On the Möbius function

by

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1. Let $\mu(n)$ denote the Möbius function: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n$ is the product of $k$ different primes, $\mu(n) = 0$ if $n$ contains any factor to a power higher than the first. The well-known connection with the Riemann $\zeta$-function is the following (see e.g. [2], p. 3, (1.1.4)):

\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1.
\]

Write

\[
M(s) = \sum_{n=1}^{\infty} \mu(n).
\]

Here the most interesting question is that of the behaviour of $\max |M(n)|$ as $T \to \infty$. This problem has been studied by many mathematicians. It is known at present that

\[
M(n) = O(n \exp(-n^{\epsilon}\log x))
\]

and even slightly better estimates have been obtained.

It has been proved by Littlewood (see e.g. [1], p. 161) that the relation

\[
M(x) = O(x^{2\epsilon+\epsilon}) \quad \text{for every} \quad \epsilon > 0
\]

is equivalent to the truth of the Riemann hypothesis.

Some conjectures, in connection with the subject, should be noted. The Mertens hypothesis

\[
|M(n)| < \sqrt{n} \quad \text{for} \quad n > 1
\]

has not been proved or disproved yet (see [2], p. 320). Also slightly less drastic conjectures:

\[
M(x) = O(x^{2\epsilon})
\]

(1) See e.g. [1], p. 157, Theorem 478. Throughout this paper $\epsilon_1, \epsilon_2, \ldots$ denote numerical positive constants.
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and

\[ \int_{1}^{T} \left( \frac{M(x)}{x} \right)^{2} dx = O(\log T) \]

still remain unproved, but it has been revealed that they imply interesting and deep results in the theory of \( \zeta \)-zeros.

All this has been concerned with the upper estimation of \( \max (M(x)) \), but the lower estimation is naturally more difficult. Up to the present time only ineffectual inequalities of \( \Omega \)-type are known in this direction:

(i) If \( \Theta \) is the upper bound of the real parts of complex \( \zeta \)-zeros, then

\[ M(x) = \Omega(x^{\frac{1}{\Theta}}} \]

for every \( x > 0 \).

(ii) \( M(x) = \Omega(x^{\frac{1}{2}}) \).

The proof of (i) is a simple application of the formula

\[ \frac{1}{\zeta(s)} = \int_{1}^{\infty} \frac{M(u)}{u^{1+it}} du \]

(see [1], p. 159, (633)),

for the proof of (ii) see e.g. [1] (p. 162, see also [2], p. 317, Theorem 14.26 (B)).

It seems to me a very interesting and important problem to find some numerical lower estimate of \( \max (M(x)) \) for all sufficiently large \( x \). The problem has a certain similarity to that of the remainder term in the prime number formula.

Let

\[ \psi(x) = \sum_{n \leq x} \Lambda(n), \quad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^{n}, \quad p \text{ is prime}, \\ 0 & \text{otherwise}. \end{cases} \]

The relation \( \psi(x) \sim x \) is the prime number theorem and \( \psi(x) - x \) is the remainder term in the prime number formula. In this case Turán has proved (see [3], p. 111):

\[ \max_{1 \leq t \leq T} |\psi(x) - x| > T^{\theta_{0}} \exp \left( -\frac{1}{2\log T} \right) \]

and

\[ e_{0} = \beta_{0} + i\gamma_{0}, \quad \beta_{0} > \frac{1}{2} \]

is any complex zero of \( \zeta(s) \) and

\[ T > \max \left\{ \epsilon_{0}, \exp \left( e^{i\gamma_{0}} \right) \right\}, \quad (\epsilon_{0} \text{ an explicit numerical constant}). \]

A full analogue of Turán's estimate (1.5) is to be expected for \( M(x) \) and my first aim was to prove it. It has turned out, however, that the problem is very difficult and so I have made only the first step in this direction. The main cause of the difficulties is the problem of the multiplicity of the complex \( \zeta \)-zeros. The analytic function connected with the number-theoretical one \( (\psi(x) - x) \) is \( \frac{1}{\zeta(s)} \). The singularities of this function are simple poles at the \( \zeta \)-zeros. With the function \( M(x) \), however, the situation is quite different. The connected analytic function is here \( \frac{1}{\zeta(s)} \). The singularities are also at \( \zeta \)-zeros, but now the difficulties with non-simple zeros appear. I need here some conjectures giving the simplicity of all \( \zeta \)-zeros. Further, some information on the horizontal distribution of \( \zeta \)-zeros has proved to be necessary. All in all I shall prove an analogue of (1.5) under the assumption of (1.4), the result being the content of this paper.

2. Theorem. If

\[ \int_{1}^{T} \left( \frac{M(x)}{x} \right)^{2} dx < \alpha \log T \quad \text{for} \quad T \geq 1, \ alpha \text{ being independent of } T \]

then

\[ \max_{1 \leq t \leq T} |M(x)| > T^{\theta_{0}} \exp \left( -\frac{\log T}{\log \log T} \right) \]

for

\[ T > \max(\epsilon_{0}, \exp(300\alpha)) \]

where \( \epsilon_{0} \) is a numerical, explicitly calculable constant.

Remark. It is worth noting that (2.1) is in a certain sense an upper estimate of \( M(x) \); thus if it is not true, then this fact gives a "lower estimate" on \( M(x) \). In particular if (2.1) was false, then the Mertens hypothesis (1.2) and the relation (1.3) would be too.

The essential tool of the proof is the following lemma due to Turán (see [3], p. 56, Corollary 1):

Let \( \alpha_{1}, \alpha_{2}, \ldots, \alpha_{M} \) be complex numbers such that

\[ \max_{1 \leq r \leq M} |\alpha_{r}| = 1, \quad \min_{1 \leq r \leq M} |\alpha_{r} - \alpha_{j}| \geq \delta, \quad \alpha_{r} \neq 0, \quad j = 1, 2, \ldots, M. \]

If \( \alpha_{r} \) is a non-negative integer, then there exists an integer \( r \) such that

\[ m-1 < r < m+M, \quad \sum_{j=1}^{M} \left| \frac{\alpha_{1}^{r} + \alpha_{2}^{r} + \ldots + \alpha_{M}^{r}}{\alpha_{r}^{m-1}} \right| \geq M^{m-1}. \]

3. Before the proof of the proper theorem I wish to examine the consequences of (2.1) with respect to \( \alpha_{r} \), in order to obtain the inequality

(2.3).
First of all one could easily show that (2.1) implies the truth of the
Riemann hypothesis (compare [1], p. 154, Theorem 477). Thus, in the
proof we can use the following lemmas:
(A) For every \( T \geq \sigma_0 \) there exists \( \tau_T \), \( T < \tau_T < T+1 \), such that
\[
\left| \frac{1}{\zeta(\sigma+i\tau_T)} \right| < T^4 \quad \text{for} \quad -1 < \sigma \leq 2\quad \text{(see [2], p. 303, Theorem 14.16)}.
\]
Besides:
\[
\left| \frac{1}{\zeta(1+i\tau)} \right| \leq \frac{\sigma_0}{(|\tau|+1)^{22}}, \quad -\infty < \tau < +\infty \quad (\cdot).
\]
(B) All \( \zeta \)-zeros are simple (see [2], p. 329, Theorem 14.29 (A)).
Further, the following lemma holds:
(C) Let \( \varphi_1, \varphi_2 \) denote different non-trivial \( \zeta \)-zeros. Then
\[
|\varphi_1 - \varphi_2| \geq 1.15 \sqrt{6} \max_{\varphi \in [1,2]} |\log |\varphi||^{\frac{1}{4}}.
\]
This lemma — and even a stronger one — has been proved in [2],
(p. 326, Theorem 14.31), but without the explicit dependence on \( \sigma \).
Therefore I will repeat here some details of this proof in order to obtain
an explicit form.
First of all, the inequality
\[
(3.1) \quad |\zeta''(\xi)| \leq V_n|\xi|, \quad \xi \text{ being any complex } \zeta \text{-zero},
\]
follows from (2.1) analogously to [3].
Now let \( \frac{1}{2} + it_1, \frac{1}{2} + it_2 \) be consecutive complex \( \zeta \)-zeros, \( 0 < \gamma < \gamma' \).
Then
\[
O = \int_{\gamma'}^{\gamma} \zeta''(\xi) d\xi = (\gamma'-\gamma) \zeta''(\frac{1}{2} + it_1) + \int_{\gamma'}^{\gamma} (\gamma'-\xi) \zeta''''(\frac{1}{2} + it_1) d\xi
\]
and by (3.1)
\[
(\gamma'-\gamma) < 2V_n \int_{\gamma'}^{\gamma} (\gamma'-\xi) \zeta''''(\frac{1}{2} + it_1) d\xi < V_n(\gamma'-\gamma) \max_{\gamma < \xi < \gamma'} |\zeta''''(\frac{1}{2} + it_1)|.
\]
Using this, and using further the inequality
\[
|\zeta''''(\frac{1}{2} + it_1)| \leq \frac{1}{\pi^3} \int_{0}^{\infty} |\zeta''''(\frac{1}{2} + it + it^2)| dt
\]
and a trivial one,
\[
|\zeta''(\xi)| \leq 7.6|\xi|^4 \quad \text{for} \quad \sigma \geq \frac{1}{2}, \quad 1 \geq \gamma_0,
\]
(1) From the functional equation for \( \zeta(\xi) \).

we obtain, taking \( r = 1/t \)
\[
\max_{0 < |t| < \gamma'} |\zeta''(\frac{1}{2} + it)| \leq 15\gamma'^2
\]
and the result follows.

4. Turning now to the proof of the Theorem, assume \( T > \sigma_0 \) such that
all further inequalities, holding for large \( T \), are satisified.
Put
\[
K_0 = 10 \log T + 10 \log \log T, \quad N_0 = \log^{10} T \cdot (\log \log T)^3 \quad \text{(thus } K_0 > N_0) > 3).
\]
There exists a number \( L > 2 \) such that
\[
L^{K_0} < L^{K_0 + N_0} \leq T < L^{K_0 + N_0 + 1} < L^{K_0}.
\]
Hence
\[
\varphi_0 < \log^{10} T \leq L \leq \log^{10} T.
\]
Let \( t_L \) be the number given by (A), then
\[
L < t_L < L + 1.
\]
Let \( k \) be an integer satisfying the inequalities:
\[
K_0 < k+1 \leq K_0 + N_0 \quad (< 2K_0).
\]
Put further
\[
\eta = 1/\log L^{K_0 + 1}.
\]
Consider the integral
\[
J(T) = \frac{1}{2\pi i} \int_{1-\frac{1}{2} + it}^{1+\frac{1}{2} + it} \frac{T^s}{s} ds.
\]
In virtue of (1.1)
\[
J(T) = \sum_{n=1}^{\infty} \mu(n) \int_{1-\frac{1}{2} + it}^{1+\frac{1}{2} + it} \frac{T^s}{s} ds = \sum_{n=1}^{\infty} \mu(n) \frac{\log T}{n} \frac{1}{s^{1/2} + \frac{1}{2}} + O \left( \sum_{n=1}^{\infty} \frac{T}{k} \frac{1}{k^{1+1/2}} \right).
\]
Further
\[
L^{k+1} > L^{K_0} = L^{K_0 + N_0 + 1} / L^{K_0 + 1} > T \exp \left(- \frac{1}{4} \log \log T \right)
\]
and
\[
\sum_{n=1}^{\infty} \frac{T}{k} \frac{1}{k^{1+1/2}} = O \left( \exp \left( \frac{1}{2} \log \log T \right) \right).
\]
Hence
\[
|J(T)| \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{k^{1+1/2}} \frac{T}{n} \right| + \epsilon \exp \left( \frac{1}{2} \log \log T \right).
\]
By partial summation we obtain
\[ |J(T)| \leq \varepsilon_0 \exp\left(2\log T + \frac{\log^2 T}{k_1} \right) \max |M(\omega)|.\]

5. In order to obtain the lower estimate of \(|J(T)|\) consider the rectangle \(1 + t \pm it_L, -1 \pm it_L\) and apply Cauchy's theorem for the function
\[ \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)}. \]

Let \( \zeta \) be any complex \( \zeta \)-zero. By (B)
\[ \rho_{t, \zeta} \left( \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)} \right) \leq \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)}. \]

Let (I), (II) denote the horizontal segments of the rectangle and (III) its left side. In virtue of (A) we obtain
\[ \left\lfloor \frac{1}{2\pi i} \int_{t_0}^{t_1} \frac{1}{\zeta(t)} \, dt \right\rfloor \leq \frac{1}{2\pi} \int_{t_0}^{t_1} \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)} \, dt \leq \varepsilon_0 \exp(\log T). \]

and similarly
\[ \left\lfloor \frac{1}{2\pi i} \int_{t_0}^{t_1} \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)} \, dt \right\rfloor \leq \varepsilon_0 \exp(\log T). \]

Further
\[ \left\lfloor \frac{1}{2\pi i} \int_{t_0}^{t_1} \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)} \, dt \right\rfloor \leq \frac{T^{a-1}}{2\pi} \int_{t_0}^{t_1} \frac{T^{a-1}}{t^{z_1}} \frac{1}{\zeta(t)} \, dt \leq \varepsilon_0 \frac{1}{T}. \]

Cauchy's theorem then gives
\[ J(T) = \sum_{\sigma = \sigma_1}^{T^a} \frac{1}{\zeta(t)} + \sum_{n = 1}^{T^a} \frac{1}{\zeta(t)} + O(\exp(\log T)). \]

Clearly (compare [3], p. 117)
\[ \left\lfloor \frac{1}{2\pi i} \int_{t_0}^{t_1} \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)} \, dt \right\rfloor < \frac{1}{2\pi} \int_{t_0}^{t_1} \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)} \, dt < \varepsilon_0 \exp(\log T). \]

(5.1), (5.2) and (4.1) give
\[ \log^2 T \max |M(\omega)| \geq \sum_{|\sigma| < T^a} \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)} - \varepsilon_0 \exp\left(\frac{\log T}{\sqrt{\log T}}\right). \]

6. Now we shall estimate
\[ \sum_{|\sigma| < T^a} \frac{T^a}{t^{z_1}} \frac{1}{\zeta(t)}. \]

Let \( \xi \) be \( \xi \)-zero of the least positive ordinate (it is known that \( \xi = 1 + \frac{1}{2} \cdot 14, 13 \ldots \) (see [2], p. 330)).

Write
\[ Z = \sum_{|\sigma| < T^a} \left( \frac{T^a}{\zeta(t)} \right)^{1/2}, \]

whence
\[ \sum_{|\sigma| < T^a} \frac{T^a}{\zeta(t)} = \frac{T^a}{\zeta(t)} Z. \]

Denoting by \( \xi_1, \xi_2, \ldots, \xi_M \) all \( \xi \)-zeros lying in the rectangle \(|\sigma| < \sigma < 1, \sigma < 1, m = [K_\xi] \)

and apply Turán's lemma.

In virtue of (C) and using (3.3) we obviously obtain for \( \mu \neq \nu \)
\[ |\sigma - \xi| = |\sigma - \xi| |\zeta(\sigma)| > \frac{1}{1 + \beta \sigma} > \frac{1}{1 + \beta P log \log T} \]

Further
\[ M \leq \log^{10} T \log \log T. \]

By a suitable choice of \( k \) we obtain
\[ |Z| > \left( \sum_{\mu = 1}^{M} |\beta_\mu| \right)^{1/2} \frac{1}{2^{kM} (\log log T)^{10/11}} \]

\[ > |\beta_\mu| \log T \exp(\log \log T)^{10/11} (\log \log T)^{10/11} \]
\[ > 1/\exp(\log T) \quad (\text{since } |\beta_\mu| = 1/\zeta(\xi) = \xi_\mu). \]

By (6.1)
\[ \sum_{|\sigma| < T^a} \frac{T^a}{\zeta(t)} \leq \frac{T^a}{\zeta(t)} \exp\left( -\frac{\log T}{\log log T} \right) \leq \frac{T^a}{\zeta(t)} \exp\left( -\frac{\log T}{\log log T} \right). \]

Further
\[ \frac{\log^2 T}{k^2} < \exp\left( \frac{\log T}{\log T} \right) \quad (\text{compare [3], p. 119}) \]
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(6.3), (6.2) and (6.3) give

\[ \max_{1 < n < T} |M(n)| > T^{1/2} \exp \left( - \frac{\log T}{\sqrt{\log \log T}} \right). \]

Remark 2. Note that the Theorem gives some lower estimates in the theory of Farey fractions. Let \( r_n, 1 < r < A(n) \), denote the \( r \)th Farey fraction for the number \( n, A(n) = \sum_{q=1}^{\infty} \varphi(q), \varphi(n) = r_n \varphi(n). \)

Then we have

\[ \sum_{n=1}^{\infty} |\varphi(n)| = O\left( \sqrt{n} \right) \]

is false, or

\[ \max_{1 < n < N} \sum_{1 < n < N} |\varphi(n)| \geq \frac{1}{2N} N^{3/2} \exp \left( - \frac{\log N}{\sqrt{\log \log N}} \right) \]

(for \( N \) sufficiently large) is true.

For the proof it suffices to apply the inequality

\[ |M(n)| \leq 2n \sum_{n=1}^{\infty} |\varphi(n)| \quad \text{(see [1], p.169)}. \]

References


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Quadratische Formen und Modulfunktionen

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Einleitung


In den vergangenen 5 Jahren habe ich die Spuren der genannten Darstellungen der Modulfunktionen unter gewissen vereinfachenden Voraussetzungen bestimmt. Mit dieser Kenntnis hat man jetzt ein sicher arbeitendes Verfahren in der Hand, die Anzahl der Darstellungen einer Zahl durch eine definierende quadratische Form formellmäßig zu berechnen, falls die Form nicht allzu kompliziert ist. Es wäre aber ein Irrtum zu glauben, daß hiermit das angeschnittene Problem dem endgültigen Abschluß nahe gebracht worden wäre. Ich weise in § 4 vielmehr auf 7 offene Probleme hin, die teilweise recht umfangreich sind, und deren Bearbeitung mit den heutigen Hilfsmitteln möglich erscheint.