[8] — *Probability methods in some problems of analysis and number theory*, Bull. Amer. Math. Soc. 55 (1949), p. 641-665.

[9] И. П. Кубилюс, *Вероятностные методы в теории чисел*, Успехи Мат. наук XI 2 (68) (1956), p. 31-66.

[10] W. J. LeVeque, *On the size of certain number-theoretic functions*, Trans. Amer. Math. Soc. 66 (1949), p. 440-463.

[11] A. Rényi, *On the density of certain sequences of integers*, Publ. Inst. Math. Belgrade 8 (1955), p. 157-162.

[12] L. G. Sathe, *On a problem of Hardy on the distribution of integers having a given number of prime factors, I-IV*, J. Indian Math. Soc. 17 (1953), p. 63-82, 83-141, 18 (1954), p. 27-42, 43-81.

[13] I. J. Schoenberg, *On asymptotic distributions of arithmetical functions*, Trans. Amer. Math. Soc. 39 (1936), p. 315-330, formula (28), p. 326.

[14] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford 1951.

[15] P. Turán, *Az egész számok primosztóinak számáról*, Mat. és Fiz. Lapok 41 (1934), p. 103-130 (in Hungarian).

[16] — *On a theorem of Hardy and Ramanujan*, J. London Math. Soc. 9 (1934), p. 274-276.

[17] — *Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan*, J. London Math. Soc. 11 (1936), p. 125-133.

[18] A. Wintner, *Prime divisors and almost periodicity*, J. Math. and Phys. 21 (1942), p. 52-56.

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The inhomogeneous minimum of quadratic forms of signature zero

by

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1. Minkowski proved that, if L_1, L_2 are linear forms in x, y of determinant Δ, then, given any x^*, y^*, we can find $(x, y) \equiv (x^*, y^*) \pmod 1$ so that

$$|L_1 L_2| \leqslant \tfrac{1}{4}|\Delta|.$$

He conjectured that a similar result remained true for the product of n linear forms; but this has been proved only for $n = 3$ and $n = 4$.

The result proved by Minkowski may be restated in terms of quadratic forms: *If $Q_2(x, y)$ is an indefinite binary quadratic form of determinant D, then, given any x^*, y^*, we can find $x \equiv x^*$ and $y \equiv y^*$ (mod 1) so that*

$$|Q_2(x, y)| \leqslant |\tfrac{1}{4}D|^{1/2}.$$

Put in this way, the result may be generalized in a different way, as follows.

Given a quadratic form Q_r in r variables $x_1, \ldots, x_r$, we define the inhomogeneous minimum $M_I(Q_r)$ by

$$M_I(Q_r) = \sup_{x_1^*, \ldots, x_n^*} \Big\{ \inf_{x_i \equiv x_i^* (\mathrm{mod}\, 1)} [Q_r(x_1, \ldots, x_r)] \Big\}.$$

Then the natural generalization for quadratic forms of Minkowski's result is: "*If Q_r is any indefinite quadratic form in r variables of determinant $D \neq 0$, then*

$$M_I(Q_r) \leqslant |\tfrac{1}{4}D|^{1/r}."$$

By giving an example of an indefinite ternary form with $M_I(Q_3) = |\tfrac{27}{100}D|^{1/3}$, Davenport [4] showed that such a wide generalization is false. However, if we restrict ourselves to forms of signature zero the conjecture is valid; I will prove

Theorem 1. *Let Q_{2n} be any indefinite quadratic form in $2n$ variables, with signature zero and determinant $D \neq 0$. Then*

$$M_I(Q_{2n}) \leqslant |\tfrac{1}{4}D|^{1/2n}.$$

Equality is necessary if Q_{2n} is equivalent to a multiple of the form

$$R_{2n} = \sum_{i=1}^{n-1} x_{2i}x_{2i-1} + 2x_{2n}x_{2n-1}.$$

The proof of the theorem is quite typical of its kind; first we show that Q_{2n} represents an indefinite binary form of fairly small discriminant; after this "reduction", we prove the theorem by induction on n, using lemmas on inhomogeneous approximation to a given number by means of binary forms. In contrast to the similar problem for the product of linear forms, the reduction we need is reasonably easily performed; and so the whole situation is far simpler. In order to reduce the form, we have to divide into cases; I will consider separately (i) incommensurable forms that represent zero, (ii) forms that do not represent zero, and (iii) rational forms that represent zero, and I prove separate theorems for these three cases, from which Theorem 1 may readily be reassembled.

THEOREM 2. *Let Q_r be an indefinite non-singular quadratic form in r variables which has incommensurable coefficients. Suppose that either*

(i) *$r \geqslant 3$ and Q_r represents arbitrarily small non-zero values*

or

(ii) *$r \geqslant 4$ and Q_r represents zero properly.*

Then $M_I(Q_r) = 0$.

THEOREM 3. *Let Q_{2n} be a quadratic form in $2n$ variables of signature zero and determinant $D \neq 0$, and suppose that Q_{2n} does not represent zero properly. Then*

$$M_I(Q_{2n}) \leqslant |\tfrac{1}{4}D|^{1/2n}\min[1, (\tfrac{5}{6})^{(n-4)/3}].$$

THEOREM 4. *Let Q_{2n} be a rational quadratic form of signature zero and determinant $D \neq 0$, that represents zero properly. Then*

$$M_I(Q_{2n}) \leqslant |\tfrac{1}{4}D|^{1/2n}.$$

Equality is necessary in Theorem 4 for the form R_{2n} referred to in the statement of Theorem 1, for this has determinant $(\frac{1}{2})^{2n-2}$, and is at least $\frac{1}{2}$ whenever $x_1, \ldots, x_{2n-2}$ are all integers and x_{2n-1}, x_{2n} are both congruent to $\frac{1}{2}$.

Meyer's theorem tells us that all indefinite rational forms in at least five variables represent zero, and in fact it is probable that all indefinite incommensurable forms in at least five variables have homogeneous minimum zero (Davenport [5] has proved this for forms in a large number of variables, subject to certain restrictions on the signature); thus, for $2n \geqslant 6$, any forms to which Theorem 3 applies will satisfy the hypotheses of Theorem 2. If this is so, Theorem 2 tells us that the inhomogeneous

minimum is zero; and so it is likely that all forms of signature zero in at least six variables with non-zero inhomogeneous minimum come within the scope of Theorem 4.

Davenport [4] showed that the inhomogeneous minimum of indefinite ternary quadratic forms is isolated — one would expect that this would be true for our $2n$-ary forms of signature zero. In fact, I have shown that this is so — but I do not propose to give the proof. I note that Theorems 2 and 3, when applicable, are definitely stronger than Theorem 1, so that (at any rate for large n) in order to show that the minimum whose existence is proved in Theorem 1 is isolated we need only consider rational forms that represent zero.

In Section 2 I prove Theorem 2, which is rather easy. Then in Section 3 I prove the approximation results that I need, and in Sections 4 and 5 I will prove Theorems 3 and 4 respectively.

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2. For the proof of Theorem 2, we need the following lemma, due to Blaney [2].

LEMMA 1. *There is a constant C_r, depending only on r, such that given any real indefinite quadratic form Q_r of determinant $D \neq 0$, and any real numbers $x_1^*, \ldots, x_r^*$, there is a point $(x_1, \ldots, x_r)$ with each $x_i \equiv x_i^*$ (mod 1) such that*

$$0 < Q_r(x_1, \ldots, x_r) < C_r|D|^{1/r}.$$

Using this lemma, we will now prove

THEOREM 2. *Let Q_r be an indefinite non-singular quadratic form in r variables which has incommensurable coefficients. Suppose that either*

(i) *$r \geqslant 3$ and Q_r represents arbitrarily small non-zero values,*

or

(ii) *$r \geqslant 4$ and Q_r represents zero properly.*

Then $M_I(Q_r) = 0$.

By considering $-Q_r$ instead of Q_r if necessary, we may suppose that Q_r has non-negative signature; denote the determinant of Q_r by D. Oppenheim has shown [7] that an incommensurable form in at least 4 variables which represents zero properly takes arbitrarily small non-zero values, and [8] that an indefinite form in at least 3 variables which takes arbitrarily small non-zero values takes them with both signs. Hence, the second case of the theorem is included in the first, and we may suppose that Q_r takes small positive values. Suppose then that Q_r takes the value

$\eta > 0$ at a primitive lattice-point. By a unimodular integral transformation, we may suppose that this lattice-point is $(1, 0, \ldots, 0)$; we can thus write Q_r in the form

$$Q_r(x_1, \ldots, x_r) = \eta(x_1 + a_{12}x_2 + \ldots)^2 - Q_{r-1}(x_2, \ldots, x_r)$$

where Q_{r-1} is an indefinite binary form whose determinant is in modulus equal to $|D/\eta|$. By Lemma 1, given any $x_2^*, \ldots, x_r^*$, we can find $x_2, \ldots, x_r$ congruent to them mod 1 so that

$$0 < Q_{r-1} < |CD/\eta|^{1/(r-1)},$$

where C depends only on r. We can now choose $x_1 \equiv x_1^*$ (mod 1) so that

$$|(x_1 + a_{12}x_2 + \ldots) - \eta^{-1/2}Q_{r-1}^{1/2}| \leqslant 1.$$

Then

$$|Q_r| \leqslant \eta + 2\eta^{1/2}Q_{r-1}^{1/2} < \eta + 2\eta^{1/2}|CD/\eta|^{1/2(r-1)}.$$

Since $r \geqslant 3$, the right hand side tends to zero with η. But η may be made as small as we like, so the theorem follows.

3. We will now prove a few lemmas on approximation by means of binary forms; Lemmas 4 and 5 are the ones which will be applied later, Lemma 2 and 3 are just steps in the proof of Lemma 4.

LEMMA 2. *Let φ be an indefinite binary quadratic form of determinant $-d$. Then, given any x^*, y^* and any μ with $|\mu| \geqslant d^{1/2}$, we can find $(x, y) \equiv (x^*, y^*)$ so that*

$$|\varphi(x, y) + \mu| \leqslant d^{1/4}|\mu|^{1/2}.$$

Proof. Dividing through by a constant, we may suppose that $d = 1$ and $\mu \geqslant 0$, and so $\mu \geqslant 1$.

Let e be a positive value taken by φ at a primitive lattice point; we can certainly ensure that $e \leqslant 2$. By an integral unimodular transformation, we may assume that e is taken at $(1, 0)$, so that

$$\varphi(x, y) = e(x + hy)^2 - e^{-1}y^2.$$

We must show how to choose x, y mod 1 so that $|\varphi + \mu| \leqslant \mu^{1/2}$. First, we will choose $y > 0$ as small as possible to ensure that

$$e^{-1}y^2 - \mu \geqslant \tfrac{1}{4}e - \mu^{1/2}.$$

Write $H = e^{-1}y^2 - \mu$ for short. We can certainly ensure that $[(\mu + \tfrac{1}{4}e - \mu^{1/2})e]^{1/2} \leqslant y < [\ldots]^{1/2} + 1$, and so we can ensure that

$$\tfrac{1}{4}e - \mu^{1/2} \leqslant H < e^{-1} + \tfrac{1}{4}e - \mu^{1/2} + 2e^{-1}[(\mu + \tfrac{1}{4}e - \mu^{1/2})e]^{1/2}.$$

Write $x + hy = z$, so that it remains to choose $z \equiv x^* + hy$ to minimise $|ez^2 - H|$. We know that $H \geqslant \tfrac{1}{4}e - \mu^{1/2}$, so by taking $|z|$ as small as possible we can certainly ensure that $|ez^2 - H| \leqslant \mu^{1/2}$ when H is negative. Hence, we need only consider the positive values of H.

For $0 < H \leqslant \tfrac{1}{2}e$, we can obviously ensure $|ez^2 - H| \leqslant \tfrac{1}{2}e \leqslant 1 \leqslant \mu^{1/2}$.

For $H \geqslant \tfrac{1}{2}e$, we choose z so that $z - \tfrac{1}{2} \leqslant [e^{-1}H - \tfrac{1}{4}]^{1/2} \leqslant z + \tfrac{1}{2}$; then $|ez^2 - H| \leqslant e[e^{-1}H - \tfrac{1}{4}]^{1/2}$. This implies that

$$|\varphi + \mu| = |ez^2 - H| \leqslant \mu^{1/2},$$

so long as $eH - \tfrac{1}{4}e^2 \leqslant 1 - \mu^{1/2}e + 2[(\mu + \tfrac{1}{4}e - \mu^{1/2})e]^{1/2} \leqslant \mu$. But the inequality on the right is $4e(\mu + \tfrac{1}{4}e - \mu^{1/2}) \leqslant (\mu - 1 + \mu^{1/2}e)^2$, which is $(\mu-1)^2 + e^2(\mu - 1) + 2e\mu^{1/2}(\mu^{1/2} - 1)^2 \geqslant 0$, which is certainly so since $\mu \geqslant 1$ by hypothesis.

This proves the lemma.

To complete the proof of Lemma 4 we will also need the following result, which is quoted from Blaney [3]. [The first part of this lemma is due to Davenport].

LEMMA 3. *Let φ be a binary quadratic form of determinant -1, and let $\lambda > 1$ be a constant. Then, given any x^*, y^*, we can find $(x, y) \equiv (x^*, y^*)$ so that*

$$-\tfrac{1}{2}\lambda^{-1/2} \leqslant \varphi(x, y) \leqslant \tfrac{1}{2}\lambda^{1/2}.$$

Further, if $\lambda > 3$, we can satisfy the stronger inequalities

$$\frac{-2}{\sqrt{(\lambda+1)(\lambda+9)}} \leqslant \varphi(x, y) \leqslant \frac{2\lambda}{\sqrt{(\lambda+1)(\lambda+9)}}.$$

LEMMA 4. *Let φ be an indefinite binary form of determinant $-d$. Then, for any x^*, y^* and any μ, we can find $(x, y) \equiv (x^*, y^*)$ so that*

$$|\varphi(x, y) + \mu| \leqslant \max[2^{-1/2}d^{1/2}, d^{1/4}|\mu|^{1/2}].$$

Proof. As in Lemma 2, we may suppose that $d = 1$ and $\mu \geqslant 0$; and after Lemma 2 it only remains to consider the case $\mu < 1$. In this case $\mu - \mu^{1/2}$ is negative.

Case 1. $\mu \geqslant \tfrac{1}{2}$. We want to ensure that

$$(*) \qquad -\mu^{1/2} + \mu \leqslant \varphi \leqslant \mu + \mu^{1/2}.$$

Write $(\mu + \mu^{1/2})/(\mu^{1/2} - \mu) = \lambda$; then, since $\mu > \tfrac{1}{4}$, $\lambda > 3$. Hence we may apply Lemma 3, and so we can certainly ensure that $(*)$ is satisfied so long as

$$\mu + \sqrt{\mu} \geqslant \frac{2\lambda}{\sqrt{(\lambda+1)(\lambda+9)}}.$$

Substituting for λ in terms of μ and simplifying, this condition is

$$1 \geqslant 2/\sqrt{[(\sqrt{\mu} + \mu) + (\sqrt{\mu} - \mu)][(\sqrt{\mu} + \mu) + 9(\sqrt{\mu} - \mu)]},$$

which is

$$\sqrt{\mu}(5\sqrt{\mu} - 4\mu) \geqslant 1.$$

Write $t = \mu^{1/2}$. We now simply need to show that $5t^2-4t^3-1 \geqslant 0$ when $2 \geqslant 2t^2 \geqslant 1$, and this is clear since $5t^2-4t^3-1 = (1-t)(4t^2-t-1)$.

Case 2. $\mu \leqslant \frac{1}{2}$. This time, we must ensure that

$$-2^{-1/2}+\mu \leqslant \varphi \leqslant \mu+2^{-1/2}.$$

By the first part of Lemma 3, we can certainly do this so long as

$$(2^{-1/2}-\mu)(2^{-1/2}+\mu) \geqslant \tfrac{1}{4}.$$

This condition is simply $\frac{1}{4} \geqslant \mu^2$, which is so since $0 \leqslant \mu \leqslant \frac{1}{2}$.

Lemma 5. *Denote the binary form x^2+xy-y^2 by P_2. Given any x^*, y^*, and any μ, we can find $(x, y) \equiv (x^*, y^*)$ so that*

$$|P_2(x,y)+\mu| \leqslant \max[|\mu|^{2/3}, 1].$$

Proof. Since P_2 is equivalent to its negative, we may suppose that $\mu \geqslant 0$. We now proceed more or less as in Lemma 4, noting that $P_2(x,y) = (x+\frac{1}{2}y)^2-\frac{5}{4}y^2$.

If $\mu \leqslant \frac{3}{4}$, we first choose y so that $|y| \leqslant \frac{1}{2}$, and then we choose x so that $|x+\frac{1}{2}y| \leqslant \frac{1}{2}$; this leads to $|P_2+\mu| \leqslant 1$, as required.

If $\frac{3}{4} \leqslant \mu \leqslant \frac{6}{5}$, we first choose y so that $H = \frac{5}{4}y^2-\mu$ is as small as possible but greater than $-\frac{3}{4}$; thus, we choose y so that

$$\sqrt{\tfrac{4}{5}(\mu-\tfrac{3}{4})} \leqslant y \leqslant 1+\sqrt{\tfrac{4}{5}(\mu-\tfrac{3}{4})}.$$

We then have

$$-\tfrac{3}{4} \leqslant H \leqslant \tfrac{1}{2}+\sqrt{5(\mu-\tfrac{3}{4})} \leqslant 2;$$

and so we can choose $z = x+\frac{1}{2}y \mod 1$ so that $|z^2-H| \leqslant 1$, that is, $|P_2+\mu| \leqslant 1$.

If $\frac{6}{5} \leqslant \mu \leqslant (\frac{5}{4})^{3/2}$, we can ensure that $|P_2+\frac{6}{5}| \leqslant 1$ as above; we then have $|P_2+\mu| \leqslant \mu-\frac{1}{5}$, which leads to $|P_2+\mu| \leqslant \mu^{2/3}$ since $\mu-\frac{1}{5} \leqslant \mu^{2/3}$ for $0 \leqslant \mu \leqslant (\frac{5}{4})^{3/2}$.

If $(\frac{5}{4})^{3/2} \leqslant \mu$, we simply apply Lemma 2 to the form P_2 of determinant $-\frac{5}{4}$, and find that we can ensure that $|P_2+\mu| \leqslant (\frac{5}{4})^{1/4}\mu^{1/2} \leqslant \mu^{2/3}$.

This completes the proof of the lemma.

4. We now prove

Theorem 3. *Let Q_{2n} be a quadratic form of signature zero and determinant D, which does not represent zero. Then*

$$M_I(Q_{2n}) \leqslant |\tfrac{1}{4}D|^{1/2n}\cdot\min[1, (\tfrac{5}{6})^{(n-4)/3}].$$

To prove this theorem, we will first show that Q_{2n} represents a binary form of reasonably small determinant. When this has been done, we will apply the lemmas of the previous section to prove the theorem for $2n \geqslant 4$ by induction on n; the case $2n = 2$ has of course been proved by Minkowski. If Q_{2n} represents arbitrarily small values, the theorem and more follows from Theorem 2, so we may suppose throughout this section that $|Q_{2n}|$ is bounded below.

Lemma 6. *Suppose that the lower bound of the non-negative values represented by Q_{2n} is $b > 0$, and that Q_{2n} represents a value v, where $b \leqslant v < \frac{17}{16}b$. Then, after a unimodular integral transformation, we can write*

$$Q_{2n} = v(x_1+a_2x_2+\ldots)^2-Q_{2n-1},$$

where Q_{2n-1} represents no value in modulus less than $\frac{1}{4}v$.

Proof. Since $v < 2b$, Q_{2n} must represent v at a primitive point, so we can certainly put Q_{2n} into this shape. Now, suppose on the contrary that we can choose $x_2, \ldots, x_{2n}$, not all zero, so that $|Q_{2n-1}| < \frac{1}{4}v$. If $-\frac{1}{4}v < Q_{2n-1} \leqslant 0$, we simply choose x_1 so that $|x_1+a_2x_2+\ldots| \leqslant \frac{1}{2}$, and then

$$0 \leqslant -Q_{2n-1} \leqslant Q_{2n} \leqslant \tfrac{1}{4}v-Q_{2n-1} < \tfrac{1}{2}v < b,$$

contrary to the definition of b. If on the other hand $0 < Q_{2n-1} < \frac{1}{4}v$, then by taking $2^m x_2, \ldots, 2^m x_{2n}$ for $x_2, \ldots, x_{2n}$ if necessary we may ensure that $\frac{1}{16}v \leqslant Q_{2n-1} < \frac{1}{4}v$. We then choose x_1 so that $\frac{1}{2} \leqslant |x_1+a_2x_2+\ldots| \leqslant 1$, and then $0 = \frac{1}{4}v-\frac{1}{4}v < Q_{2n} = v(x_1+\ldots)^2-Q_{2n-1} \leqslant v-\frac{1}{16}v = \frac{15}{16}v < b$, again a contradiction. This proves the lemma.

I now quote two results, from Oppenheim [6] and Barnes [1] respectively.

Lemma 7 (Oppenheim). *Suppose that Q_4 is a quaternary quadratic form of determinant $D > 0$ and signature zero. Then either Q_4 is equivalent to one of eight special forms enumerated by Oppenheim [6], or $|Q_4|$ represents a value less than $(\frac{1}{10}D)^{1/4}$. The eight exceptional forms are*

$$P_4 = t^2+xt-x^2+zt-xy+y^2+yz-z^2,$$

and seven others, all of which may be reduced to the shape

$$Q_4 = \psi(x_1+\ldots, x_2+\ldots)+Q_2(x_3, x_4);$$

ψ is an indefinite binary form with determinant in modulus at most $|\frac{4}{9}D|^{1/2}$.

Lemma 8 (Barnes). *If the ternary quadratic form Q_3 of signature $+1$ and determinant $D < 0$ does not represent zero, it represents a positive value less than or equal to $|\frac{4}{3}D|^{1/3}$.*

We note that the form P_4 mentioned in Lemma 7 has determinant $\frac{9}{4}$, and represents the indefinite binary form $P_2 = t^2+xt-x^2$ of determinant $-\frac{5}{4} = -\frac{5}{6}\sqrt{\frac{9}{4}}$.

From these results, we deduce

LEMMA 9. *Let Q_4 be a quadratic form of determinant $D > 0$ and signature zero, and suppose that $|Q_4|$ is boundet below. Then either Q_4 is equivalent to the special form P_4, or else we can reduce Q_4 to the shape*

$$Q_4 = \psi(x_1+\ldots, x_2+\ldots)+Q_2(x_3, x_4);$$

ψ is an indefinite binary form of determinant in modulus less than $|\frac{4}{7}D|^{1/2}$.

Proof. Let the lower bound of values represented by $|Q_4|$ be b; we have supposed that $b > 0$. Then, given any ε such that $0 < \varepsilon < \frac{1}{16}$, Q_4 represents a value v such that $b \leqslant |v| < b(1+\varepsilon)$, and taking $-Q_4$ for Q_4 if necessary we may suppose that $v > 0$. By Lemma 6, we may make a unimodular integral transformation and write $Q_4 = v(x_1+\ldots)^2-Q_3$, where Q_3 does not represent arbitrarily small values. Let the lower bound of the positive values represented by Q_3 be a; then in a similar way we may write $Q_3 = u(x_2\ldots)^2-Q_2$, where $a \leqslant u < a(1+\varepsilon)$.

We thus have

$$Q_4(x_1, \ldots, x_4) = \psi(x_1+a_{13}x_3+a_{14}x_4, x_2+\ldots)+Q_2(x_3, x_4),$$

where ψ is an indefinite binary form of determinant $\delta_1 = -uv$, so that $|\delta_1| < ab(1+\varepsilon)^2$, and Q_2 does not represent arbitrarily small values; write δ_2 for the determinant of Q_2.

We note that a similar reduction may be applied to forms in more than four variables, to put them into the shape $Q_{2n} = \psi_1+Q_{2n-2}$.

By Lemma 7, $b^4 < \frac{1}{10}|\delta_1\delta_2|$ unless Q_4 is equivalent to one of eight particular forms; further, applying Lemma 8 to Q_3, we have $a^3 \leqslant \frac{4}{3}u|\delta_2|$, and so $a^2 \leqslant \frac{4}{3}|\delta_2|(1+\varepsilon)$. Thus, except in the particular cases already mentioned, we have $b^3 < \frac{1}{10}a|\delta_2|(1+\varepsilon)^2$ and $a^2 \leqslant \frac{4}{3}|\delta_2|(1+\varepsilon)$, and so $a^3b^3 < \frac{16}{90}|\delta_2|^3(1+\varepsilon)^4$, $|\delta_1|^3 < \frac{16}{90}|\delta_2|^3(1+\varepsilon)^{10}$. By taking ε small enough, we may ensure that $\frac{16}{90}(1+\varepsilon)^{10} < (\frac{4}{7})^3$, and so ψ has determinant in modulus less than $|\frac{4}{7}D|^{1/2}$, and Q_4 has the shape asserted in the lemma.

On the other hand, all the forms other than P_4 excepted in Lemma 7 have this shape. This completes the proof of the lemma.

LEMMA 10. *Let Q_6 be a quadratic form of determinant D and signature zero, with $|Q_6|$ boundet below. Then we may write Q_6 in the shape*

$$Q_6(x_1, \ldots, x_6) = \psi_1(x_1+a_{13}x_3+\ldots, x_2+\ldots)+Q_4(x_3, \ldots, x_6),$$

where either the form ψ_1 is an indefinite binary form with determinant in modulus less than $|\frac{3}{13}D|^{1/3}$, or the form ψ_1 is a multiple of $P_2 = x^2+xy-y^2$ with determinant in modulus at most $\frac{25}{36}|D|^{1/3}$, and where $|Q_4|$ is bounded below.

Proof. As in Lemma 9, we can certainly write Q_6 in the shape ψ_1+Q_4, where Q_4 does not represent arbitrarily small values. We can now apply the process again to Q_4, and obtain

$$Q_6 = \psi_1(x_1+\ldots, x_2+\ldots)+\psi_2(x_3+\ldots, x_4+\ldots)+\psi_3(x_5, x_6).$$

We write $\delta_1, \delta_2, \delta_3$ for the determinants of ψ_1, ψ_2, ψ_3.

We now distinguish two cases.

First, if $Q_6(x_1, \ldots, x_4, 0, 0) = [\psi_1+\psi_2]_{x_5=x_6=0}$ is not equivalent to P_4, we have $|\delta_1| < \frac{4}{7}|\delta_2|$ as in Lemma 9; and even if $\psi_2+\psi_3$ is equivalent to P_4, we still have $|\delta_2| \leqslant \frac{25}{36}|\delta_3|$. Hence,

$$|\delta_1|^3 < \tfrac{16}{49}\cdot\tfrac{25}{36}|\delta_1\delta_2\delta_3| < \tfrac{3}{13}|\delta_1\delta_2\delta_3| = \tfrac{3}{13}|D|.$$

On the other hand, if $Q_6(x_1, \ldots, x_4, 0, 0)$ is equivalent to P_4, then $\psi_1(x_1, x_2)$ will be equivalent to P_2, and $|\delta_1| = \frac{25}{36}|\delta_2|$. As before, $|\delta_2| \leqslant \frac{25}{36}|\delta_3|$, and so $|\delta_1|^3 \leqslant (\frac{5}{6})^6|\delta_1\delta_2\delta_3| = (\frac{5}{6})^6|D|$.

LEMMA 11. *Let Q_{2n} be a quadratic form in at least 8 variables of determinant D and signature zero, with $|Q_{2n}|$ bounded below. Then we may write Q_{2n} in the shape $Q_{2n} = \psi_1+Q_{2n-2}$, where ψ_1 is an indefinite binary form of determinant in modulus at most $(\frac{5}{6})^{n-1}|D|^{1/n}$, and $|Q_{2n-2}|$ is bounded below.*

(Note that $(\frac{5}{6})^{n-1}|D|^{1/n} < |\frac{1}{8}D|^{1/n}$ for $n \geqslant 4$.)

Proof. As in Lemmas 9 and 10, we can write

$$Q_{2n} = \psi_1+\psi_2+\psi_3+\ldots+\psi_n,$$

where $\psi_1, \ldots, \psi_n$ are indefinite binary forms with determinants $\delta_1, \ldots, \delta_n$. Then, as in Lemma 10, we can ensure that

$$|\delta_i| \leqslant \tfrac{25}{36}|\delta_{i+1}| \quad \text{for each} \quad i = 1, \ldots, n.$$

Hence, $|\delta_1|^n \leqslant (\frac{5}{6})^{n^2-n}|\delta_1\ldots\delta_n| = (\frac{5}{6})^{n^2-n}|D|$.

We can now prove Theorem 3. We will first deal with the cases $2n = 4$ and $2n = 6$, and then we will prove the theorem for $2n \geqslant 8$ by induction on n. The theorem is certainly true for $2n = 2$, having been proved by Minkowski.

Suppose then that Q_4 is a quaternary quadratic form of determinant D and signature zero that does not represent zero. By Lemma 9 we can put Q_4 into the shape

$$Q_4(x_1, \ldots, x_4) = \psi(x_1+a_{13}x_3+a_{14}x_4, x_2+\ldots)+Q_2(x_3, x_4),$$

where ψ is an indefinite binary form of determinant $-d$, say, with $d^2 \leqslant \frac{25}{36}D$, and Q_2 is a form of signature zero and determinant $(-D/d)$.

By the case $2n=2$, given x_3^*, x_4^*, we can find x_3, x_4 congruent to them so that $|Q_2| \leqslant |\frac{1}{4}D/d|^{1/2}$. We have $|\frac{1}{4}D/d|^{1/2} > \frac{1}{2}d^{1/2}$, and so we may apply Lemma 4 with $\mu = Q_2$, $\varphi = \psi$ to find $(z_1, z_2) \equiv (x_1^* + a_{13}x_3 + + a_{14}x_4, x_2^* + \ldots)$ so that

$$|\psi(z_1, z_2) + Q_2| \leqslant \max[2^{-1/2}d^{1/2}, d^{1/4}|Q_2|^{1/2}] \leqslant d^{1/4}|\tfrac{1}{4}D/d|^{1/4} = |\tfrac{1}{4}D|^{1/4}.$$

Thus, we have shown how to choose $(x_1, \ldots, x_4) \equiv (x_1^*, \ldots, x_4^*)$ so that $|Q_4| \leqslant |\frac{1}{4}D|^{1/4}$. This proves that $M_I(Q_4) \leqslant |\frac{1}{4}D|^{1/4}$, which is the assertion of the theorem for $2n = 4$.

We now prove the theorem for $2n = 6$. Let Q_6 be a senary form of determinant D and signature zero that does not represent zero. Then by Lemma 10, we can write $Q_6 = \psi + Q_4$ where ψ is an indefinite binary form of determinant $-d$, say, and Q_4 is a quaternary form of signature zero and determinant $-D/d$ that does not represent zero. We have to distinguish two cases, corresponding to the alternatives in Lemma 10; in case 1, $d < |\frac{3}{13}D|^{1/3}$, and in case 2, $\psi = \nu P_2$, with $d = \frac{5}{4}\nu^2$ and $d \leqslant \frac{25}{36}|D|^{1/3}$. In either case, we first apply the theorem with $2n = 4$ to Q_4, so as to choose $x_3, \ldots, x_6$ to ensure $|Q_4| \leqslant |\frac{1}{4}D/d|^{1/4}$.

Case 1. $|\frac{1}{4}D/d|^{1/4} > |\frac{13}{12}d^2|^{1/4} > d^{1/2}$, and so we may apply Lemma 4 with $\mu = Q_4$, and $\varphi = \psi$. We can thus choose x_1, x_2 to ensure that

$$|Q_6| = |\psi + Q_4| \leqslant \max[2^{-1/2}d^{1/2}, d^{1/4}|Q_4|^{1/2}] \leqslant d^{1/4}|\tfrac{1}{4}D/d|^{1/8} \leqslant d^{1/6}|\tfrac{1}{4}D/d|^{1/6}$$
$$= |\tfrac{1}{4}D|^{1/6}.$$

This is all that we need.

Case 2. In this case, $Q_6 = \nu(P_2 + \nu^{-1}Q_4)$, where by homogeneity we may suppose that $\nu = 1$. Then $d = \frac{5}{4}\nu^2 \leqslant \frac{25}{36}|D|^{1/3}$, so $|D| > 4$. We have already ensured that $|Q_4| \leqslant |\frac{1}{4}D/d|^{1/4}$, and so by Lemma 5 we may choose x_1, x_2 mod 1 so as to ensure that

$$|Q_6| = |P_2 + Q_4| \leqslant \max[1, |\tfrac{1}{4}D/d|^{1/6}] < |\tfrac{1}{4}D|^{1/6}.$$

This completes the proof for $2n = 6$.

We now prove the theorem for $2n \geqslant 8$; we will prove, by induction on n, that if Q_{2n} is any form in $2n$ variables with signature zero and determinant D that does not represent zero, then $M_I(Q_{2n}) < (\frac{5}{6})^{(n-4)/3}|\frac{1}{4}D|^{1/2n}$. We have proved that this is so when $2n = 6$, so our induction starts. Suppose then that the result has been proved for forms in $2n-2$ variables, and let Q_{2n} be a form as above. By Lemma 11, we may write

$$Q_{2n} = \psi(x_1 + a_{13}x_3 + \ldots, x_2 + \ldots) + Q_{2n-2}(x_3, \ldots, x_{2n}),$$

where ψ is an indefinite binary quadratic form of determinant $-d$, say, with $|d|^n \leqslant (\frac{5}{6})^{n^2-n}|D|$, and Q_{2n-2} is of determinant $-D/d$ and satisfies the conditions of the theorem. By our induction hypothesis, we can choose $x_3, \ldots, x_{2n}$ (mod 1) to ensure that

$$|Q_{2n-2}| \leqslant (\tfrac{5}{6})^{(n-5)/3}|\tfrac{1}{4}D/d|^{1/2(n-1)}.$$

Hence, we can apply Lemma 4 with $\varphi = \psi$ and $\mu = Q_{2n-2}$ to ensure that

$$|\psi + Q_{2n-2}| \leqslant \max[2^{-1/2}d^{1/2}, d^{1/4}(\tfrac{5}{6})^{(n-5)/6}|\tfrac{1}{4}D/d|^{1/4(n-1)}].$$

Now, $d^{1/4} \leqslant (\frac{5}{6})^{n/4}|D/d|^{1/4(n-1)} < (\frac{5}{6})^{(n-5)/6}|\frac{1}{4}D/d|^{1/4(n-1)}$, since $n \geqslant 4$ and $\frac{1}{4} > (\frac{5}{6})^8$, and so

$$|\psi + Q_{2n-2}| \leqslant d^{1/4}(\tfrac{5}{6})^{(n-5)/6}|\tfrac{1}{4}D/d|^{1/4(n-1)} = d^{(n-2)/4(n-1)}(\tfrac{5}{6})^{(n-5)/6}|\tfrac{1}{4}D|^{1/4(n-1)}$$
$$\leqslant (\tfrac{5}{6})^{(n-5)/6+(n-2)/6}|\tfrac{1}{4}D|^{1/4(n-1)}|D|^{(n-2)/4n(n-1)} < (\tfrac{5}{6})^{(n-4)/3}\cdot|\tfrac{1}{4}D|^{1/2n},$$

as required, since $n \geqslant 4$ and $\frac{1}{4} > (\frac{5}{6})^8$ as before.

This completes the proof of Theorem 3.

I have made no effort to prove best possible estimates for $M_I(Q_{2n})$ in Theorem 3; in fact, the results I have given can be improved quite easily for all $2n \geqslant 4$.

5. In this final section we will prove

THEOREM 4. *Let Q_{2n} be a rational form of signature zero and determinant $D \neq 0$ that represents zero properly. Then*

$$M_I(Q_{2n}) \leqslant |\tfrac{1}{4}D|^{1/2n}.$$

To prove Theorem 4, we need a reduction lemma; while I am about it, I will prove a stronger result than is actually needed for the proof of our theorem. The shape of the reduction described by the lemma makes it clear that the minimum asserted by Theorem 4 may easily be isolated.

LEMMA 12. *Let Q_{2n} be a rational quadratic form of determinant $D \neq 0$ and signature zero that represents zero. Then we may transform Q_{2n} into the shape*

$$Q_{2n}(x_1, \ldots, x_{2n})$$
$$= H_1(x_1 + a_2^1x_2 + a_3^1x_3 + \ldots + a_{2n}^1x_{2n})x_2 + H_2(x_3 + a_4^2x_4 + \ldots)x_4 + \ldots +$$
$$+ H_m(x_{2m-1} + a_{2m}^m x_{2m} + \ldots)x_{2m} + Q_{2n-2m},$$

where $-\frac{1}{2} \leqslant a_j^i \leqslant \frac{1}{2}$ and $0 < H_i$ for $i = 1, \ldots, m$ and $j = 2i, \ldots, 2n$,

H_i is an integer multiple of H_{i-1} for $i = 2, \ldots, m$,

and either $m = n$, in which case Q_{2n-2m} is omitted and $D = (-1)^m\prod_{i=1}^{m}(\frac{1}{2}H_i)^2$, *or else $m = n-1$ or $m = n-2$, in which case Q_{2n-2m} is a binary or quaternary form of determinant $\Delta = D/(-1)^m \prod_{i=1}^{m} (\frac{1}{2}H_i)^2$ and signature zero that does not represent zero.*

Proof. Q_{2n} represents zero, so it represents zero at a primitive point; after a transformation, we may suppose that $Q_{2n}(1, 0, \ldots, 0) = 0$. Thus,

$$Q_{2n} = x_1(\text{linear form in } x_2, \ldots, x_{2n}) + (\text{quadratic form in } x_2, \ldots, x_{2n}).$$

Since Q_{2n} is non-singular, there is at least one term involving x_1, so we may suppose that there is a term in x_1x_2. Thus, we may write

$$(*) \qquad Q_{2n} = h(x_1 + a_2x_2 + \ldots)(x_2 + b_3x_3 + \ldots) + Q_{2n-2}(x_3, \ldots, x_{2n}),$$

where h, a_i, b_i are appropriate rational constants, and Q_{2n-2} is a rational form of signature zero and determinant $-4D/h^2$. By absorbing integral multiples of $x_3, \ldots, x_{2n}$ into x_2, we may suppose that each $|b_i| \leqslant \frac{1}{2}$, and taking a suitable sign for x_1, we may suppose that $h > 0$. It may be possible to express Q_{2n} in the shape $(*)$ in more than one way; let H_1 be the lower bound of the possible h. Since Q_{2n} is rational, H_1 is obviously attained, and so non-zero; we pick on a definite expression

$$Q_{2n} = H_1(x_1 + \ldots)(x_2 + b_3x_3 + \ldots + b_{2n}x_{2n}) + Q_{2n-2},$$

and assert that, when H_1 is minimal, all the b_i must vanish. In fact, if a b_i fails to vanish, it is a fraction, and g. c. d. $(1, b_3, \ldots, b_{2n}) = \gamma$, say, is at most $\frac{1}{2}$. We can then find a unimodular integral transformation not involving x_1 which replaces $(x_2 + b_3x_3 + \ldots)$ by γx_2. Since $\gamma \leqslant \frac{1}{2}$, this contradicts our assumption that H_1 is minimal. We have thus

$$Q_{2n} = H_1(x_1 + a_2^1x_2 + a_3^1x_3 + \ldots + a_{2n}^1x_{2n})x_2 + Q_{2n-2}(x_3, \ldots, x_{2n}).$$

By absorbing integer multiples of $x_2, \ldots, x_{2n}$ into x_1 we may ensure that $|a_j^1| \leqslant \frac{1}{2}$ for $j = 2, \ldots, 2n$.

We may now repeat the process until we are left with a form Q_{2n-2m} that does not represent zero; by Meyer's theorem, Q_{2n-2m} is at most a quaternary form. We have thus put Q_{2n} into the shape postulated in the lemma; since the relations between the determinants are obvious, it only remains to prove that H_i divides H_{i+1} for $i = 1, \ldots, m-1$. For this, it will suffice to prove that H_1 divides H_2 if $m \geqslant 2$. Now by the minimal property of H_1, $a_3^1 = 0$, for otherwise $x_3, x_2, a_3^1H_1$ may take the roles of x_1, x_2, H_1. Thus, if r is any integer, we have

$$H_1[(x_1 + rx_3) + a_2^1(x_2 - x_4) + a_3^1x_3 + a_4^1x_4 + \ldots](x_2 - x_4) + H_2(x_3 + \ldots)x_4 + \ldots$$
$$= (H_2 - rH_1 - a_3^1H_1)x_3x_4 + H_1(r + a_3^1)x_3x_2 + (\text{terms not involving } x_3),$$

and so, again by the minimal property of H_1, $H_1 \leqslant |H_2 - rH_1 - a_3^1H_1| = |H_2 - rH_1|$ whenever $H_2 - rH_1 \neq 0$. It follows that H_1 divides H_2; so the proof is complete.

For our actual application, we need the following corollary:

Corollary 13. *If Q_{2n} is a rational quadratic form of determinant $D \neq 0$ and signature zero that represents zero, then we can put Q_{2n} into the shape*

$$Q_{2n} = \psi(x_1 + \ldots, x_2 + \ldots) + Q_{2n-2}(x_3, \ldots, x_{2n}),$$

where either $\psi = H(x_1 + \ldots)x_2$ with $(\frac{1}{4}H)^{2n} \leqslant |\frac{1}{4}D|$, or else ψ is an indefinite binary form whose determinant $-d$ satisfies $|d|^n \leqslant |\frac{1}{4}D|$.

It is easy to deduce this corollary from the lemma. If $m = n$ in the lemma, then $(\frac{1}{4}H_1)^{2n} \leqslant \prod_{i=1}^{n} (\frac{1}{4}H_i)^2 \leqslant \frac{1}{4} \prod_{i=1}^{n} (\frac{1}{2}H_i)^2 = |\frac{1}{4}D|$, and we have the first case of the corollary. If $m \neq n$, then Q_{2n-2m} is a form in $2n-2m$ variables that does not represent zero, and so by Lemma 9 it represents a binary form φ whose determinant δ satisfies $|\delta| \leqslant |\Delta|^{1/(n-m)}$; then $|\delta|^{n-m}(\frac{1}{2}H_1)^{2m} \leqslant |D|$, and so either $|\delta|^n \leqslant |\frac{1}{4}D|$ and the corollary is true with $\psi = \varphi$, or else $(\frac{1}{4}H_1)^{2n} \leqslant |\frac{1}{4}D|$ and we may take $\psi = H_1(x_1 + \ldots)x_2$.

It is now an easy matter to complete the proof of the theorem. We must prove that, if Q_{2n} is our form, and $x_1^*, \ldots, x_{2n}^*$ is any $2n$-tuple of real numbers, then we can find $x_1, \ldots, x_{2n}$ congruent to $x_1^*, \ldots, x_{2n}^*$ so that

$$|Q_{2n}(x_1, \ldots, x_{2n})| \leqslant |\tfrac{1}{4}D|^{1/2n}.$$

Note that Theorem 1 is obtained immediately by combining Theorems 2, 3 and 4; we have already proved Theorems 2 and 3, so, when we prove Theorem 4 by induction, we may assume not merely Theorem 4 but the whole strength of Theorem 1 for forms in $2n-2$ variables.

We split into cases according to the alternatives in Corollary 13. First, suppose that $Q_{2n} = \psi + Q_{2n-2}$, where ψ has determinant $-d$, $|d|^n \leqslant |\frac{1}{4}D|$. By Theorem 1, we can pick $x_3, \ldots, x_{2n}$ mod 1 so that $|Q_{2n-2}| \leqslant |\frac{1}{4}D/d|^{1/(2n-2)}$. We may now apply Lemma 4 with $\mu = Q_{2n-2}$, to choose x_1, x_2 mod 1 so that

$$|Q_{2n}| = |\psi + Q_{2n-2}| \leqslant \max[d^{1/2}, d^{1/4}|\tfrac{1}{4}D/d|^{1/4(n-1)}]$$
$$= \max[d^{1/2}, d^{(n-2)/4(n-1)}|\tfrac{1}{4}D|^{1/4(n-1)}] \leqslant |\tfrac{1}{4}D|^{1/2n},$$

since $d^n \leqslant |\frac{1}{4}D|$. This is all that is required.

In the other case, we have

$$Q_{2n} = H(x_1 + \ldots)x_2 + Q_{2n-2},$$

where $(\frac{1}{4}H)^{2n} \leqslant |\frac{1}{4}D|$. As before, we first pick $x_3, \ldots, x_{2n}$ mod 1 so that $|Q_{2n-2}| \leqslant |D/H^2|^{1/(2n-2)}$. We have now to pick $x_2 \equiv x_2^*$ and $x_1 \equiv x_1^*$. First, suppose that $x_2^* \not\equiv 0$; we simply choose x_2 so that $|x_2| \leqslant \frac{1}{2}$, and then x_1 so that $|Q_{2n}| \leqslant |\frac{1}{2}Hx_2| \leqslant |\frac{1}{4}H| \leqslant |\frac{1}{4}D|^{1/2n}$. If on the other hand $x_2^* \equiv 0$, we can certainly ensure that $|Q_{2n}| \leqslant |D/H^2|^{1/(2n-2)}$ by simply taking $x_2 = 0$, and we can ensure that $|Q_{2n}| \leqslant |\frac{1}{2}H|$ by taking $x_2 = 1$ and then choosing x_1. Thus, we can certainly ensure that

$$|Q_{2n}| \leqslant \min[|\tfrac{1}{2}H|, |D/H^2|^{1/(2n-2)}] \leqslant |\tfrac{1}{4}D|^{1/2n}.$$

This completes the proof of the theorem in all cases.

References

[1] E. Barnes, *Non-negative values of quadratic forms*, Proc. London Math. Soc. (3) 5 (1955), p. 185-196.

[2] H. Blaney, *Indefinite quadratic forms in n variables*, J. London Math. Soc. 23 (1948), p. 153-160.

[3] — *Some asymmetric inequalities*, Proc. Cambridge Philos. Soc. 46 (1950), p. 359-376.

[4] H. Davenport, *Non-homogeneous ternary quadratic forms*, Acta Math. 80 (1948), p. 65-95.

[5] — *Indefinite quadratic forms in many variables*, Mathematika 3 (1956), p. 81-101.

[6] A. Oppenheim, *Minima of quaternary quadratic forms of signature zero*, Proc. London Math. Soc. (2) 37 (1934), p. 63-81.

[7] — *Value of quadratic forms III*, Monats. Math. 57 (1953), p. 97-101.

[8] — *Value of quadratic forms I*, Quart. J. Math. Oxford Ser. (2) 4 (1953), p. 54-59.

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