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On the Möbius function

by

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1. Let $\mu(n)$ denote the Möbius function: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k different primes, $\mu(n) = 0$ if n contains any factor to a power higher than the first. The well-known connection with the Riemann ζ -function is the following (see e.g. [2], p. 3, (1.1.4)):

$$(1.1) \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1.$$

Write

$$M(x) = \sum_{n \leq x} \mu(n).$$

Here the most interesting question is that of the behaviour of $\max_{1 \leq n \leq T} |M(n)|$ as $T \rightarrow \infty$. This problem has been studied by many mathematicians. It is known at present that

$$M(x) = O(x \exp(-c_1 \sqrt{\log x})) \quad (*)$$

and even slightly better estimates have been obtained.

It has been proved by Littlewood (see e.g. [1], p. 161) that the relation

$$M(x) = O(x^{1/2+\varepsilon}) \quad \text{for every } \varepsilon > 0$$

is equivalent to the truth of the Riemann hypothesis.

Some conjectures, in connection with the subject, should be noted. The Mertens hypothesis

$$(1.2) \quad |M(n)| < \sqrt{n} \quad \text{for } n > 1$$

has not been proved or disproved yet (see [2], p. 320). Also slightly less drastic conjectures:

$$(1.3) \quad M(x) = O(x^{1/2})$$

(*) See e.g. [1], p. 157. Theorem 478. Throughout this paper c_1, c_2, \dots denote numerical positive constants.

and

$$(1.4) \quad \int_1^T \left(\frac{M(x)}{x} \right)^2 dx = O(\log T)$$

still remain unproved, but it has been revealed that they imply interesting and deep results in the theory of ζ -zeros.

All this has been concerned with the upper estimation of $\max_{1 \leq x \leq T} |M(x)|$.

The corresponding problem of the lower estimation is naturally more difficult. Up to the present time only ineffective inequalities of Ω -type are known in this direction:

(i) If Θ is the upper bound of the real parts of complex ζ -zeros, then

$$M(x) = \Omega(x^{\theta-\varepsilon}) \quad \text{for every } \varepsilon > 0.$$

$$(ii) \quad M(x) = \Omega(x^{1/2}).$$

The proof of (i) is a simple application of the formula

$$\frac{1}{\zeta(s)} = s \int_1^\infty \frac{M(u)}{u^{s+1}} du \quad (\text{see [1], p. 159, (633)}),$$

for the proof of (ii) see e.g. [1] (p. 162, see also [2], p. 317, Theorem 14.26 (B)).

It seems to me a very interesting and important problem to find some numerical lower estimate of $\max_{1 \leq x \leq T} |M(x)|$ for all sufficiently large T . The problem has a certain similarity to that of the remainder term in the prime number formula. Let

$$\psi(x) = \sum_{n \leq x} \Lambda(n), \quad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, p - \text{prime}, \\ 0 & \text{otherwise.} \end{cases}$$

The relation $\psi(x) \sim x$ is the prime number theorem and $\psi(x) - x$ is the remainder term in the prime number formula. In this case Turán has proved (see [3], p. 111):

$$(1.5) \quad \max_{1 \leq x \leq T} |\psi(x) - x| > T^{\beta_0} \exp \left(-21 \frac{\log T}{\sqrt{\log \log T}} \right)$$

if $\varrho_0 = \beta_0 + i\gamma_0$, $\beta_0 \geq \frac{1}{2}$ is any complex zero of $\zeta(s)$ and

$$T > \max(c_2, \exp(e^{60 \log^2 \varrho_0})) \quad (c_2 \text{ an explicit numerical constant}).$$

A full analogue of Turán's estimate (1.5) is to be expected for $M(x)$ and my first aim was to prove it. It has turned out, however, that the problem is very difficult and so I have made only the first step in this

direction. The main cause of the difficulties is the problem of the multiplicity of the complex ζ -zeros. The analytic function connected with the number-theoretical one ($\psi(x) - x$) is $-(\zeta(s) + \frac{\zeta'}{\zeta}(s))$. The singularities of this function are simple poles at the ζ -zeros. With the function $M(x)$, however, the situation is quite different. The connected analytic function is here $1/\zeta(s)$, the singularities are also at the ζ -zeros, but now the difficulties with non-simple zeros appear. I need here some conjecture giving the simplicity of all ζ -zeros. Further, some information on the horizontal distribution of ζ -zeros has proved to be necessary. All in all I shall prove an analogue of (1.5) under the assumption of (1.4), the result being the content of this paper.

2. THEOREM. If

$$(2.1) \quad \int_1^T \left(\frac{M(x)}{x} \right)^2 dx < a \log T \quad \text{for } T \geq 1, a \text{ being independent of } T$$

then

$$(2.2) \quad \max_{1 \leq x \leq T} |M(x)| > T^{1/2} \exp \left(- \frac{\log T}{\sqrt{\log \log T}} \right)$$

for

$$(2.3) \quad T > \max(c_0, \exp(300a))$$

where c_0 is a numerical, explicitly calculable constant.

Remark 1. It is worth noting that (2.1) is in a certain sense an upper estimate of $M(x)$; thus if it is not true, then this fact gives a „lower estimate” on $M(x)$. In particular if (2.1) was false, then the Mertens hypothesis (1.2) and the relation (1.3) would be too.

The essential tool of the proof is the following lemma due to Turán (see [3], p. 56, Corollary 1):

Let z_1, z_2, \dots, z_M be complex numbers such that

$$\max_{1 \leq j \leq M} |z_j| = 1, \quad \min_{\mu \neq \nu} |z_\mu - z_\nu| \geq \delta, \quad z_j \neq 0, j = 1, 2, \dots, M.$$

If m is a non-negative integer, then there exists an integer r such that

$$m+1 \leq r \leq m+M, \quad \frac{|b_1 z_1^r + b_2 z_2^r + \dots + b_M z_M^r|}{\sum_{j=1}^M |b_j| |z_j|^r} \geq \frac{\delta^{M-1}}{M^{2M}}.$$

3. Before the proof of the proper theorem I wish to examine the consequences of (2.1) with respect to a , in order to obtain the inequality (2.3).

First of all one could easily show that (2.1) implies the truth of the Riemann hypothesis (compare [1], p. 154, Theorem 477). Thus, in the proof we can use the following lemmas:

(A) For every $T \geq c_3$ there exists t_T , $T < t_T < T+1$, such that

$$\left| \frac{1}{\zeta(\sigma+it_T)} \right| < T^{c_4} \quad \text{for } -1 \leq \sigma \leq 2 \quad (\text{see [2], p. 303, Theorem 14.16}).$$

Besides:

$$\left| \frac{1}{\zeta(-1+it)} \right| \leq \frac{c_5}{(|t|+1)^{3/2}}, \quad -\infty < t < +\infty \quad (2).$$

(B) All ζ -zeros are simple (see [2], p. 322, Theorem 14.29 (A)).

Further, the following lemma holds:

(C) Let ρ_1, ρ_2 denote different non-trivial ζ -zeros. Then

$$|\rho_1 - \rho_2| \geq 1/15\sqrt{a}(\max_{i=1,2} |\rho_i|)^4$$

This lemma — and even a stronger one — has been proved in [2], (p. 326, Theorem 14.31), but without the explicit dependence on a . Therefore I will repeat here some details of this proof in order to obtain an explicit form.

First of all, the inequality

$$(3.1) \quad |1/\zeta'(\rho)| \leq \sqrt{a}|\rho|, \quad \rho \text{ being any complex } \zeta\text{-zero},$$

follows from (2.1) analogously to [2].

Now let $\frac{1}{2}+iy, \frac{1}{2}+iy'$ be consecutive complex ζ -zeros, $0 < y < y'$. Then

$$O = \int_y^{y'} \zeta'(\frac{1}{2}+it) dt = (\gamma' - \gamma) \zeta'(\frac{1}{2}+iy) + i \int_y^{\gamma'} (\gamma' - t) \zeta''(\frac{1}{2}+it) dt$$

and by (3.1)

$$(\gamma' - \gamma)/\gamma < 2\sqrt{a} \left| \int_y^{\gamma'} (\gamma' - t) \zeta''(\frac{1}{2}+it) dt \right| < \sqrt{a}(\gamma' - \gamma)^2 \max_{\gamma \leq t \leq \gamma'} |\zeta''(\frac{1}{2}+it)|.$$

Using this, and using further the inequality

$$|\zeta''(\frac{1}{2}+it)| \leq \frac{1}{\pi r^2} \int_0^{2\pi} |\zeta(\frac{1}{2}+it+re^{i\theta})| d\theta$$

together with a trivial one,

$$|\zeta(s)| \leq 7,5t^{3/4} \quad \text{for } \sigma \geq \frac{1}{4}, t \geq 14,$$

(2) From the functional equation for $\zeta(s)$.

we obtain, taking $r = 1/t$

$$\max_{\gamma \leq t \leq \gamma'} |\zeta''(\frac{1}{2}+it)| \leq 15\gamma'^3$$

and the result follows.

4. Turning now to the proof of the Theorem, assume $T \geq c_6$ such that all further inequalities, holding for large T , are satisfied.

Put

$$K_0 = 10 \frac{\log T}{\log \log T}, \quad N_0 = \log^{1/10} T \cdot (\log \log T)^2 \quad (\text{thus } K_0 > N_0 \geq 3).$$

There exists a number $L > 2$ such that

$$L^{K_0} < L^{K_0+N_0} \leq T < L^{K_0+N_0+1} < L^{3K_0}.$$

Hence

$$c_3 < \log^{1/30} T \leq L \leq \log^{1/10} T.$$

Let t_L be the number given by (A), then

$$L < t_L < L+1.$$

Let k be an integer satisfying the inequalities:

$$K_0 < k+1 \leq K_0+N_0 \quad (< 2K_0).$$

Put further

$$\eta = 1/\log L^{k+1}.$$

Consider the integral

$$J(T) = \frac{1}{2\pi i} \int_{1+\eta-iL}^{1+\eta+iL} \frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)} ds.$$

In virtue of (1.1)

$$\begin{aligned} J(T) &= \sum_{n=1}^{\infty} \mu(n) \int_{1+\eta-iL}^{1+\eta+iL} \frac{(T/n)^s}{s^{k+1}} ds \\ &= \sum_{n \leq T} \frac{\mu(n)}{k!} \log^k \frac{T}{n} + O\left(\sum_{n=1}^{\infty} \frac{T}{k} \cdot \frac{1}{n^{1+\eta}} \cdot \frac{1}{t_L^k}\right). \end{aligned}$$

Further

$$L^{k+1} > L^{K_0} \Rightarrow L^{K_0+N_0+1}/L^{N_0+1} > T \exp(-\frac{1}{2}\sqrt{\log T})$$

and

$$\sum_{n=1}^{\infty} \frac{T}{k} \cdot \frac{1}{n^{1+\eta}} \cdot \frac{1}{t_L^k} = O(\exp(\sqrt{\log T})).$$

Hence

$$|J(T)| \leq \left| \sum_{n \leq T} \frac{\mu(n)}{k!} \log^k \frac{T}{n} \right| + c_7 \exp(\sqrt{\log T}).$$

By partial summation we obtain

$$|J(T)| \leq c_7 \exp(\sqrt{\log T}) + \frac{\log^k T}{k!} \max_{1 \leq x \leq T} |M(x)|.$$

5. In order to obtain the lower estimation of $|J(T)|$ consider the rectangle $1+\eta \pm it_L, -1 \pm it_L$ and apply Cauchy's theorem for the function

$$\frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)}.$$

Let ϱ be any complex ζ -zero. By (B)

$$\operatorname{res}_{s=\varrho} \frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)} = \frac{T^\varrho}{\varrho^{k+1}} \cdot \frac{1}{\zeta'(\varrho)}.$$

Let (I), (II) denote the horizontal segments of the rectangle and (III) its left side. In virtue of (A) we obtain

$$\left| \frac{1}{2\pi i} \int_{(I)} \frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)} ds \right| \leq \frac{1}{2\pi} \int_{-1}^{1+\eta} \frac{T^\sigma}{t_L^{k+1}} t_L^{\sigma_4} d\sigma \leq c_8 \exp(\sqrt{\log T})$$

and similarly

$$\left| \frac{1}{2\pi i} \int_{(II)} \frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)} ds \right| \leq c_8 \exp(\sqrt{\log T}).$$

Further

$$\left| \frac{1}{2\pi i} \int_{(III)} \frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)} ds \right| \leq c_5 \frac{T^{-1}}{2\pi} \int_{-\log T}^{\log T} \frac{(|t|+1)^{-3/2}}{(t^2+1)^{(k+1)/2}} dt \leq c_9 \cdot \frac{1}{T}.$$

Cauchy's theorem then gives

$$(5.1) \quad J(T) = \sum_{|\varrho| < t_L} \frac{T^\varrho}{\varrho^{k+1}} \cdot \frac{1}{\zeta'(\varrho)} + \operatorname{res}_{s=0} \frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)} + O(\exp(\sqrt{\log T})).$$

Clearly (compare [3], p. 117)

$$(5.2) \quad \left| \operatorname{res}_{s=0} \frac{T^s}{s^{k+1}} \cdot \frac{1}{\zeta(s)} \right| = \left| \frac{1}{k!} \cdot \frac{d^k}{ds^k} \left(T^s \frac{1}{\zeta(s)} \right) \Big|_{s=0} \right| < \exp \left(\frac{\log T}{\sqrt{\log \log T}} \right).$$

(5.1), (5.2) and (4.1) give

$$(5.3) \quad \frac{\log^k T}{k!} \max_{1 \leq x \leq T} |M(x)| \geq \left| \sum_{|\varrho| < t_L} \frac{T^\varrho}{\varrho^{k+1}} \cdot \frac{1}{\zeta'(\varrho)} \right| - c_{10} \exp \left(\frac{\log T}{\sqrt{\log \log T}} \right).$$

6. Now we shall estimate

$$\sum_{|\varrho| < t_L} \frac{T^\varrho}{\varrho^{k+1}} \cdot \frac{1}{\zeta'(\varrho)}.$$

Let ϱ_1 be ζ -zero of the least positive ordinate (it is known that $\varrho_1 = \frac{1}{2} + i \cdot 14, 13 \dots$ (see [2], p. 330)).

Write

$$Z = \sum_{|\varrho| < t_L} \frac{T^{\varrho-\varrho_1}}{\zeta'(\varrho)} \cdot \left(\frac{\varrho_1}{\varrho} \right)^{k+1},$$

whence

$$(6.1) \quad \sum_{|\varrho| < t_L} \frac{T^\varrho}{\varrho^{k+1}} \cdot \frac{1}{\zeta'(\varrho)} = \frac{T^{\varrho_1}}{\varrho_1^{k+1}} Z.$$

Denoting by $\varrho_1, \varrho_2, \dots, \varrho_M$ all ζ -zeros lying in the rectangle $|\varrho| < t_L$, $0 < \sigma < 1$, put

$$z_\mu = \frac{\varrho_1}{\varrho_\mu}, \quad b_\mu = \frac{T^{\varrho_\mu-\varrho_1}}{\zeta'(\varrho_\mu)}, \quad \mu = 1, 2, \dots, M, \quad m = [K_0]$$

and apply Turán's lemma.

In virtue of (C) and using (2.3) we obviously obtain for $\mu \neq \nu$

$$|z_\mu - z_\nu| = \frac{|\varrho_1|}{|\varrho_\mu \varrho_\nu|} \cdot |\varrho_\mu - \varrho_\nu| > \frac{1}{15\sqrt{a} t_L^6} > \frac{1}{16\sqrt{a} \log^{6/10} T} \geq \frac{1}{\log^{11/10} T}.$$

Further

$$M \leq \log^{1/10} T (\log \log T)^2.$$

By a suitable choice of k we obtain

$$\begin{aligned} |Z| &\geq \left(\sum_{j=1}^M |b_j| |z_j|^{k+1} \right) \cdot \frac{1}{2^M M (\log T)^{11M/10}} \\ &\geq |b_1| \log T / \exp \left(\frac{11}{10} \log^{1/10} T (\log \log T)^2 (\log 2 + \log \log T) \right) \\ &\geq 1/\exp(\sqrt{\log T}) \quad (\text{clearly } |b_1| = |1/\zeta'(\varrho_1)| = c_{11}). \end{aligned}$$

By (6.1)

$$\begin{aligned} (6.2) \quad \left| \sum_{|\varrho| < t_L} \frac{T^\varrho}{\varrho^{k+1}} \cdot \frac{1}{\zeta'(\varrho)} \right| &\geq \frac{T^{1/2}}{20^{k+1}} \exp(-\sqrt{\log T}) \\ &> T^{1/2} \exp \left(-c_{12} \frac{\log T}{\log \log T} \right). \end{aligned}$$

Further

$$(6.3) \quad \frac{\log^k T}{k!} < \exp \left(\frac{1}{2} \cdot \frac{\log T}{\sqrt{\log \log T}} \right) \quad (\text{compare [3], p. 119}).$$

(5.3), (6.2) and (6.3) give

$$\max_{1 \leq x \leq T} |M(x)| > T^{1/2} \exp\left(-\frac{\log T}{\sqrt{\log \log T}}\right).$$

Remark 2. Note that the Theorem gives some lower estimates in the theory of Farey fractions. Let $r_v^{(n)}$, $1 \leq v \leq A(n)$, denote the v th Farey fraction for the number n , $A(n) = \sum_{q=1}^N \varphi(q)$, $\delta_v^{(n)} = r_v^{(n)} - v/A(n)$.

Then we have

Either $\sum_{v=1}^{A(N)} |\delta_v^{(N)}| = O(\sqrt{N})$ is false, or

$$\max_{1 \leq n \leq N} \sum_{v=1}^{A(n)} |\delta_v^{(n)}| \geq \frac{1}{2\pi} N^{1/2} \exp\left(-\frac{\log N}{\sqrt{\log \log N}}\right)$$

(for N sufficiently large) is true.

For the proof it suffices to apply the inequality

$$|M(n)| \leq 2\pi \sum_{v=1}^{A(n)} |\delta_v^{(n)}| \quad (\text{see [1], p.169].})$$

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Quadratische Formen und Modulfunktionen

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Einleitung

Zu den ältesten Themen der analytischen Zahlentheorie gehört die Frage nach der Anzahl der Darstellungen von natürlichen Zahlen durch definite quadratische Formen und speziell die Behandlung dieser Frage mit Hilfe von Modulfunktionen. Ungeachtet der langen Geschichte, die dieses Thema gehabt hat, verdankt man die am weitesten reichenden Entdeckungen über die Zusammenhänge von Modulfunktionen und definiten quadratischen Formen den Arbeiten von E. Hecke der Jahre 1937-1941, auf deren zusammenfassende Darstellung [9] wir hier mehrfach Bezug nehmen müssen. Hecke zeigte, daß sich die fraglichen Darstellungsanzahlen aus den Koeffizienten gewisser Darstellungen der seit langem wohlbekannten Modularkorrespondenzen berechnen lassen. Durch diese Tatsache wird die Theorie der quadratischen Formen mit der Algebra der algebraischen Funktionenkörper verbunden, was unter den Händen Heckes und seiner Schülern zu zahlreichen schönen Anwendungen führte.

In den vergangenen 3 Jahren habe ich die Spuren der genannten Darstellungen der Modularkorrespondenzen unter gewissen vereinfachten Voraussetzungen bestimmt. Mit dieser Kenntnis hat man jetzt ein sicher arbeitendes Verfahren in der Hand, die Anzahl der Darstellungen einer Zahl durch eine definite quadratische Form formelmäßig zu berechnen, falls die Form nicht allzu kompliziert ist. Es wäre aber ein Irrtum zu glauben, daß hiermit das angeschnittene Problem dem endgültigen Abschluß nahe gebracht worden wäre. Ich weise in § 4 vielmehr auf 7 offene Probleme hin, die teilweise recht umfangreich sind, und deren Bearbeitung mit den heutigen Hilfsmitteln möglich erscheint.

Der Zweck dieser Abhandlung ist es, die Ergebnisse früherer Arbeiten zusammenzutragen, sie zu erläutern und zu ergänzen. Auf eine ausführliche Wiederholung der Beweise darf verzichtet werden. In § 1 bringen wir nur einen neuen und allgemeiner gültigen Beweis für die Existenz des Normalintegraden 3. Gattung.