $\boldsymbol{C}$ for every $\sigma\in\mathrm{Aut}(\boldsymbol{C})$.

Note that every matrix $(A^+(n)-a_n^\sigma \mathbf{1}_d)$ (resp. $(A^-(n)-a_n^\sigma \mathbf{1}_d)$) has the coefficients in the field $Q_F^\sigma = \boldsymbol{Q}(a_1^\sigma, a_2^\sigma, a_3^\sigma, \ldots)$. Hence the equations $(3.1)_1^+$ (resp. $(3.1)_1^-$) have a non trivial solution

$$(x_1^+, x_2^+, \ldots, x_d^+) = (\alpha_1, \alpha_2, \ldots, \alpha_d) \ (\text{resp. } (x_1^-, x_2^-, \ldots, x_d^-) = (\beta_1, \beta_2, \ldots, \beta_d))$$

with the coefficients in the Q^F. Since the $A^\pm(n)$ $(n \in \boldsymbol{Z})$ has the coefficients in $\boldsymbol{Z}$, the vector $(x_1^+, x_2^+, \ldots, x_d^+) = (\alpha_1^\sigma, \alpha_2^\sigma, \ldots, \alpha_d^\sigma)$ (resp. $(x_1^-, x_2^-, \ldots$ $\ldots, x_d^-) = (\beta_1^\sigma, \beta_2^\sigma, \ldots, \beta_d^\sigma)$) becomes a non trivial solution of the equations $(3.1)_\sigma^+$ (resp. $(3.1)_\sigma^-$) for every $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. Namely there exists a unique complex number c_σ^+ (resp. c_σ^-) such that

$$(3.2)_\sigma^+ \qquad F^\sigma = c_\sigma^+ \times (\alpha_1^\sigma U_1^+ + \alpha_2^\sigma U_2^+ + \ldots + \alpha_d^\sigma U_d^+)$$

(resp.

$$(3.2)_\sigma^- \qquad F^\sigma = c_\sigma^- \times (\beta_1^\sigma U_1^- + \beta_2^\sigma U_2^- + \ldots + \beta_d^\sigma U_d^-)).$$

We shall show that $(3.2)_\sigma^\pm$ assert our Theorem 0.2 (ii) (c). Then it is easy to see, from $(3.2)_\sigma^\pm$, that

$$c_\sigma^+ = c_{\sigma'}^+ \ (\text{resp. } c_\sigma^- = c_{\sigma'}^-) \quad \text{if} \quad \sigma|_{Q_F} = \sigma'|_{Q_F} \ (\sigma \text{ and } \sigma' \in \mathrm{Aut}(\boldsymbol{C}))$$

and

$$(c_\sigma^+)^J = c_{\sigma J}^+ \ (\text{resp. } (c_\sigma^-)^J = -c_{\sigma J}^-)$$

for the complex conjugation J and any $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ since $(U_t^\pm)^J = \pm U_t$ for all $t \in \boldsymbol{Z}$ with $1 \leqslant t \leqslant d$. This is a proof of Theorem 0.2 (ii) (a) and (b).

Now we shall prove Theorem 0.2 (ii) (c). For each $\mathbf{g} \in \Gamma(1)$ and $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, set

$$W(F, \mathbf{g}, \sigma)^\pm = \Big((2c_1^\pm)^{-1}\Big(\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(F) \pm \int_{i\infty}^{\mathbf{tgt}(i\infty)} \mathbf{D}(F)^E\Big)\Big)^\sigma - \\ - \Big((2c_\sigma^\pm)^{-1}\Big(\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(F^\sigma) \pm \int_{i\infty}^{\mathbf{tgt}(i\infty)} \mathbf{D}(F^\sigma)^E\Big)\Big).$$

Since each U_j^+ (resp. U_j^-) is a cusp form in $\Phi^{-1}\big(j_1\big(H_P^1(\Gamma(1), \eta_w, \boldsymbol{Z})\big)\big)$, there exists a $(w+1)m$ dimensional real column vector B_j^+ (resp. B_j^-) for each $j \in \boldsymbol{Z}$ with $1 \leqslant j \leqslant d$ such that

$$(3.3) \qquad \mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(U_j^\pm)(z) + \big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big) B_j^\pm \in \boldsymbol{Z}^{(w+1)m}$$

for all $\mathbf{g} \in \Gamma(1)$. Since $U_j^+ \in S_{w+2}^{\boldsymbol{R}}(\Gamma_1(N))$ and $U_j^- \in \sqrt{-1}S_{w+2}^{\boldsymbol{R}}(\Gamma_1(N))$, changing the variable of integration from z to $-\bar{z}$, we have:

$$(3.4) \qquad \overline{\int_{i\infty}^{\mathbf{g}(i\infty)} (U_j^\pm(z)\,dz_w)} \circ \mathbf{h} = \pm \varrho_w(\mathbf{t}) \int_{i\infty}^{\mathbf{tgt}(i\infty)} (U_j^\pm(z)\,dz_w) \circ (\mathbf{tht})$$

for all $\mathbf{g}$ and $\mathbf{h} \in \Gamma(1)$. Hence

$$(3.5) \qquad \mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(U_j^\pm)(z) = \tfrac{1}{2}\Big(\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(U_j^\pm)(z) \pm \int_{i\infty}^{\mathbf{tgt}(i\infty)} \mathbf{D}(U_j^\pm)^E(z)\Big)$$

for all $\mathbf{g} \in \Gamma(1)$. Set

$$B(\sigma)^+ = \sum_{j=1}^d \alpha_j^\sigma\big(B_j^+ - (B_j^+)^\sigma\big) \quad \text{and} \quad B(\sigma)^- = \sum_{j=1}^d \beta_j^\sigma\big(B_j^- - (B_j^-)^\sigma\big).$$

By $(3.2)_\sigma^+$ and (3.5),

$$W(F, \mathbf{g}, \sigma)^+ = \sum_{j=1}^d \alpha_j^\sigma\Big(\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(U_j^+)\Big)^\sigma - \sum_{j=1}^d \alpha_j^\sigma\Big(\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(U_j^+)\Big).$$

By (3.3),

$$(3.6)^+ \quad W(F, \mathbf{g}, \sigma)^+ = \sum_{j=1}^d \alpha_j^\sigma\Big(\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(U_j^+) + \big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big)B_j^+\Big)^\sigma - \\ - \sum_{j=1}^d \alpha_j^\sigma\Big(\mathrm{Re}\int_{i\infty}^{\mathbf{g}(i\infty)} \mathbf{D}(U_j^+) + \big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big)B_j^+\Big) - \\ - \sum_{j=1}^d \alpha_j^\sigma\big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big)\big((B_j^+)^\sigma - B_j^+\big) \\ = \big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big)B(\sigma)^+ \quad (\mathbf{g} \in \Gamma(1)).$$

In the same way, from $(3.2)_\sigma^-$, (3.3) and (3.5), we obtain:

$$(3.6)^- \qquad W(F, \mathbf{g}, \sigma)^- = \big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big)B(\sigma)^- \quad (\mathbf{g} \in \Gamma(1)).$$

We shall prove below that

$$(3.7)^\pm \qquad \big(\mathbf{1}_{(w+1)m} - \eta_w(\mathbf{g})\big)B(\sigma)^\pm = \mathbf{0} \quad \text{for all } \mathbf{g} \in \Gamma(1) \text{ and } \sigma \in \mathrm{Aut}(\boldsymbol{C}).$$

Let K_1 be the matrix defined in Lemma 2.5. Set

$$\eta_w^*(\mathbf{g}) = K_1^{-1}\eta_w(\mathbf{g})K_1 \quad (\mathbf{g} \in \Gamma(1)),$$

$$K_1^{-1}B(\sigma)^+ = {}^t(\lambda_{1,0}^+, \lambda_{1,1}^+, \ldots, \lambda_{1,w}^+, \lambda_{2,0}^+, \lambda_{2,1}^+, \ldots, \lambda_{2,w}^+, \ldots, \lambda_{m,0}^+, \ldots, \lambda_{m,w}^+)$$

and

$$K_1^{-1}B(\sigma)^- = {}^t(\lambda_{1,0}^-, \lambda_{1,1}^-, \ldots, \lambda_{1,w}^-, \lambda_{2,0}^-, \lambda_{2,1}^-, \ldots, \lambda_{2,w}^-, \ldots, \lambda_{m,0}^-, \ldots, \lambda_{m,w}^-)$$

for an arbitrarily fixed $\sigma \in \mathrm{Aut}(\boldsymbol{C})$. From $(3.6)^\pm$, we have:

$$(3.8)^\pm \qquad K_1^{-1}W(F, \mathbf{g}, \sigma)^\pm = \big(\mathbf{1}_{(w+1)m} - \eta_w^*(\mathbf{g})\big)K_1^{-1}B(\sigma)^\pm \quad (\mathbf{g} \in \Gamma(1)).$$

Proof of $(3.7)^{\pm}$.

Case 1 (weight $w+2\geqslant 3$). Let k be an integer with $1\leqslant k\leqslant m$ and $\mathbf{g}$ be an element of $g_k^{-1}\Gamma_1(N)g_k$. From $(3.8)^{\pm}$, we obtain:

$$(3.9)^{\pm}\quad \Big((2c_1^{\pm})^{-1}\Big(\int_{i\infty}^{\mathbf{g}(i\infty)} F|_{w+2}[g_k](z)\,dz_w \pm \varrho_w(\mathbf{t})\int_{i\infty}^{\mathbf{t}\mathbf{g}\mathbf{t}(i\infty)} F|_{w+2}[\mathbf{t}g_k\mathbf{t}](z)\,dz_w\Big)\Big)^{\sigma} - \Big((2c_\sigma^{\pm})^{-1}\Big(\int_{i\infty}^{\mathbf{g}(i\infty)} F^{\sigma}|_{w+2}[g_k](z)\,dz_w \pm \varrho_w(\mathbf{t})\int_{i\infty}^{\mathbf{t}\mathbf{g}\mathbf{t}(i\infty)} F^{\sigma}|_{w+2}[\mathbf{t}g_k\mathbf{t}](z)\,dz_w\Big)\Big) = \big(\mathbf{1}_{w+1}-\varrho_w(\mathbf{g})\big)\,{}^t(\lambda_{k,0}^{\pm},\lambda_{k,1}^{\pm},\lambda_{k,2}^{\pm},\ldots,\lambda_{k,w}^{\pm})$$

for all $\mathbf{g}\in g_k^{-1}\Gamma_1(N)g_k$. Put *the left side of* (3.9) $= M_k(F,\mathbf{g})^{\pm}$. Now set $\mathbf{g}=\begin{pmatrix}1&N\\0&1\end{pmatrix}$ in $(3.9)^{\pm}$. Then $\mathbf{g}(i\infty)=i\infty$ and

$$(3.10)^{\pm}\qquad \mathbf{0}=\Big(\mathbf{1}_{w+1}-\varrho_w\Big(\begin{pmatrix}1&N\\0&1\end{pmatrix}\Big)\Big)\,{}^t(\lambda_{k,0}^{\pm},\lambda_{k,1}^{\pm},\ldots,\lambda_{k,w}^{\pm}).$$

Using the definition of the representation ϱ_w, we solve $(3.10)^{\pm}$ directly. Then we obtain easily:

$$(3.10.1)^{\pm}\qquad \lambda_{k,0}^{\pm}=\lambda_{k,1}^{\pm}=\lambda_{k,2}^{\pm}=\ldots=\lambda_{k,w-1}^{\pm}=0.$$

Only $\lambda_{k,w}^{+}$ and $\lambda_{k,w}^{-}$ remain unknown. Now we shall show that $\lambda_{k,w}^{\pm}=0$. Let p be an odd rational prime with $p\equiv 1 \pmod N$ and $a_p^{\sigma}\neq 1+p^{w+1}$. Such a prime p exists by Lemma 1.12. Put

$$\nu_u=\begin{pmatrix}1&Nu\\0&p\end{pmatrix}\quad\text{for}\quad u\in\mathbf{Z}\ \text{with}\ 0\leqslant u\leqslant p-1\quad\text{and}\quad \nu_p=\begin{pmatrix}p&0\\0&1\end{pmatrix}.$$

Note that $\nu_u\mathbf{g}\nu_u^{-1}\in\Gamma(N)$ for all $\mathbf{g}\in\Gamma(pN)$ and all $u\in\mathbf{Z}$ with $0\leqslant u\leqslant p$. Put $u(\mathbf{g})=\nu_u\mathbf{g}\nu_u^{-1}$ for every $u\in\mathbf{Z}$ with $0\leqslant u\leqslant p$. We need:

LEMMA 3.11. *Let p be an odd prime with $p\equiv 1 \pmod N$ and $\mathbf{g}$ be an element of $\Gamma(pN)$. We obtain:*

$$(3.11.1)^{\pm}\qquad M_k(F|T_{w+2}(p),\mathbf{g})^{\pm}=p^{w+1}\Big(\sum_{u=0}^{p}\varrho_w(\nu_u)^{-1}M_k(F,u(\mathbf{g}))^{\pm}\Big).$$

Proof of Lemma 3.11. Recall

$$F|T_{w+2}(p)(z)=p^{w/2}\sum_{u=0}^{p}F|_{w+2}[\nu_u](z)=p^{w/2}\sum_{u=0}^{p}F|_{w+2}[\mathbf{t}\nu_u\mathbf{t}](z).$$

Hence

$$M_k(F|T_{w+2}(p),\mathbf{g})^{\pm}=p^{w/2}\sum_{u=0}^{p}\Big(\Big((2c_1^{\pm})^{-1}\Big(\int_{i\infty}^{\mathbf{g}(i\infty)}F|_{w+2}[\nu_u g_k](z)\,dz_w \pm \varrho_w(\mathbf{t})\int_{i\infty}^{\mathbf{t}\mathbf{g}\mathbf{t}(i\infty)}F|_{w+2}[\mathbf{t}\nu_u g_k\mathbf{t}](z)\,dz_w\Big)\Big)^{\sigma} - \Big((2c_\sigma^{\pm})^{-1}\Big(\int_{i\infty}^{\mathbf{g}(i\infty)}F^{\sigma}|_{w+2}[\nu_u g_k](z)\,dz_w \pm \varrho_w(\mathbf{t})\int_{i\infty}^{\mathbf{t}\mathbf{g}\mathbf{t}(i\infty)}F^{\sigma}|_{w+2}[\mathbf{t}\nu_u g_k\mathbf{t}](z)\,dz_w\Big)\Big)\Big).$$

By Lemma 1.11 and the fact that the group $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$ with $\Gamma(N)\subset\Gamma_1(N)$, for every $\mathbf{x}\in\Gamma(1)$,

$$\sum_{u=0}^{p}\big(F|_{w+2}[\nu_u\mathbf{x}]-F|_{w+2}[\mathbf{x}\nu_u]\big)=0,\qquad \sum_{u=0}^{p}\big(F^{\sigma}|_{w+2}[\nu_u\mathbf{x}]-F^{\sigma}|_{w+2}[\mathbf{x}\nu_u]\big)=0,$$

$$\sum_{u=0}^{p}\big(F|_{w+2}[\mathbf{t}\nu_u\mathbf{x}\mathbf{t}]-F|_{w+2}[\mathbf{t}\mathbf{x}\nu_u\mathbf{t}]\big)=0,$$

$$\sum_{u=0}^{p}\big(F^{\sigma}|_{w+2}[\mathbf{t}\nu_u\mathbf{x}\mathbf{t}]-F^{\sigma}|_{w+2}[\mathbf{t}\mathbf{x}\nu_u\mathbf{t}]\big)=0.$$

Replace the variable z of the integrations

$$\int_{i\infty}^{\mathbf{g}(i\infty)}F|_{w+2}[g_k\nu_u](z)\,dz_w\quad\text{and}\quad\int_{i\infty}^{\mathbf{g}(i\infty)}F^{\sigma}|_{w+2}[g_k\nu_u](z)\,dz_w$$

(resp.

$$\int_{i\infty}^{\mathbf{t}\mathbf{g}\mathbf{t}(i\infty)}F|_{w+2}[\mathbf{t}g_k\nu_k\mathbf{t}](z)\,dz_w\quad\text{and}\quad\int_{i\infty}^{\mathbf{t}\mathbf{g}\mathbf{t}(i\infty)}F^{\sigma}|_{w+2}[\mathbf{t}g_k\nu_u\mathbf{t}](z)\,dz_w)$$

by $\nu_u^{-1}(z)$ (resp. $\mathbf{t}\nu_u^{-1}\mathbf{t}(z)$). Note that $\nu_u^{-1}(i\infty)=\mathbf{t}\nu_u^{-1}\mathbf{t}(i\infty)=i\infty$, $\nu_u\mathbf{g}(i\infty)=\nu_u\mathbf{g}\nu_u^{-1}(i\infty)=u(\mathbf{g})(i\infty)$ and $\mathbf{t}\nu_u\mathbf{g}\mathbf{t}(i\infty)=\mathbf{t}\nu_u\mathbf{g}\nu_u^{-1}\mathbf{t}(i\infty)=\mathbf{t}u(\mathbf{g})\mathbf{t}(i\infty)$ $(\mathbf{g}\in\Gamma(pN))$. Then we obtain:

$$M_k(F|T_{w+2}(p),\mathbf{g})^{\pm} = p^{w+1}\sum_{u=0}^{p}\Big(\Big((2c_1^{\pm})^{-1}\Big(\int_{i\infty}^{u(\mathbf{g})(i\infty)}F|_{w+2}[g_k](z)\,\varrho_w(\nu_u)^{-1}dz_w \pm \varrho_w(\mathbf{t})\int_{i\infty}^{\mathbf{t}u(\mathbf{g})\mathbf{t}(i\infty)}F|_{w+2}[\mathbf{t}g_k\mathbf{t}](z)\,\varrho_w(\mathbf{t}\nu_u^{-1}\mathbf{t})\,dz_w\Big)\Big) - \Big((2c_\sigma^{\pm})^{-1}\Big(\int_{i\infty}^{u(\mathbf{g})(i\infty)}F^{\sigma}|_{w+2}[g_k](z)\,\varrho_w(\nu_u)^{-1}dz_w \pm \varrho_w(\mathbf{t})\int_{i\infty}^{\mathbf{t}u(\mathbf{g})\mathbf{t}(i\infty)}F^{\sigma}|_{w+2}[\mathbf{t}g_k\mathbf{t}](z)\,\varrho_w(\mathbf{t}\nu_u^{-1}\mathbf{t})\,dz_w\Big)\Big)\Big)$$

$$= p^{w+1}\Big(\sum_{u=0}^{p}\varrho_w(\nu_u)^{-1}M_k(F,u(\mathbf{g}))^{\pm}\Big).\ \blacksquare$$

We continue the proof of $(3.7)^{\pm}$. Note that $F|T_{w+2}(p)=a_pF$ and $F^{\sigma}|T_{w+2}(p)=a_p^{\sigma}F^{\sigma}$. From $(3.9)^{\pm}$ and $(3.11.1)^{\pm}$, we obtain:

$$(3.12)^{\pm}\qquad \Big(a_p^{\sigma}\big(\mathbf{1}_{w+1}-\varrho_w(\mathbf{g})\big)-\Big(\sum_{u=0}^{p}p^{w+1}\varrho_w(\nu_u)^{-1}\big(\mathbf{1}_{w+1}-\varrho_w(u(\mathbf{g}))\big)\Big)\Big)\times{}^t(\lambda_{k,0}^{\pm},\lambda_{k,1}^{\pm},\ldots,\lambda_{k,w}^{\pm})=\mathbf{0}\qquad(\mathbf{g}\in\Gamma(pN)).$$

Now set $\mathbf{g} = \begin{pmatrix} 1 & 0 \\ pN & 1 \end{pmatrix}$. Then we have:

$$\nu_u^{-1}u(\mathbf{g}) = \begin{cases} \begin{pmatrix} 1 & -Nu/p \\ pN & (1-N^2pu)/p \end{pmatrix} & \text{for } u \in \mathbf{Z} \text{ with } 0 \leqslant u \leqslant p-1, \\ \begin{pmatrix} 1/p & 0 \\ N & 1 \end{pmatrix} & \text{for } u = p, \end{cases}$$

$$(3.13) \quad \begin{cases} \text{the } (1, w+1) \text{ component of the } \varrho_w(\mathbf{g}) = p^w N^w, \\ \text{the } (1, w+1) \text{ component of the } \varrho_w(\nu_u^{-1}u(\mathbf{g})) = p^{w-1}N^w, \\ \qquad\qquad (0 \leqslant u \leqslant p-1) \\ \text{the } (1, w+1) \text{ component of the } \varrho_w(\nu_p^{-1}p(\mathbf{g})) = p^{-1}N^w, \\ \text{the } (1, w+1) \text{ component of the } \varrho_w(\nu_u^{-1}) = 0. \end{cases}$$

Now compute the first component of the vector of the left side of $(3.12)^{\pm}$ using the results $(3.10.1)^{\pm}$ and (3.13). Then we obtain:

$$\left(-a_p^\sigma p^w N^w + p^{w+1}(p^{w-1}N^w p + p^{-1}N^w)\right)\lambda_{k,w}^{\pm}.$$

Namely $p^w N^w(-a_p^\sigma + p^{w+1} + 1)\lambda_{k,w}^{\pm}$.

From $(3.12)^{\pm}$ and $a_p^\sigma \neq 1 + p^{w+1}$, we obtain $\lambda_{k,w}^{\pm} = 0$. Hence we have ${}^t(\lambda_{k,0}^{\pm}, \lambda_{k,1}^{\pm}, \ldots, \lambda_{k,w}^{\pm}) = \mathbf{0}$. Since k is an arbitrary integer with $1 \leqslant k \leqslant m$, we have $K_1^{-1}B(\sigma)^{\pm} = \mathbf{0}$. $(3.7)^{\pm}$ are proved for the case of $w+2 \geqslant 3$.

Case 2 (weight $w+2 = 2$). Set $B(\sigma)^{\pm} = K_1^{-1}B(\sigma)^{\pm} = {}^t(\lambda_1^{\pm}, \lambda_2^{\pm}, \ldots, \lambda_m^{\pm})$ (namely ${}^t(\lambda_1^{\pm}, \lambda_2^{\pm}, \ldots, \lambda_m^{\pm}) = {}^t(\lambda_{1,0}^{\pm}, \lambda_{2,0}^{\pm}, \ldots, \lambda_{m,0}^{\pm})$) and $\eta_0\ (= \eta_0^*) = \eta$ which is the representation defined by $\eta(\mathbf{g})\mathbf{D}(f) = \mathbf{D}(f) \circ \mathbf{g}\ (f \in S_2(\Gamma_1(N)),\ \mathbf{g} \in \Gamma(1))$. Let p be an odd prime with $p \equiv 1 \pmod N$ and $a_p^\sigma \neq 1+p$ and $\nu_u\ (0 \leqslant u \leqslant p)$ be the same as defined in Case 1. (Such a prime exists by Lemma 1.12.) By Lemma 1.11 and the fact: $\Gamma(N) \lhd \Gamma(1)$, every element $\mathbf{g}$ of $\Gamma(1)$ induces the permutation $\tilde{\mathbf{g}}$ on $\{0, 1, 2, \ldots, p\}$ which is defined by $\Gamma(N)\mathbf{g}^{-1}\nu_u\mathbf{g} = \Gamma(N)\nu_{\tilde{\mathbf{g}}(u)}$. Hence there exist $\gamma(u, \mathbf{g}) \in \Gamma(N)$ for $u \in \mathbf{Z}$ with $0 \leqslant u \leqslant p$ and $\mathbf{g} \in \Gamma(1)$ such that $\mathbf{g}^{-1}\nu_u\mathbf{g} = \gamma(u, \mathbf{g})\nu_{\tilde{\mathbf{g}}(u)}$. We have

$$\sum_{u=0}^{p} (F|_2[g_j\nu_u] - F|_2[\nu_u g_j]) = 0.$$

Recall

$$F|T_2(p) = \sum_{u=0}^{p} F|_2[\nu_u] = \sum_{u=0}^{p} F|_2[\mathbf{t}\nu_u\mathbf{t}].$$

Hence for each $j \in \mathbf{Z}$ with $1 \leqslant j \leqslant m$, we obtain:

$$\int_{z=i\infty}^{\mathbf{g}(i\infty)} \left(\left(F|T_2(p)\right)(z)\,dz\right) \circ g_j = \sum_{u=0}^{p} \int_{\nu_u(i\infty)}^{\nu_u\mathbf{g}(i\infty)} F|_2[g_j](z)\,dz$$
$$= \sum_{u=0}^{p} \int_{\nu_u(i\infty)}^{\mathbf{g}\gamma(u,\mathbf{g})\nu_{\tilde{\mathbf{g}}(u)}(i\infty)} F|_2[g_j](z)\,dz = \sum_{u=0}^{p} \int_{i\infty}^{\mathbf{g}\gamma(u,\mathbf{g})(i\infty)} F|_2[g_j](z)\,dz$$

and, in the same way,

$$\int_{z=i\infty}^{\mathbf{tgt}(i\infty)} \left(\left(F|T_2(p)(z)\right)dz\right) \circ g_j = \sum_{u=0}^{p} \int_{i\infty}^{\mathbf{t}\mathbf{g}\gamma(u,\mathbf{g})\mathbf{t}(i\infty)} F|_2[\mathbf{t}g_j\mathbf{t}](z)\,dz.$$

Hence we obtain:

$$W(F|T_2(p), \mathbf{g}, \sigma)^{\pm} = \sum_{u=0}^{p} W(F, \mathbf{g}\cdot\gamma(u, \mathbf{g}), \sigma)^{\pm}.$$

Since $F|T_2(p) = a_p F$ and $F^\sigma|T_2(p) = a_p^\sigma F^\sigma$, by $(3.8)^{\pm}$,

$$(3.14)^{\pm} \qquad a_p^\sigma(\mathbf{1}_m - \eta(\mathbf{g}))B(\sigma)^{\pm} = \sum_{u=0}^{p}\left(\mathbf{1}_m - \eta(\mathbf{g}\cdot\gamma(u, \mathbf{g}))\right)B(\sigma)^{\pm}.$$

Since $\gamma(u, \mathbf{g}) \in \Gamma(N) \subset \Gamma_1(N)$, we have $\eta(\gamma(u, \mathbf{g})) = \mathbf{1}_m$. Hence by $(3.14)^{\pm}$,

$$(3.15)^{\pm} \qquad (a_p^\sigma - 1 - p)(\mathbf{1}_m - \eta(\mathbf{g}))B(\sigma)^{\pm} = \mathbf{0} \quad \text{and} \quad (\mathbf{1}_m - \eta(\mathbf{g}))B(\sigma)^{\pm} = \mathbf{0} \quad (\mathbf{g} \in \Gamma(1)).$$

In $(3.15)^{\pm}$, set $\mathbf{g} = g_j$ for each $j \in \mathbf{Z}$ with $2 \leqslant j \leqslant m$. The first row vector of $(\mathbf{1}_m - \eta(g_j))$ is $(1, 0, \ldots, 0, -1, 0, \ldots, 0)$ where -1 is the jth component. Then the first component of the left side of the vector $(3.15)^{\pm}$ is equal to $\lambda_1^{\pm} - \lambda_j^{\pm}$, which is 0 by $(3.15)^{\pm}$. Hence we obtain: $\lambda_1^+ = \lambda_2^+ = \ldots = \lambda_m^+$ (resp. $\lambda_1^- = \lambda_2^- = \ldots = \lambda_m^-$). Since $\eta(\mathbf{g})$ $(\mathbf{g} \in \Gamma(1))$ is a permutation matrix, we obtain:

$$(\mathbf{1}_m - \eta(\mathbf{g}))B(\sigma)^{\pm} = \mathbf{0} \quad \text{for all } \mathbf{g} \in \Gamma(1).$$

$(3.7)^{\pm}$ is proved for the case of $w+2 = 2$.

By $(3.6)^{\pm}$, $(3.7)^{\pm}$ assert that $W(F, \mathbf{g}, \sigma)^{\pm} = \mathbf{0}$ for all $\mathbf{g} \in \Gamma(1)$ and all $\sigma \in \mathrm{Aut}(\mathbf{C})$. Set $\mathbf{g} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $W(F, \mathbf{g}, \sigma)^{\pm}$. Then we obtain Theorem 0.2 (ii) (c).

Now consider all the automorphisms $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q}_F)$. Then Theorem 0.2 (i) follows from the Galois theory, Theorem 0.2 (ii) (a) and Theorem 0.2 (ii) (c). ■

We give two corollaries of Theorem 0.2. For $l \in \mathbf{Z}$ with $0 \leqslant l \leqslant w$, $x \in \mathbf{Q}$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$, set

$$\mathbf{P}_{l,\sigma}^{\pm}(x) = (c_\sigma^{\pm})^{-1}\left(\int_0^{i\infty} F^\sigma(z+x)z^l dz \pm (-1)^{l+1}\int_0^{i\infty} F^\sigma(z-x)z^l dz\right).$$

Corollary 3.16 (of Theorem 0.2). *For all $x \in \mathbf{Q}$, $l \in \mathbf{Z}$ with $0 \leqslant l \leqslant w$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$, we have:*

$$\left(\mathbf{P}_{l,1}^{+}(x)\right)^\sigma = \mathbf{P}_{l,\sigma}^{+}(x) \in Q_F^\sigma \quad \textit{and} \quad \left(\mathbf{P}_{l,1}^{-}(x)\right)^\sigma = \mathbf{P}_{l,\sigma}^{-}(x) \in Q_F^\sigma.$$

For the proof we need:

LEMMA 3.17. *For $x \in \boldsymbol{Q}$, $l \in \boldsymbol{Z}$ with $0 \leqslant l \leqslant w$ and $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, set*

$$H^{\pm}_{l,\sigma}(x) = (c^{\pm}_{\sigma})^{-1}\Big(\int_{x}^{i\infty} F^{\sigma}(z)z^{l}dz \pm (-1)^{l+1}\int_{-x}^{i\infty} F^{\sigma}(z)z^{l}dz\Big).$$

We have:

$$H^{\pm}_{l,\sigma}(x) \in Q^{\sigma}_{F} \quad \text{and} \quad (H^{\pm}_{l,1}(x))^{\sigma} = H^{\pm}_{l,\sigma}(x).$$

Proof. This is a direct consequence of $W(F, \mathbf{g}, \sigma)^{\pm} = 0$ $(3.7)^{\pm}$, since the set $\{\mathbf{g}(i\infty) \mid \mathbf{g} \in \Gamma(1)\}$ coincides with $\boldsymbol{Q} \cup \{i\infty\}$. ∎

Proof of Corollary 3.16.

$$\mathbf{P}^{\pm}_{l,\sigma}(x) = \sum_{j=0}^{\infty} \binom{l}{j} (-x)^{l-j} H^{\pm}_{l,\sigma}(x). \; ∎$$

Let ψ be a primitive Dirichlet character, $m(\psi)$ be its conductor and $G(\psi)$ be its Gauss sum $\big(= \sum_{n=1}^{m(\psi)} \psi(n)\exp\big(2\pi\sqrt{-1}n/m(\psi)\big)\big)$. Set $F \otimes \psi(z) = \sum_{n=1}^{\infty} \psi(n) a_n q^n$, where $q = \exp(2\pi\sqrt{-1}z)$ and $\psi(n) = 0$ if $(n, m(\psi)) \neq 1$. Let $\boldsymbol{Q}(\psi)$ be the field generated over $\boldsymbol{Q}$ by the values which ψ takes.

COROLLARY 3.18 (of Theorem 0.2) (cf. Shimura [23], [24]). *Take the superscript on $c^{\pm}_{\sigma}$ as in the formula $\psi(-1) = \pm(-1)^{l+1}$. We have:*

$$\Big((c^{\pm}_{1} G(\psi))^{-1} \int_{0}^{i\infty} F \otimes \psi(z) z^{l} dz\Big)^{\sigma} = \big(c^{\pm}_{\sigma} G(\psi^{\sigma})\big)^{-1} \int_{0}^{i\infty} F^{\sigma} \otimes \psi^{\sigma}(z) z^{l} dz \in Q^{\sigma}_{F} \cdot \boldsymbol{Q}(\psi^{\sigma})$$

for every $l \in \boldsymbol{Z}$ with $0 \leqslant l \leqslant w$ and $\sigma \in \mathrm{Aut}(\boldsymbol{C})$.

Proof. It is well known (see e.g. Lemma 9.4 in Manin [12] or Weil [25]) that

$$F \otimes \psi(z) = \frac{G(\psi)}{m(\psi)} \sum_{b \,(\mathrm{mod}\, m(\psi))} \psi^{-1}(-b) F\Big(z + \frac{b}{m(\psi)}\Big).$$

Hence

$$\begin{aligned}
&(G(\psi^{\sigma}))^{-1} \int_{0}^{i\infty} (F^{\sigma} \otimes \psi^{\sigma})(z) z^{l} dz \\
&= \big(m(\psi^{\sigma})\big)^{-1} \sum_{b \,(\mathrm{mod}\, m(\psi^{\sigma}))} (\psi^{\sigma})^{-1}(-b) \int_{0}^{i\infty} F^{\sigma}\Big(z + \frac{b}{m(\psi^{\sigma})}\Big) z^{l} dz \\
&= \big(m(\psi^{\sigma})\big)^{-1} \sum_{0 < b < m(\psi^{\sigma})/2} (\psi^{\sigma})^{-1}(-b) \Big(\int_{0}^{i\infty} F^{\sigma}\Big(z + \frac{b}{m(\psi^{\sigma})}\Big) z^{l} dz + \\
&\qquad + (\psi^{\sigma})^{-1}(-1) \int_{0}^{i\infty} F^{\sigma}\Big(z - \frac{b}{m(\psi^{\sigma})}\Big) z^{l} dz\Big).
\end{aligned}$$

Now Corollary 3.18 follows from Lemma 3.17. ∎

4. On p-adic measures. In this section we shall generalize the results on p-adic measures in Manin [16] to the case of Neben-typus primitive forms. Let F be a primitive form in $S_{w+2}(N, \chi)$, p be a rational prime, L be a finite extension of $\boldsymbol{Q}_p$ and Δ_0 (> 0) be a rational integer with $(\Delta_0, p) = 1$. Put

$$F(z) = \sum_{n=1}^{\infty} a_n q^n \quad (a_1 = 1,\ q = \exp(2\pi\sqrt{-1}z)), \qquad \omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

$$\Delta = \Delta_0 p \ \text{ if } p \geqslant 3, \qquad \Delta = 4\Delta_0 \ \text{ if } p = 2 \quad \text{and} \quad \boldsymbol{Z}_{\Delta} = \text{proj-}\lim_{m} \big(\boldsymbol{Z}/(\Delta p^m)\big).$$

Following (8.4) in Manin [15] we call a finitely additive function μ of the open and closed subsets of $\boldsymbol{Z}^{*}_{\Delta}$ with values in the field L a L-measure μ on $\boldsymbol{Z}^{*}_{\Delta}$. The following proposition is a generalization of Lemma 9.4 in Manin [15]. Let $\boldsymbol{Q}_{\Delta}$ be the set of rational numbers whose denominators divide Δp^m for all $m \geqslant 1$.

PROPOSITION 4.1. *Let $\mathscr{R} : \boldsymbol{Q} \to L$ be a function with the following properties: for some A and $B \in L$ and all $x \in \boldsymbol{Q}_{\Delta}$*

$$\mathscr{R}(x+1) = \mathscr{R}(x) \quad \text{and} \quad \sum_{k=0}^{p-1} \mathscr{R}\big((x+k)/p\big) = A \times \mathscr{R}(x) + B \times \mathscr{R}(px).$$

Let ψ be a Dirichlet character which takes its values in L and ϱ denote a root of the equation $\varrho^2 = \psi(p)\big(A\varrho + Bp\psi(p)\big)$, with $\varrho \neq 0$. Then there exists a $L(\varrho)$ valued measure μ on $\boldsymbol{Z}^{*}_{\Delta}$ such that for all non negative $m \in \boldsymbol{Z}$ and all $a \in \boldsymbol{Z}$

$$\mu\big(a + (\Delta p^m)\big) = \varrho^{-m}\psi(\Delta p^m)\mathscr{R}\big(a/(\Delta p^m)\big) + B\varrho^{-m-1}\psi(\Delta p^{m+1})\mathscr{R}\big(a/(\Delta p^{m-1})\big).$$

Here (Δp^m) denotes $\Delta p^m \boldsymbol{Z}_{\Delta}$.

Proof. By (24) in Manin [15], it is sufficient to show

$$\mu(I_{a,m}) = \sum_{b \equiv a \,(\mathrm{mod}\, \Delta p^m)} \mu(I_{b,m+1}) \quad (\text{for all } a \in \boldsymbol{Z} \text{ and all } 0 \leqslant m \in \boldsymbol{Z})$$

where $I_{a,m}$ denotes $a + \Delta p^m \boldsymbol{Z}_{\Delta}$. We compute as follows.

$$\begin{aligned}
&\sum_{k=0}^{p-1} \mu\big(a + \Delta p^m k + (\Delta p^{m+1})\big) \\
&\quad = \sum_{k=0}^{p-1} \Big(\varrho^{-m-1}\psi(\Delta p^{m+1})\mathscr{R}\big((a + \Delta p^m k)/(\Delta p^{m+1})\big) + \\
&\qquad + B\varrho^{-m-2}\psi(\Delta p^{m+2})\mathscr{R}\big((a + \Delta p^m k)/(\Delta p^m)\big)\Big) \\
&\quad = \varrho^{-m-1}\psi(\Delta p^{m+1})\big(A \times \mathscr{R}\big(a/(\Delta p^m)\big) + B \times \mathscr{R}\big(a/(\Delta p^{m-1})\big) + \\
&\qquad + \varrho^{-m-2}\psi(\Delta p^{m+2})Bp \times \mathscr{R}\big(a/(\Delta p^m)\big)\big)
\end{aligned}$$

$$= \varrho^{-m-2}\psi(\Delta p^m)\times(\psi(p)A\varrho+\psi(p)^2Bp)\mathscr{R}(a/(\Delta p^m))+ +\varrho^{-m-1}\psi(\Delta p^{m+1})B\times\mathscr{R}(a/(\Delta p^{m-1}))$$
$$= \varrho^{-m}\psi(\Delta p^m)\mathscr{R}(a/(\Delta p^m))+B\varrho^{-m-1}\psi(\Delta p^{m+1})\mathscr{R}(a/(\Delta p^{m-1}))$$
$$= \mu(a+(\Delta p^m)). \blacksquare$$

It is well known that $F|_{w+2}[\omega_N]$ is a cusp form in $S_{w+2}(N,\bar{\chi})$ and satisfies $F|_{w+2}[\omega_N]|T_{w+2}(n) = \bar{a}_n F|_{w+2}[\omega_N]$ for all positive $n\in \mathbf{Z}$ with $(n,N)=1$ (see e.g. Proposition 3.57 in Shimura [22]). Set

$$\tilde{\mathbf{P}}^{\pm}_{l,\sigma}(x) = N^{w/2}(c_\sigma^{\pm})^{-1}\Big(\int_0^{i\infty} F^\sigma|_{w+2}[\omega_N](z+x)z^l dz \pm \pm(-1)^{w+l+1}\int_0^{i\infty} F^\sigma|_{w+2}[\omega_N](z-x)z^l dz\Big)$$

for $x\in \mathbf{Q}$, $l\in \mathbf{Z}$ with $0\leqslant l\leqslant w$ and $\sigma\in \mathrm{Aut}(C)$. Changing the variable of the integrations in this formula we obtain easily:

$$\tilde{\mathbf{P}}^{\pm}_{l,\sigma}(x) = N^{w-l-1}(-1)^{w+1}(c_\sigma^{\pm})^{-1}\Big(\int_{-1/(Nx)}^{0} F^\sigma(z)z^{w-l}(1+Nxz)^l dz \pm \pm(-1)^{w+l+1}\int_{1/(Nx)}^{0} F^\sigma(z)z^{w-l}(1-Nxz)^l dz\Big).$$

By this and Lemma 3.18,

$$\tilde{\mathbf{P}}^{\pm}_{l,\sigma}(x) = (\tilde{\mathbf{P}}^{\pm}_{l,1}(x))^\sigma \in Q_F^\sigma. \tag{4.2.3}$$

Multiply the following two formulas by z^l $(0\leqslant l\leqslant w)$ and integrate them from 0 to $i\infty$.

$$a_pF(z) = \sum_{v=0}^{p-1} p^{-1}F((z\pm v)/p)+p^{w+1}\chi(p)F(pz)$$

and

$$\bar{a}_pF|_{w+2}[\omega_N](z) = \sum_{v=0}^{p-1} p^{-1}F|_{w+2}[\omega_N]((z\pm v)/p)+p^{w+1}\overline{\chi(p)}F|_{w+2}[\omega_N](pz).$$

Then we obtain easily:

$$a_p\mathbf{P}^{\pm}_{l,1}(x) = \sum_{v=0}^{p-1} p^l\mathbf{P}^{\pm}_{l,1}((x+v)/p)+p^{w-l}\chi(p)\mathbf{P}^{\pm}_{l,1}(px) \tag{4.2.4}$$

and

$$\bar{a}_p\tilde{\mathbf{P}}^{\pm}_{l,1}(x) = \sum_{v=0}^{p-1} p^l\tilde{\mathbf{P}}^{\pm}_{l,1}((x+v)/p)+p^{w-l}\overline{\chi(p)}\tilde{\mathbf{P}}^{\pm}_{l,1}(px). \tag{4.2.5}$$

We fix an embedding ι of the algebraic closure $\bar{Q}$ $(\supset Q_F)$ of Q into $\bar{Q}_p$ once for all and identify the elements of $\bar{Q}$ with those of $(\bar{Q})^\iota$ by this ι. For each $l\in \mathbf{Z}$ with $0\leqslant l\leqslant w$, let $\mu_l^{\pm}$ (resp. $\tilde{\mu}_l^{\pm}$) be the p-adic measures on $\mathbf{Z}_\Delta^*$, constructed by Proposition 4.1, (4.2.4) and (4.2.5), which are defined by:

$$\mu_l^{\pm}(a+(\Delta p^m)) = \varrho_l^{-m}\mathbf{P}^{\pm}_{l,1}(a/(\Delta p^m))-p^{w-2l}\chi(p)\varrho_l^{-m-1}\mathbf{P}^{\pm}_{l,1}(a/(\Delta p^{m-1})) \quad (\varrho_l^2 = p^{-l}a_p\varrho_l-p^{w+1-2l}\chi(p)), \tag{4.2.6}$$

$$\tilde{\mu}_l^{\pm}(a+(\Delta p^m)) = \tilde{\varrho}_l^{-m}\chi(\Delta p^m)\tilde{\mathbf{P}}^{\pm}_{l,1}(a/(\Delta p^m))-p^{w-2l}\overline{\chi(p)}\tilde{\varrho}_l^{-m-1}\chi(\Delta p^{m+1})\tilde{\mathbf{P}}^{\pm}_{l,1}(a/(\Delta p^{m-1})) \quad (\tilde{\varrho}_l^2 = \bar{a}_p\chi(p)p^{-l}\tilde{\varrho}_l-p^{w+1-2l}\overline{\chi(p)}\chi(p)^2). \tag{4.2.7}$$

In these formulae, we may put $\varrho=\varrho_0$, $\tilde{\varrho}=\tilde{\varrho}_0$, $\varrho_l=p^{-l}\varrho$ and $\tilde{\varrho}_l=p^{-l}\tilde{\varrho}$ for $l\in \mathbf{Z}$ with $0\leqslant l\leqslant w$. The following theorem is a generalization of Theorem 3 in Manin [16].

THEOREM 4.2. *Let l be an integer with $0\leqslant l\leqslant w$, m be a non negative integer and ψ be a primitive Dirichlet character* mod Δp^m. *Then we obtain:*

$$\frac{\Delta p^m}{c_1^{\pm}G(\psi)}\int_0^{i\infty} F\otimes\psi(z)z^l dz = \frac{\varrho^m}{p^{lm}}\int_{\mathbf{Z}_\Delta^*}\psi^{-1}(-a)\,d\mu_l^{\pm}(a) \tag{4.2.1}$$

where the superscripts $\pm$ on $c_1^{\pm}$ and $d\mu_l^{\pm}$ are taken as in the formula $\psi(-1) = \pm(-1)^{l+1}$.

(4.2.2)
$$\frac{\Delta p^m\chi(\Delta p^m)}{c_1^{\pm}G(\psi)}\int_0^{i\infty}((F|_{w+2}[\omega_N])\otimes\psi)(z)z^l dz = \frac{\tilde{\varrho}^m}{N^{w/2}p^{lm}}\int_{\mathbf{Z}_\Delta^*}\psi^{-1}(-a)\,d\tilde{\mu}_l^{\pm}(a)$$

where the superscripts $\pm$ on $c_1^{\pm}$ and $d\tilde{\mu}_l^{\pm}$ are taken as in the formula $\psi(-1) = \pm(-1)^{w+l+1}$.

Proof. The proof goes in a similar way to that of Theorem 3 in Manin [16]. We prove (4.2.2). ((4.2.1) is proved in the same way.) By the proof of Corollary 3.18,

$$\textit{the left side of } (4.2.2) = \frac{\chi(\Delta p^m)}{N^{w/2}}\sum_{b\,(\mathrm{mod}\,\Delta p^m)}\psi^{-1}(-b)\tilde{\mathbf{P}}^{\pm}_{l,1}(b/(\Delta p^m)).$$

On the other hand,

$$\int_{\mathbf{Z}_\Delta^*}\psi^{-1}(-a)\,d\tilde{\mu}_l^{\pm}(a) = \sum_{a\,(\mathrm{mod}\,\Delta p^m)}\psi^{-1}(-a)\tilde{\mu}_l^{\pm}(a+(\Delta p^m))$$
$$= \sum_{a\,(\mathrm{mod}\,\Delta p^m)}\psi^{-1}(-a)\big(\tilde{\varrho}_l^{-m}\chi(\Delta p^m)\tilde{\mathbf{P}}^{\pm}_{l,1}(a/(\Delta p^m)) - -p^{w-2l}\overline{\chi(p)}\tilde{\varrho}_l^{-m-1}\chi(\Delta p^{m+1})\tilde{\mathbf{P}}^{\pm}_{l,1}(a/(\Delta p^{m-1}))\big).$$

Since $\tilde{\mathbf{P}}^{\pm}_{l,1}(x+1) = \tilde{\mathbf{P}}^{\pm}_{l,1}(x)$ and ψ is primitive,

$$\int_{Z_\Delta^*} \psi^{-1}(-a)\, d\tilde{\mu}_l^{\pm}(a) = \sum_{a \,(\mathrm{mod}\, \Delta p^m)} \psi^{-1}(-a)\tilde{\varrho}_l^{-m}\chi(\Delta p^m)\tilde{\mathbf{P}}^{\pm}_{l,1}\big(a/(\Delta p^m)\big)$$

$$= \tilde{\varrho}^{-m}p^{lm}\chi(\Delta p^m)\sum_{a \,(\mathrm{mod}\, \Delta p^m)} \psi^{-1}(-a)\tilde{\mathbf{P}}^{\pm}_{l,1}\big(a/(\Delta p^m)\big). \ \blacksquare$$

In case of $p \nmid N$, we have $\overline{a}_p\chi(p) = a_p$ (see e.g. Proposition 3.56 in Shimura [22]). Hence we may assume $\varrho = \tilde{\varrho}$. The following theorem is a generalization of Theorem 4 and Lemma 8 in Manin [16].

THEOREM 4.3. *Assume* $\varrho = \tilde{\varrho}$ *and* $\mathrm{ord}_p\varrho < 1$.

(i) $d\mu_l^{\pm}(a) = (-\Delta)^{-l}a^l d\mu_0^{\pm}(a)$ $(0 \leqslant l \leqslant w)$.

(ii) $d\tilde{\mu}_l^{\pm}(a) = (-\Delta)^{-l}a^l d\tilde{\mu}_0^{\pm}(a)$ $(0 \leqslant l \leqslant w)$.

(iii) *Let* $\Delta_0 p$ *be prime to* N. *Then we have:*

$$(4.3.1)\qquad d\mu_0^{\pm}\big(-1/(Na)\big) = (-1)^{w+1}a^w d\tilde{\mu}_0^{\pm}(a) \quad \text{on } Z_\Delta^*.$$

Moreover let ψ *be a Dirichlet character:* $Z_\Delta^* \to L(\varrho)^*$. *Then we have:*

$$(4.3.2)\qquad \int_{Z_\Delta^*}\psi^{-1}(-a)\, a^l d\mu_0^{\pm}(a) = (-1)^{w+l+1}\psi(N)N^{-l}\int_{Z_\Delta^*}\psi(a)\,a^{w-l}d\tilde{\mu}_0^{\pm}(a).$$

For the proof of Theorem 4.3, we need:

LEMMA 4.4. *For* $x \in Q$, $l \in Z$ *with* $0 \leqslant l \leqslant w$ *and* $\sigma \in \mathrm{Aut}(C)$, *set*

$$\tilde{H}^{\pm}_{l,\sigma}(x) = N^{w/2}(c_\sigma^{\pm})^{-1}\Big(\int_x^{i\infty} F^\sigma|_{w+2}[\omega_N](z)z^l dz \pm$$
$$\pm(-1)^{l+1+w}\int_{-x}^{i\infty} F^\sigma|_{w+2}[\omega_N](z)z^l dz\Big).$$

Then $\tilde{H}^{\pm}_{l,\sigma}(x)$ *(resp.* $H^{\pm}_{l,\sigma}(x)$ *in Lemma* 3.17) *is contained in the finitely generated* Z*-module*

$$M_\sigma^{\pm} = \sum_{k=0}^{w}\sum_{j=1}^{m} Z\bigg((c_\sigma^{\pm})^{-1}\Big(\int_0^{i\infty} F^\sigma|_{w+2}[g_j](z)z^l dz \pm$$
$$\pm(-1)^{l+1}\int_0^{i\infty} F^\sigma|_{w+2}[\mathrm{t}g_j\mathrm{t}](z)z^l dz\Big)\bigg).$$

Proof. By replacing the variable z of the integrations by $-N^{-1}z^{-1}$, we obtain

$$\tilde{H}^{\pm}_{l,\sigma}(x) = (-1)^{l-w-1}N^{w-l}(c_\sigma^{\pm})^{-1}\Big(\int_0^{-1/(Nx)} F^\sigma(u)u^{w-l}du \pm$$
$$\pm(-1)^{w-l+1}\int_0^{1/(Nx)} F^\sigma(u)u^{w-l}du\Big).$$

Let $-1/(Nx) = b_n/d_n,\ b_{n-1}/d_{n-1}, \ldots, b_1/d_1,\ b_0/d_0 = 0/1$ be the successive convergents obtained by the continued fraction of a rational number $-1/(Nx)$. (We may assume $-1/(Nx) > 0$.) Set

$$\mathbf{g}_k = \begin{pmatrix} b_k & (-1)^{k-1}b_{k-1} \\ d_k & (-1)^{k-1}d_{k-1}\end{pmatrix} \quad (1 \leqslant k \leqslant n).$$

It is well known that all the $\mathbf{g}_k$ are elements of $SL(2, Z)$. Then we have:

$$(4.4.1)\qquad \int_0^{-1/(Nx)} F^\sigma(u)u^{w-l}du \pm (-1)^{w-l+1}\int_0^{1/(Nx)} F^\sigma(u)u^{w-l}du$$
$$= \sum_{k=1}^{n}\Big(\int_{\mathbf{g}_k(0)}^{\mathbf{g}_k(i\infty)} F^\sigma(u)u^{w-l}du \pm (-1)^{w-l+1}\int_{\mathrm{t}\mathbf{g}_k\mathrm{t}(0)}^{\mathrm{t}\mathbf{g}_k\mathrm{t}(i\infty)} F^\sigma(u)u^{w-l}du\Big)$$
$$= \sum_{k=1}^{n}\Big(\int_0^{i\infty} F^\sigma|_{w+2}[\mathbf{g}_k](z)\big(b_k z + (-1)^{k-1}b_{k-1}\big)^{w-l}\big(d_k z + (-1)^{k-1}d_{k-1}\big)^l dz \pm$$
$$\pm(-1)^{w-l+1}\int_0^{i\infty} F^\sigma|_{w+2}[\mathrm{t}\mathbf{g}_k\mathrm{t}](z)\big(b_k z + (-1)^k b_{k-1}\big)^{w-l}\big(-d_k z + (-1)^{k-1}d_{k-1}\big)^l dz\Big).$$

Here use the binomial expansions. Note that

$$\big((-1)^{k-1}\big)^{w-l-i}\big((-1)^{k-1}\big)^{l-j}\big((-1)^k\big)^{w-l-i}(-1)^j\big((-1)^{k-1}\big)^{l-j}(-1)^{w-l+1} = (-1)^{i+j+1}.$$

Hence the left side of (4.4.1) is contained in the module $(c_\sigma^{\pm})M_\sigma^{\pm}$. $\blacksquare$

Define an equivalence relation $\sim$ between the sequences $\{\{A_n\}_{n=0}^{\infty} \mid A_n \in \overline{Q}_p\}$ by:

$$A_n \sim B_n \Leftrightarrow \mathrm{ord}_p(A_m - B_m) \to +\infty \quad \text{when} \quad m \to +\infty.$$

Proof of Theorem 4.3 (ii). (The proof of Theorem 4.3 (i) is almost the same as this. We omit it.) It is easy to see

$$\tilde{\mathbf{P}}^{\pm}_{l,\sigma}(x) = \sum_{j=0}^{l}\binom{l}{j}(-x)^{l-j}\tilde{H}_{j,\sigma}(x).$$

We compute as follows.

$$\tilde{\mu}_l^{\pm}\big(a + (\Delta p^m)\big) = \varrho^{-m}p^{lm}\chi(\Delta p^m)\tilde{\mathbf{P}}^{\pm}_{l,1}\big(a/(\Delta p^m)\big) -$$
$$-p^{w-2l}\overline{\chi(p)}\,\varrho^{-m-1}p^{l(m+1)}\chi(\Delta p^{m+1})\tilde{\mathbf{P}}^{\pm}_{l,1}\big(a/(\Delta p^{m-1})\big)$$
$$= \sum_{j=0}^{l}\binom{l}{j}\varrho^{-m}p^{lm}\chi(\Delta p^m)\big(-a/(\Delta p^m)\big)^{l-j}\tilde{H}^{\pm}_{j,1}\big(a/(\Delta p^m)\big) -$$
$$-\sum_{j=0}^{l}\binom{l}{j}p^{w-2l}\overline{\chi(p)}\,\varrho^{-m-1}p^{l(m+1)}\chi(\Delta p^{m+1})\big(-a/(\Delta p^{m-1})\big)^{l-j}\tilde{H}^{\pm}_{j,1}\big(a/(\Delta p^{m-1})\big).$$

By Lemma 4.4 and the assumption $\mathrm{ord}_p \varrho < 1$, for $j \in \mathbf{Z}$ with $1 \leqslant j \leqslant l$,

$$\mathrm{ord}_p\Big(\varrho^{-m} p^{lm} \chi(\Delta p^m)\big(-a/(\Delta p^m)\big)^{l-j} \tilde{H}^{\pm}_{j,1}\big(a/(\Delta p^m)\big)\Big) \to +\infty$$

and

$$\mathrm{ord}_p\Big(\varrho^{-m-1} p^{l(m+1)} \chi(\Delta p^{m+1})\big(-a/(\Delta p^{m-1})\big)^{l-j} \tilde{H}^{\pm}_{j,1}\big(a/(\Delta p^{m-1})\big)\Big) \to +\infty$$

when $m \to +\infty$.

Hence we obtain

$$\begin{aligned}\tilde{\mu}^{\pm}_l\big(a+(\Delta p^m)\big) \sim\ & \varrho^{-m}\chi(\Delta p^m)\tilde{H}^{\pm}_{0,1}\big(a/(\Delta p^m)\big)(-a)^l \Delta^{-l} - \\ & -p^w \overline{\chi(p)}\, \varrho^{-m-1}\chi(\Delta p^{m+1})\tilde{H}^{\pm}_{0,1}\big(a/(\Delta p^{m-1})\big)(-a)^l \Delta^{-l} \\ =\ & \tilde{\mu}^{\pm}_0\big(a+(\Delta p^m)\big) \times (-a)^l \Delta^{-l}\end{aligned}$$

since we have $\tilde{H}^{\pm}_{0,1}(y) = \tilde{\mathbf{P}}^{\pm}_{0,1}(y)$ $(y \in \mathbf{Q})$. Namely we have

$$d\tilde{\mu}^{\pm}_l(a) = (-\Delta)^{-l} a^l d\tilde{\mu}^{\pm}_0(a). \ \blacksquare$$

Proof of Theorem 4.3 (iii). (This proof goes in a similar way to that of Theorem 4 in Manin [16].) Let a and a' be rational integers with $Naa' \equiv -1 \pmod{\Delta p^m}$. There is an integer U such that $Naa' - U\Delta p^m = -1$. Put $\mathbf{g} = \begin{pmatrix} U & a' \\ Na & \Delta p^m \end{pmatrix}$, which is an element of the group $\Gamma_0(N)$. We compare $\mu^{\pm}_0\big(a'+(\Delta p^m)\big)$ with $\tilde{\mu}^{\pm}_0\big(a+(\Delta p^m)\big)$ and show that

$$\mathrm{ord}_p\Big(\mu^{\pm}_0\big(a'+(\Delta p^m)\big) - (-1)^{w+1} a^w \tilde{\mu}^{\pm}_0\big(a+(\Delta p^m)\big)\Big) \to +\infty$$

when $m \to +\infty$. We have

$$\text{(4.3.3)}\quad \begin{aligned} c^{\pm}_1 \mathbf{P}^{\pm}_{0,1}\big(a'/(\Delta p^m)\big) &= \int\limits_{a'/(\Delta p^m)}^{i\infty} F(z)\,dz \mp \int\limits_{-a'/(\Delta p^m)}^{i\infty} F(z)\,dz \\ &= \int\limits_{\mathbf{g}(0)}^{\mathbf{g}(-\Delta p^m a^{-1} N^{-1})} F(z)\,dz \mp \int\limits_{\mathbf{tgt}(0)}^{\mathbf{tgt}(\Delta p^m a^{-1} N^{-1})} F(z)\,dz \\ &= \int\limits_0^{-\Delta p^m a^{-1} N^{-1}} (F|_{w+2}[\mathbf{g}])(z)(Naz+\Delta p^m)^w dz \mp \\ &\qquad \mp \int\limits_0^{\Delta p^m a^{-1} N^{-1}} (F|_{w+2}[\mathbf{tgt}])(z)(-Naz+\Delta p^m)^w dz \\ &\sim \int\limits_0^{-\Delta p^m a^{-1} N^{-1}} \chi(\Delta p^m) F(z) N^w a^w z^w dz \mp \\ &\qquad \mp (-1)^w \int\limits_0^{\Delta p^m a^{-1} N^{-1}} \chi(\Delta p^m) F(z) N^w a^w z^w dz. \end{aligned}$$

Here replace the variable z of the integrations by $-N^{-1}z^{-1}$. We have:

(4.3.4) *the right side of* (4.3.3)

$$\begin{aligned} &= (-1)^{w+1} N^{w/2} a^w \chi(\Delta p^m)\Big(\int\limits_{a\Delta^{-1}p^{-m}}^{i\infty} (F|_{w+2}[\omega_N])(z)\,dz \pm \\ &\qquad \pm (-1)^{w+1} \int\limits_{-a\Delta^{-1}p^{-m}}^{i\infty} (F|_{w+2}[\omega_N])(z)\,dz\Big) \\ &= (-1)^{w+1} a^w \chi(\Delta p^m) \tilde{\mathbf{P}}^{\pm}_{0,1}\big(a/(\Delta p^m)\big) c^{\pm}_1. \end{aligned}$$

From (4.3.4), we obtain:

$$\text{(4.3.5)}\quad \begin{aligned} \mu^{\pm}_0\big(a'+(\Delta p^m)\big) &= \varrho^{-m}\mathbf{P}^{\pm}_{0,1}\big(a'/(\Delta p^m)\big) - \varrho^{-m-1} p^w \chi(p) \mathbf{P}^{\pm}_{0,1}\big(a'/(\Delta p^{m-1})\big) \\ &\sim (-1)^{w+1} a^w \Big(\varrho^{-m}\chi(\Delta p^m)\tilde{\mathbf{P}}^{\pm}_{0,1}\big(a/(\Delta p^m)\big) - \\ &\qquad - \varrho^{-m-1} p^w \chi(\Delta p^m) \tilde{\mathbf{P}}^{\pm}_{0,1}\big(a/(\Delta p^{m-1})\big)\Big) \\ &= (-1)^{w+1} a^w \tilde{\mu}^{\pm}_0\big(a+(\Delta p^m)\big). \end{aligned}$$

(4.3.5) implies (4.3.1). By changing the variable of the integration of the left side of (4.3.2) by $-1/(Na)$, we obtain from (4.3.1):

$$\begin{aligned} \int\limits_{\mathbf{Z}^*_\Delta} \psi^{-1}(-a) a^k d\mu^{\pm}_0(a) &= \int\limits_{\mathbf{Z}^*_\Delta} \psi^{-1}(N^{-1}a^{-1})(-N^{-1}a^{-1})^k d\mu^{\pm}_0(-N^{-1}a^{-1}) \\ &= (-1)^{w+k+1}\psi(N) N^{-k} \int\limits_{\mathbf{Z}^*_\Delta} \psi(a) a^{w-k} d\tilde{\mu}^{\pm}_0(a). \ \blacksquare \end{aligned}$$

Theorem 4.3 is regarded as a p-adic analogue for the usual complex functional equation satisfied by the zeta functions associated with the functions F and $F|_{w+2}[\omega_N]$ (cf. Hecke [9], Shimura [22]).

Note added in proof. We explain precisely the method printed in the parentheses at lines 6 and 7, page 24 of this paper. (i) Replace $\mathfrak{Z}^{(x)}$ by $\mathfrak{Z}^{(x)} - \mathfrak{B}_x$ for each $x \in \mathbf{Z}$ with $1 \leqslant x \leqslant r-1$. Then $\mathfrak{B}_x$ (resp. $\mathfrak{B}_{r-1}$) is changed into $\varrho_w(\tau)\mathfrak{B}_{x+1}$ (resp. $\mathbf{0}$) for each $x \in \mathbf{Z}$ with $1 \leqslant x \leqslant r-2$. (ii) Carry out (i) inductively $(r-1)$ times. [Hence, for each $x \in \mathbf{Z}$ with $1 \leqslant x \leqslant r-1$, the original $\mathfrak{Z}^{(x)}$ (resp. $\mathfrak{B}_x$) is replaced by $\mathfrak{Z}^{(x)} - \sum_{j=x}^{r-1} \varrho_w(\tau)^{j-x}\mathfrak{B}_j$ (resp. $\mathbf{0}$).]

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Received on 1.6.1977
and in revised form on 26.1.1979 (949)

ACTA ARITHMETICA
XL (1981)

An additive problem in the theory of numbers

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1. Introduction. Vinogradov (cf. [5]) proved that every sufficiently large odd integer N can be written as

$$N = p^{(1)}+p^{(2)}+p^{(3)},$$

where $p^{(i)}$'s are odd primes. Here we shall prove

THEOREM. *Let k be an integer $\geqslant 2$. Let $\delta_1, \delta_2, \ldots, \delta_k$ be positive numbers satisfying $\delta_1+\delta_2+\ldots+\delta_k = 1$. Then every sufficiently large odd integer N can be written as*

$$N = n^{(1)}+n^{(2)}+n^{(3)},$$

where $n^{(i)} = p_1^{(i)}p_2^{(i)}\ldots p_k^{(i)}$ with some odd primes $p_j^{(i)}$'s satisfying $p_j^{(i)} \leqslant N^{\delta_j}$ for $j = 1, 2, \ldots, k$ and for $i = 1, 2, 3$.

In fact, we shall prove using Hardy–Littlewood's circle method

$$\sum_{N=n^{(1)}+n^{(2)}+n^{(3)}} \Big(\prod_{i=1}^{3}\prod_{j=1}^{k}\log p_j^{(i)}\Big) = \frac{1}{((k-1)!)^3}\mathfrak{S}(N)\tilde{r}_k(N)+O\big(N^2(\log N)^{-A}\big),$$

where

$$\mathfrak{S}(N) = \prod_{p\nmid N}\Big(1+\frac{1}{(p-1)^3}\Big)\prod_{p\mid N}\Big(1-\frac{1}{(p-1)^2}\Big),$$

$$\tilde{r}_k(N) = \sum_{N=h_1+h_2+h_3}\Big(\log\frac{N}{h_1}\Big)^{k-1}\Big(\log\frac{N}{h_2}\Big)^{k-1}\Big(\log\frac{N}{h_3}\Big)^{k-1},$$

p runs over primes, h_j's are positive integers and A is a sufficiently large constant. We remark that there are smaller N's which cannot be written as in our theorem.