On the so-called density-hypothesis in the theory of the zeta-function of Riemann

by

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§ 1. Introduction

1. If \( w = u + is \), then the zeta-function of Riemann is for \( u > 1 \) defined by

\[
\zeta(w) = \sum_{n=1}^{\infty} \frac{1}{n^w}.
\]

As is well known, \((w-1)\zeta(w)\) is an integral-function, which has an infinity of zeros, called “non-trivial” zeros, in the vertical strip \( 0 < u < 1 \), which according to Riemann have a mysterious connection with the prime-numbers. Denoting by \( N(T) \) the number of these non-trivial zeros in the parallelogram

\[
0 < u < 1, \quad 0 < v < T,
\]

we have according to Riemann-Mangoldt\(^{(1)}\) for \( T \geq 2 \)

\[
N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi} \sim \gamma \log T,
\]

where \( \gamma \) and later \( \gamma_1, \ldots \) are positive numerical constants (if some of them depend upon small parameters \( \varepsilon \) or \( \delta \), the dependence will always be explicitly stated). The famous unproved conjecture of Riemann asserts that \( \zeta(w) \neq 0 \) for \( u > \frac{1}{2} \). Recently it has been realized that many of its consequences in the number-theory could have been deduced from “density-theorems” which assert that in parallelograms

\[
u \geq a, \quad 0 < v < T, \quad \frac{1}{2} \leq a \leq 1,
\]

the number of zeros of \( \zeta(w) \) is “not too large”. More exactly, \( N(a, T) \)

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\(^{(1)}\) See e.g. [3], p. 181, the name of Mangoldt not being mentioned.
denoting the number of zeros in the parallelogram (1.1.4), this sort of theorems assert that

\[ N(a, T) < T^{\frac{1}{15}} \log^2 T \]

for \( T \geq c_1 \) uniformly for \( \frac{1}{2} \leq a \leq 1 \). Comparing (1.1.5) with (1.1.3) we get at once

\[ c_1 \geq 2. \]

After the pioneering works of Bohr-Landau (see [3], p. 197) and Carlson (see [3], p. 197) the best results were achieved by Ingham (see [3], p. 197) who proved (1.1.5) with

\[ c_1 = \frac{41}{45}. \]

Moreover he proved the inequality

\[ N(a, T) < c_2 T^{\frac{1}{45} + c_0} \log^2 T, \]

where \( T \geq 2 \) and

\[ \lambda(a) = \begin{cases} \frac{41}{45}, & \text{if } \frac{50}{61} \leq a \leq 1, \\ \frac{3}{2 - a}, & \text{if } 1 \leq a \leq \frac{50}{61}. \end{cases} \]

These results are superseded only in the neighbourhood of \( a = 1 \); I have proved \(^{(7)}\) that for a certain (small) \( c_2 \) we have for

\[ 1 - c_0 \leq a \leq 1, \quad T \geq c_0 \]

the inequality

\[ N(a, T) < T^{\frac{1}{6} + \log \log T} \log^2 T. \]

2. The proof of the inequality

\[ N(a, T) < T^{\frac{1}{6} + \log \log T}, \quad T \geq c_0(a), \quad \frac{1}{2} \leq a \leq 1 \]

would be very important in the analytical theory of numbers; e.g., we could derive from it a proof of the longstanding conjecture

\[ p_{n+1} - p_n < p_n^{\log \log n}, \quad n > p_0(a), \]

where \( p_n \) denotes the \( n \)th prime. This is called \(^{(8)}\) "the density-hypothesis."

\(^{(7)}\) See [4]. In an essentially enlarged (Chinese version) (1.1.10) is replaced by

\[ N(a, T) < T^{\frac{1}{6} + \log(1 + (a - 1))}, \]

for (1.1.9).

\(^{(8)}\) Sometimes the somewhat stronger inequality

\[ N(a, T) < c_0 T^{\frac{1}{6} + \log \log T}, \quad \frac{1}{2} \leq a \leq 1, \quad T \geq 2 \]

is called the density-hypothesis.

and is not proved so far. Ingham (see [3], p. 202) deduced it from the unproved Lindelöf-hypothesis, according to which

\[ |\zeta(s + it)| < c_0(t) |t| \]

holds for \( \frac{1}{2} < s < 1, |t| \gg 1 \). Wanting to know whether or not my methods in [4] work also "far" from the line \( s = 1 \), I gave an alternative proof\(^{(5)}\) of Ingham's results; this proof gave me the impression that they can work and that from (1.2.2) much stronger estimation of \( N(a, T) \) than (1.2.1) can be derived. In turn it seemed to me possible that the density-hypothesis (1.2.1) is deducible from results much weaker than (1.2.2). In what follows we shall show that this is indeed the case.

5. As Littlewood (see [3], p. 279) has shown, Lindelöf's conjecture (1.2.2) is equivalent to the inequality

\[ \lim_{T \to +\infty} \frac{N(a_1, T + 1) - N(a_1, T)}{\log T} = 0 \]

provided \( \frac{1}{2} < a \leq 1 \) and \( a_1 \) is fixed. Consider now the following

Conjecture A. There is a function \( g(x) \) positive and increasing for \( x > 0 \) with

\[ \lim_{x \to +\infty} g(x) = 0 \]

having the following property. Let

\[ \frac{1}{2} < x < 1, \quad 0 < \delta < \frac{1}{12}(x - \frac{1}{2}) \]

and \( x \leq a_1 \).

Then denoting by \( M(a, \delta, \beta) \) the number of zeros in the parallelogram

\[ a_1 - \delta \leq u \leq a_1, \quad |v - \tau| \leq \delta^2 \]

we have for \( \tau > c_0(x, \delta) \) the estimation

\[ M(a, \delta, \beta) < \delta g(\delta) \log^2 \frac{\tau}{2}. \]

The content of conjecture A may roughly be expressed by saying: "the concentration of zero-zeros is not too big". If Lindelöf's conjecture (1.2.2) is true, then owing to (1.3.1) the truth of conjecture A trivially follows; the converse assertion, however, is not true: (1.3.1) does not follow even with a \( c_0(a_1) \) instead of 0.

The conjecture of Lindelöf in its form (1.3.1) is not proved for any \( a_1 < 1 \). The conjecture A follows quite easily at least in the case \( a_1 = 1 \).

\(^{(5)}\) Even in a little stronger form. See [5].
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As we shall see in the Appendix, in this case the proof of the inequality

$$(1.4.5) \quad M(t, a_1, \delta < 0.71 \delta \log \frac{t}{2}$$

is easy; it requires only Jensen’s formula, the three-circle theorem and the classical inequality of Hadamard-Carathéodory.

5. In what follows we shall prove the following theorem. The truth of conjecture B implies the truth of the density-hypothesis (1.2.1).

A sketch of the proof of this theorem and a number of remarks to the conjecture B I gave in a lecture (see [5]). As to the proof of the theorem it is based not only on the methods of my book (the knowledge of which, however, is not assumed) but also on two new ideas. The starting point of my papers on the zeta-function was always the forming of appropriate identities, which connected zeta-roots with prime-numbers. But in these identities the complex variable was restricted to the half-plane $u > 1$ and therefore their full force could work only in the neighbourhood of the line $u = 1$. In this paper the starting point is another identity, which can be applied also in the critical strip. This identity (which is written, to save a step, in the form of the inequality (1.1.4)) seems to me applicable also to other aims in the theory of the zeta-function; to these I shall return elsewhere. The second idea is a simple reduction-process which enables me to replace the conjecture A by conjecture B; a more detailed description must be postponed to the first few lines of § 5.

The methods of this paper could also give the best-known estimations for $N(a, T)$ without any conjectures. E.g., before finding the new starting inequality and using as a new tool only the reduction-process I proved that for $T > \alpha$ and $1 - 2^{-12} \leq \alpha \leq 1$ the estimation

$$(1.5.1) \quad N(a, T) < T^{\log(1-\alpha)}$$

holds without any conjectures. Slight changes in the proofs of this paper and in particular in the Appendix would certainly enlarge the $\alpha$-interval in which (1.5.1) holds; we do not go into details here. I mean in particular such a change in (1.4.5) which would replace in it the constant 0.71 by another one depending on $a$ and tending to 0 with $1 - a$.

§ 2. Preliminaries to the proof

1. We shall make use of the following three inequalities, which are easy consequences of (1.1.3). For $\tau > \alpha$, we have

$$(2.1.1) \quad N(\tau) < r \log \tau,$$
Further very roughly

\[ N(r + \log \frac{1}{2} r) - N(r) > 0 \]

and for all real \( r \)-values

\[ N(r+1) - N(r) < 0.9 \log(2 + |r|). \]

2. Let \( \varepsilon \) be an arbitrarily small positive number \( < \frac{1}{4} \) and fixed. With \( \gamma(x) \) of conjecture \( B \) we determine uniquely \( \delta_1 > 0 \) by

\[ \gamma(\delta_1) \leq \varepsilon, \quad \delta_1 \leq \frac{1}{16} \varepsilon^2 \]

and let

\[ N_\varepsilon = (N_\varepsilon) = \frac{12}{\varepsilon \delta_1}. \]

Taking \( \varepsilon \) sufficiently small we have

\[ 0 < \delta_1 < \frac{1}{16}, \]

i.e.

\[ \delta_1 N_\varepsilon > 2000, \quad N_\varepsilon > 40000. \]

Further let the \( s \) of \( N(s, T) \) be restricted by

\[ \frac{1}{2} + \frac{125}{\varepsilon N_\varepsilon} \leq a \leq 1 - \max \left( 3\varepsilon + 3\varepsilon^2, \frac{6}{\varepsilon N_\varepsilon} + 3\varepsilon^2 \right) \]

and fixed. Let \( T \) be so large that

\[ T > \max \left( e^{100}, e^{1000}, \frac{1}{\log(T)} \right) < \min \left( 1, \frac{1}{N_\varepsilon}, \varepsilon \right); \]

later we shall have some more restrictions upon \( T \), but all of the type \( T > \epsilon(s) \). Further let \( k \) be an even integer \( \geq 20 \), to be determined later; at present we require only

\[ \log T \leq kN_\varepsilon \leq (1 + \epsilon) \log T. \]

Finally the complex parameter

\[ s = a + it \]

is restricted by

\[ \frac{1}{2} + \frac{2}{\log^2 T} \leq a \leq 1 - \frac{2}{\log^2 T}, \quad T \leq t \leq 2T. \]

15. Distribution of the values of an almost periodical polynomial

1. Let \( J_\alpha(s) \) be defined by

\[ J_\alpha(s) = -\frac{1}{2\pi i} \int_0^1 \left( x^{N_\varepsilon} e^{\alpha x N_\varepsilon} - e^{-\alpha x N_\varepsilon} \right) \zeta(s + iw) \frac{dw}{w}. \]

If \( A(n) \) stands for the usual Dirichlet-symbol we have the

**Lemma I.** \( J_\alpha(s) \) has also the representation

\[ J_\alpha(s) = \sum_{n \leq N_\varepsilon} \frac{A(n) R_\alpha(n, \varepsilon)}{n^s}, \]

where for the \( R_\delta(n, \varepsilon) \)-numbers the estimation

\[ |R_\delta(n, \varepsilon)| < \delta \]

holds.

**Proof.** Obviously we may insert in (3.1.1) the Dirichlet-series of \( \zeta'/\zeta \). This gives

\[ J_\alpha(s) = \sum_{n \leq N_\varepsilon} \frac{A(n)}{n^s} \frac{1}{2\pi i} \int_0^1 \left( x^{N_\varepsilon} e^{i\alpha x N_\varepsilon} - e^{-\alpha x N_\varepsilon} \right) \frac{dw}{w^s}. \]

Writing the integral in the form

\[ \int_0^1 \left( \frac{e^x - e^{-x}}{2i} \right) \frac{dx}{x} \]

we see at once, owing to the well-known integral-formula

\[ \frac{1}{2\pi i} \int_0^1 \frac{a^w}{w^s} dw = 0, \quad 0 \leq a \leq 1, \]

that each term in (3.1.5) vanishes if

\[ n \geq e^{N_\varepsilon (1 - \varepsilon)}. \]

Further an easy application of Cauchy's integral-theorem gives

\[ \frac{1}{2\pi i} \int_0^1 \left( \frac{x^{N_\varepsilon} e^{i\alpha x N_\varepsilon} - e^{-\alpha x N_\varepsilon}}{2i} \right) \frac{dx}{x^s} = \frac{1}{2\pi i} \int_0^1 \left( \frac{x^{N_\varepsilon} e^{i\alpha x N_\varepsilon} - e^{-\alpha x N_\varepsilon}}{2i} \right) \frac{dx}{x^s}, \]

and thus expanding the binom on the right we see at once that each term vanishes again for

\[ n \geq e^{N_\varepsilon (1 - \varepsilon)}. \]

owing to the well-known integral-formula

\[ \frac{1}{2\pi i} \int_0^1 \frac{a^w}{w^s} dw = 0, \quad a \geq 1. \]
Thus (3.1.2) is proved with
\[
R_d(n, \varepsilon) = \frac{1}{2\pi i} \int_{C_\varepsilon} \left( e^{xw} - e^{-xw} \right) \frac{e^{-2\pi i w}}{2\pi w} \, dw.
\]

Cauchy's integral-theorem gives again
\[
(3.1.6) \quad R_d(n, \varepsilon) = \frac{1}{2\pi i} \int_{C_\varepsilon} \left( e^{xw} - e^{-xw} \right) \frac{e^{-2\pi i w}}{2\pi w} \, dw = \frac{1}{\pi} \int_{\mathbb{R}} \left( \int_{C_\varepsilon} \left( e^{xw} - e^{-xw} \right) \frac{1}{2\pi w} \, dw \right) \sin^2 N_1 v \, dv.
\]

Hence we have
\[
|R_d(n, \varepsilon)| \leq \frac{1}{\pi^2 N_1} \int_{\mathbb{R}} \left( \int_{C_\varepsilon} \frac{1}{2\pi w} \, dw \right) \sin^2 (kN_1 - \log n) v \, dv.
\]

and thus very roughly
\[
|R_d(n, \varepsilon)| \leq \frac{1}{\pi^2 N_1} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \frac{1}{2\pi w} \, dw \right) \sin^2 \left( \frac{v}{N_1} \right) \, dv \leq \frac{1}{N_1^2} \delta_1
\]

using (2.2.2).

2. Writing
\[
U_d(\sigma) = \frac{2T}{\delta} \int_{\mathbb{R}} |J_d(\sigma + it)| \, dt
\]
we have the Lemma II. The inequality(1)
\[
U_d(\sigma) < c_d(\varepsilon) T^{\sigma - 1 + \delta_3} \log^2 T
\]
holds.

Proof. If we write shortly
\[
J_d(\sigma) = \sum_{|a_n| < 2|a_n|^{1/2}} a_n e^{2\pi i a_n(\sigma + it)}
\]
we have, by Lemma I,
\[
(3.2.1) \quad |a_n| \leq \delta_3 \log n.
\]

The usual technique gives
\[
U_d(\sigma) < T \sum_n |a_n|^2 \frac{1}{m^2} + 4 \sum_{m,n} \frac{|a_m||a_n|}{(mn)^2} \log (m/n)^2,
\]
i. e., using (3.2.1) and (2.2.7), we obtain
\[
U_d(\sigma) < 8 \delta_1^2 \sum_{|a_n| < 2|a_n|^{1/2}} \log^2 n \cdot n^{-1} \delta_1^2 + 4 \delta_1^2 \sum_{m,n} \frac{\log m \log n}{(mn)^2} \log (m/n)^2
\]
\[
< 2T \delta_1^2 (kN_1)^2 \sum_{|a_n| < 2|a_n|^{1/2}} n^{-1} \delta_1^2 + 16 \delta_1^2 (kN_1)^2 \sum_{m,n} \frac{1}{(mn)^2} \log (m/n)^2.
\]

Owing to (2.2.3), this evidently gives
\[
U_d(\sigma) < c_d(\varepsilon) T^{\sigma - 1 + \delta_3} \log^2 T
\]
using (2.2.2).
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Lemma III. For \( T > c_0(s) \) with the exception of at most

\[
\frac{3}{2} \sum_{n=1}^{T^{\frac{1}{2}}-1} \frac{\pi(n)}{\log T} \leq c_0(s) T^{\frac{1}{2}} \log T
\]

traditional strips to each "good" \( 1 \) strips for the points

\[
\sigma_+ - i t_n = s_n, \quad 2 \leq \sigma \leq \frac{1}{2} [\log^2 T] - 3
\]

the inequality

\[
|J_s(e^{i\theta})| \leq c_0(s) T^{-s+\delta} \log^4 T
\]

holds for each permitted \( k \)-value in (2.27).

This is the assertion to which the title of this paragraph refers.

4. Connection of \( J_s(e^{i\theta}) \) with the zeros of \( \zeta(s) \)

The connection that we are going to show will be proved for all \( s \)-values with

\[
\frac{1}{2} \leq \sigma \leq 1, \quad T \leq t \leq 2T,
\]

but actually used only for the \( \sigma \)-numbers defined above. We start from the representation (3.1.1). The usual contour-integration gives at once

\[
J_s(e^{i\theta}) = \frac{e^{\pi i(s-1)\theta} e^{\frac{\pi^2 \theta^2}{4}}}{2\pi i N_1(1-e^{i\theta})} \left( 1 + \sum_{n=1}^{T^{\frac{1}{2}}-1} \frac{\pi(n)}{\log T} e^{2\pi i\theta n} e^{2\pi i\theta} \right) \frac{1}{\zeta(1-\theta)}
\]

Denoting the last integral by \( D \) we get by (2.2.4)

\[
|D| \leq \frac{e^{-\pi N_1(1+\theta)+e^{2\pi i\theta}N_1(1+\theta)}}{2\pi} \left( \frac{2\pi N_1}{1+e^{2\pi i\theta}} \right)^{\frac{3}{2}} \left( 1 - 1 + i(t+\epsilon) \right) \left( \frac{1}{i(t+\epsilon)} \right) \left( 1 - 1 + i(t+\epsilon) \right) \left( 1 - 1 + i(t+\epsilon) \right) \left( 1 - 1 + i(t+\epsilon) \right)
\]

As is well known,

\[
\left( \frac{1}{\zeta(1+i(t+\epsilon))} \right) \leq c_0 \log(2+i(t+\epsilon))
\]
from this, (4.1.2) and (2.2.7) we get

(4.1.3) \[ |D| \leq c_2 T^{(3/5)(1 - \delta)} \log T. \]

Further we have roughly

\[
\left| e^{i\pi y_0} e^{2\pi i N(1) \frac{s}{2} - e^{2\pi i N(1) \frac{s}{2}}} \right| \leq \frac{e^{2\pi N(1) \frac{s}{2}}}{T^2} \leq T^{a + b + 3(1 - \delta)} \ll T^{-\alpha},
\]

using again (2.2.7). From this, (4.1.3) and (4.1.1) we get

(4.1.4) \[ J_N(s) \ll \sum_{\varepsilon} \left| e^{\frac{1}{2} \pi (1 - \delta)} \frac{e^{2\pi i N(1) \frac{s}{2} - e^{2\pi i N(1) \frac{s}{2}}} \right| \leq c_2 T^{(3/5)(1 - \delta)} \log T. \]

Comparing this with lemma III we obtain

LEMMMA IV. For \( T > c_0 \varepsilon \), we have for each "good" \( L \)-strip and \( \varepsilon \)-numbers in (3.4.5) the inequality

(4.1.5) \[ \sum_{\varepsilon} \left| e^{\frac{1}{2} \pi (1 - \delta)} \frac{e^{2\pi i N(1) \frac{s}{2} - e^{2\pi i N(1) \frac{s}{2}}} \right| \leq c_2 (T^{a - \varepsilon + \alpha} \log T). \]

§ 5. A reduction-process

1. Let

(5.1.1) \[ \eta_1^* = \eta_1 + \delta \]

stand for a zero of \( \zeta(\varepsilon) \) in \( L \) with the greatest real part if there exists a zero in \( L \) at all; in this case we may call it an "extreme right-hand" zero in \( L \). If there are more zeros in \( L \) with the same maximal real parts, choose an arbitrary one as \( \eta_1 \) and fix it. We may vaguely call a "neighbourhood" of an \( L \)-strip those strips whose indices differ "not much" from \( j \). The aim of this paragraph is to show that omitting "not too many" "good" strips, we may reduce the study of the zeros to the study of zeros in such "good" \( L \)-strips, which contain indeed zeros and for which the \( \eta_1 \) zeros in "essentially the extreme right-hand in a big neighbourhood of \( L \)."

In order to give an exact meaning to what has been said above, we call a "block" a maximal sequence of "good" strips with consecutive indices. We speak of a "short block" if the "block" contains at most

(5.1.2) \[ 24 \log^2 T + 4 \]

consecutive strips (all of which are of course "good" strips); in the opposite case we speak about "long blocks". Since a block is bordered on both sides by "bad" strips, (3.4.4) implies at once that the total number of "blocks" and thus a fortiori that of the "short blocks" is

\[
\ll \frac{T^{3(1 - \alpha)}}{\log T}.
\]

Thus the number of "good" strips contained in "short blocks" is

(5.1.3) \[ \ll \frac{T^{3(1 - \alpha)}}{\log T (24 \log^2 T + 4)} \ll \frac{T^{3(1 - \alpha)}}{\log^2 T}. \]

Considering (3.1.3), (5.1.3) and (3.4.4) imply at once that the contribution of the "bad" strips and of the "good" strips in "short blocks" to the total number of zeros in

\[ T \leq \varepsilon \leq 2T, \]

i. e., a fortiori to the total number of zeros in the parallelogram

(5.1.4) \[ a \leq u \leq 1 \quad (T \leq u \leq 2T), \]

is at most

(5.1.5) \[ \frac{T^{3(1 - \alpha)}}{\log T}. \]

2. Hence we have to consider only the contribution of the "good" strips belonging to "long blocks" to the number of zeros in the domain

(5.1.4), i. e., to

\[ N(a, 2T) - N(a, T). \]

Consider an \( A \) "long block" and divide it starting from the \( L \)-strip with the smallest index into "sub-blocks" each containing

(5.2.1) \[ 2[3 \log^2 T] + 1 \]

consecutive strips; the last "sub-block" may contain less. The contribution of these "incomplete sub-blocks" to \( N(a, 2T) - N(a, T) \) is obviously at most

(5.2.2) \[ \frac{T^{3(1 - \alpha)}}{\log T} \left[ 2[3 \log^2 T] \right] c_1 \log (2 + 2T) \ll \frac{T^{3(1 - \alpha)}}{\log^2 T}. \]

Thus it remains to consider the contribution of those "long blocks" to \( N(a, 2T) - N(a, T) \) which consist of "sub-blocks", each containing exactly \( 2[3 \log^2 T] + 1 \) consecutive "good" strips. We remark that obviously each "long block" contains owing to (5.1.2) and (5.2.1) at least four "sub-blocks".
3. We consider the “extreme right-hand” zeros \( \zeta^*_l \) defined in (5.3.1) belonging to the different good \( f_l \)-stripes. We shall call such a \( \zeta^*_l \)-zero belonging to a “good” \( f_l \)-strip an “outstanding” one, if \( \zeta(w) \neq 0 \) in the parallelogram

\[
\begin{align*}
\eta \geq \sigma^*_l + \frac{1}{\log^2 T}, \\
|e - \sigma^*_l| \leq \log T.
\end{align*}
\]

(5.3.1)

We assert the simple

**Lemma V.** Each “sub-block” of a “long block” contains at least one “outstanding” \( \zeta^*_l \)-zero.

**Proof.** We consider an “arbitrary sub-block” \( E \) of our “long block” and consider first the strip \( f_l \) in the middle of \( E \) (\( E \) consists of an odd number of strips). Perhaps this \( f_l \)-strip does not contain any \( \zeta \)-zeros; but owing to (2.1.2) there is a \( f_l \)-index such that the \( f_l \)-strip indeed contains \( \zeta \)-zeros and for this \( f_l \)-index we have

\[
|j - j_l| \leq \log T|\log T|, 
\]

(5.3.2)

If \( \sigma^*_l \) is not an “outstanding” zero, this means that there is a \( f_l \)-index with

\[
|j - j_l| \leq \log T|\log T|,
\]

i.e., by (5.3.2), with

\[
|j_l - j| \leq 2f|\log^2 T|\log T|
\]

(5.3.3)
such that

\[
\sigma^*_l \geq \sigma^*_n + \frac{1}{\log^2 T}.
\]

(5.3.4)

If also \( \sigma^*_n \) were not an “outstanding” zero, then there would be a \( f_l \)-index with

\[
|j_l - j| \leq \log^2 T|\log T|,
\]

i.e., by (5.3.3), with

\[
|j - j| \leq 3|\log^2 T|\log T|
\]

(5.3.3)
such that

\[
\sigma^*_l \geq \sigma^*_n + \frac{1}{\log^2 T} \geq \sigma^*_n + \frac{2}{\log^2 T}.
\]

(5.3.4)

taking (5.3.4) in account. Since \( \zeta(w) \neq 0 \) for \( n \geq 1 \), this process necessarily stops within \( \log^2 T \) such steps. Hence the index of the last strip is at most

\[
\frac{1}{\log^2 T} \quad \text{or} \quad \frac{2}{\log^2 T},
\]

\( f + ([\log^2 T] + 1)[\log^2 T][\log T] < f + [2\log^2 T] < f + [3\log^2 T], \)

i.e., the corresponding \( f \)-strip still belongs to our sub-block \( E \), i.e.,

\[
(5.4.1)
\]

As to the contribution of all “kernels” we assert

**Lemma VI.** If \( \lambda \) is an upper bound of the real parts of the “outstanding” zeros lying in the “(good)” \( f_l \)-stripes of all “long blocks”, then in all “kernels” \( \zeta(w) \) does not vanish for

\[
\eta > \lambda.
\]

**Proof.** Let \( \gamma \) be the maximum of the real parts of zeros lying in the “kernels”. If \( \gamma \) is at the same time the real part of an “outstanding” zero, the proof is finished. If not, then applying the process used in the proof of Lemma V we shall find that \( \gamma \) is majorized by an “outstanding” zero of a “wing”; but this means that again

\[
\gamma \leq \lambda,
\]

from which our lemma follows.

4. As has been said, each “long block” \( A \) consists of at least four “sub-blocks”. Let us call the two “sub-blocks” containing the \( f_l \)-stripes of \( A \) with the greatest, or the smallest, indices the “wings” of our “long block” \( A \), and \( A \) without the “wings” — the “kernel” of \( A \). According to the introductory remark of this section 4 the “kernel” of \( A \) is not empty. The contribution of all “wings” to \( N(\alpha, 2T) - N(\alpha, T) \) is obviously

\[
(5.4.1)
\]

As to the contribution of all “kernels” we assert

**Lemma VI.** If \( \lambda \) is an upper bound of the real parts of the “outstanding” zeros lying in the “(good)” \( f_l \)-stripes of all “long blocks”, then in all “kernels” \( \zeta(w) \) does not vanish for

\[
\eta > \lambda.
\]

**Proof.** Let \( \gamma \) be the maximum of the real parts of zeros lying in the “kernels”. If \( \gamma \) is at the same time the real part of an “outstanding” zero, the proof is finished. If not, then applying the process used in the proof of Lemma V we shall find that \( \gamma \) is majorized by an “outstanding” zero of a “wing”; but this means that again

\[
\gamma \leq \lambda,
\]

from which our lemma follows.

5. It follows immediately from this lemma that if for \( T > c_\delta(\epsilon) \) and for the above \( \lambda \) the inequality

\[
(5.5.1)
\]

holds, then owing to (5.1.5), (5.2.5), (5.4.1) and Lemma VI the estimation

\[
(5.5.2)
\]

holds. We are going to prove in the next chapter that (5.5.1) holds for all “outstanding” zeros (i.e., not only for those belonging to the “long blocks”). Here we shall see why the reduction to the “outstanding” zeros has been so essential.
§ 6. The upper estimation of the real parts of the “outstanding” zeros

1. We consider an arbitrary “good” $L$-strip where the “extreme right-hand” zero is an “outstanding” one, i.e., with

\[ \sigma' \leq \sigma' + \frac{1}{\log T}, \quad |v - \sigma'| \leq \log T. \]

The index $j$ is fixed. Then we have the crucial

**Lemma VII.** We have

\[ \sigma' \leq a + 3^r. \]

**Proof.** If this lemma were false, then we have

\[ a + 3^r < \frac{1}{\log T}. \]

We shall apply Lemma IV with

\[ \mu = j; \]

the $r$-index is uniquely determined by requiring

\[ a = \frac{1}{\log T} = \frac{1}{\log T}. \]

Owing to (6.1.3) and (2.2.5) the condition

\[ 2 \leq r \leq 4[\log T] - 2 \]

is fulfilled for $T > c_\delta(e)$. Owing to the definition we have

\[ \left| \frac{\sigma' - T}{\log T} \right| < \frac{2}{\log T}. \]

Owing to (6.1.2) the inequality (4.1.5) can be written in the form

\[ \left| \sum_{\nu = \nu(v) > \log T} \frac{e^{\nu(v)-v\nu} - e^{\nu(v)-\nu(v)}}{2^{\nu(v)}(\nu - \nu)} \right| \leq c_\delta(e) T^{\nu(v)} \log^{2} \log T, \]

\[ + \sum_{\nu = \nu(v) > \log T} \frac{e^{\nu(v)-v\nu} - e^{-\nu(v)}}{2^{\nu(v)}(\nu - \nu)} \leq c_\delta(e) T^{\nu(v)} \log^{2} \log T, \]

where $\nu = a_i + it_i$ stand for the zeros of $\xi(w)$. By (6.1.5) the expression on the right of (6.1.7) is a fortiori

\[ \leq c_\delta(e) T^{\nu(v)} \log^{2} \log T. \]

2. We estimate the first sum on the left of (6.1.7) roughly by (2.1.3).

This implies at once that this sum is absolutely

\[ \leq 2c_\delta(e) \sum_{\nu = \log T} \log(2 + \nu - v) e^{2\nu(v)-v\nu} \frac{\nu^{1+\nu}}{(\nu)^{\nu^2} \nu^2} \]

i.e., owing to (2.2.4) and (2.2.7) for $T > c_\delta(e)$,

\[ < c_\delta(e) \frac{T^{2}}{\log T} \leq c_\delta(e) \frac{T^{2}}{(\log T)^{2} \log^{2} \log T} < \frac{1}{2}. \]

This, (6.1.7) and (6.1.8) give together for $T > c_\delta(e)$

\[ \left| \sum_{\nu = \nu(v) > \log T} \frac{e^{\nu(v)-v\nu} - e^{-\nu(v)}}{2^{\nu(v)}(\nu - \nu)} \right| \leq c_\delta(e) T^{\nu(v)} \log^{2} \log T. \]

Next we consider the contribution of the zeros with

\[ \sigma' \leq \frac{1}{\log T}, \quad |v - \sigma'| \leq \log T. \]

Since from (6.2.2) and (6.1.5)

\[ \text{Re} N_i(\nu - \nu) \leq N_i \left( \sigma' - \frac{6}{\log T} \right) \]

we have owing to (2.2.7)

\[ \left| e^{\nu(v)-v\nu} \frac{e^{\nu(v)-v\nu} - e^{-\nu(v)}}{2^{\nu(v)}(\nu - \nu)} \right| \leq \frac{e^{\nu(v)}}{6} < \frac{1}{2}. \]

hence the use of (2.1.3) implies at once that the contribution of these $\nu$'s is absolutely

\[ \leq c_\delta(e) T^{\nu(v)} \log^{2} \log T. \]

Next we consider the contribution of the zeros with

\[ \sigma' - \frac{6}{\log T} < \sigma' \leq \frac{1}{\log T}, \quad \frac{6}{\log T} |v - \sigma'| \leq \log T. \]


Since owing to (6.1.5)
\[
\left| e^{\nu_1(1-\gamma_1)} \frac{e^{2N_1(1-\gamma_1)}}{2\pi N_1(\eta - s_1)} - e^{-\nu_1(1-\gamma_1)} \frac{e^{-2N_1(1-\gamma_1)}}{2\pi N_1(\eta - s_1)} \right| \leq \left( \begin{array}{c} d^2 \nu_1 e^{\nu_1} \log \nu_1 \\ 2\pi N_1(\eta - s_1) \end{array} \right) \leq \frac{e^{\nu_1} N_1}{\eta - s_1} \leq \frac{e^{\nu_1} N_1}{\eta - s_1},
\]
we have using (2.2.7) for the absolute value of this contribution the upper bound
\[
q_1 \log T - \frac{q_1 \log T}{6^k} < q_1 T^{-2N_1/\log T}.
\]
Combining (6.2.1), (6.2.3) and (6.2.5) we obtain
\[
(6.2.6) \quad Z = \sum_{\nu_1, \gamma_1 = 0}^{\log T} \left| e^{\nu_1(1-\gamma_1)} \frac{e^{2N_1(1-\gamma_1)}}{2\pi N_1(\eta - s_1)} \right| \leq q_2(q_1 \log T) \left( T^{-2N_1/\log T} + T^{-2N_1/\log T} \right)
\]
for $T > q_2(q_1)$.

3. Until now we have not used the conjecture B. We shall use it in estimating the number $\nu_1$ of terms in $Z$. Owing to (2.2.6) this number is not decreased when we replace it by the number of zeros in the square
\[
(6.3.1) \quad \sigma^* - \frac{6}{e^{\nu_1}} \leq \sigma \leq \sigma^* + \frac{6}{e^{\nu_1}}, \quad |t - \gamma_1| \leq \frac{6}{e^{\nu_1}}.
\]
We apply the conjecture B with
\[
(6.3.2) \quad a_0 = \sigma^* + \frac{6}{e^{\nu_1}}, \quad a = \frac{12}{e^{\nu_1}} (\sigma^*), \quad x = \frac{1}{2}, \quad \tau = \gamma_1.
\]
Then owing to (2.2.6) the domain (1.4.2) is contained in the parallellogram
\[
\sigma^* + \frac{1}{[\log T]}, \quad u \leq 1, \quad |u - \gamma_1| \leq [\log T],
\]
which is indeed free of zeros of $\zeta(\omega)$ owing to the definition of the "outstanding" zeros. The squares (6.3.1) and (1.4.3) are then identical, further by (2.2.4) $0 < \beta \leq \frac{1}{2}(\eta - \frac{1}{2})$ is fulfilled as well as $\frac{1}{2} < \gamma < 1$. Finally by (6.1.3) and (2.2.5)
\[
\sigma^* > a \geq \frac{1}{2} + \frac{120}{e^{\nu_1}}
\]
is also satisfied and thus conjecture B is indeed applicable. Formula (1.4.1) gives for $\nu_1$ by (2.2.1) and (2.2.2) the upper bound
\[
(6.3.3) \quad \delta^2 \log T \left( \frac{12}{N_1} e^{\nu_1} e^{\nu_1} \log T < \frac{e^2}{N_1} \log T.
\]
Let us denote this last quantity by $L_1$ in the sequel.

4. Up to this point the integer $k$ has only been restricted by the inequality (2.2.7). We shall now estimate $Z$ from below by an appropriate choice of $k$ within the given limits (2.2.7). This will be done by the following theorem.

If \( \tau \)
\[
(6.4.1) \quad |\tau| \geq |\tau_1| \geq \ldots \geq |\tau_m|
\]
are arbitrary complex numbers, $m > 0$ arbitrary real and $n \leq L_1$, then there is an integer $\nu$ with
\[
(6.4.2) \quad m \leq \nu \leq m + L_1
\]
such that
\[
(6.4.3) \quad |\tau_1 + \tau_2 + \ldots + \tau_m| \geq \frac{L_1}{25(m + L_1)} |\tau_1|.
\]

If we choose as $z$-vectors the quantities
\[
(6.4.4) \quad e^{\pi N_1(1-\gamma_1)} \frac{e^{2N_1(1-\gamma_1)}}{2\pi N_1(\eta - s_1)}
\]
these vectors and the domain of summation are independent of $\zeta$, and hence the number of terms in the sum of (6.2.6) is independent of $\zeta$. Hence the sum in (6.2.6) is a power-sum of fixed complex numbers. We choose as $m$ of (6.4.2)
\[
(6.4.5) \quad m = \frac{1}{N_1} \log T.
\]

Owing to (6.3.3) the number of $z$'s is at most $L_1$, i.e., the interval $(m, m + L_1)$ is identical with the interval given for $\zeta$ in (2.2.7); thus $\zeta$ can be chosen as the $\nu$ of theorem (6.4.1), (6.4.2), (6.4.3). The factor $|\tau_1|$
can be estimated from below by taking the term corresponding to \( \phi = \phi^* \). Owing to (6.1.5), (2.2.7) and (2.2.8) we have

\[
(6.4.6) \quad e^{\varepsilon N_1(\phi^*-\phi)} = e^{\varepsilon N_1(\phi^*-\phi)} > e^{-(1+ \log^3 T) \log^2 T} > \frac{1}{2}.
\]

Further using the inequality valid for \( |s| \leq \frac{1}{2} \)

\[
\frac{e^{-s^2} - e^{-s^2}}{2s} = 1 + \frac{e^s}{3!} + \frac{e^s}{5!} + \ldots > 1 - \frac{|s|^2}{3}
\]

and observing that for \( T > c_0(\varepsilon) \) owing to (6.1.5) and (6.1.6)

\[
e^{\varepsilon N_1(\phi^*-\phi)} \leq e^{\varepsilon N_1} \sqrt{\frac{1}{1 + \log^2 T}} + \frac{1}{2} \log^2 T < \frac{1}{2},
\]

we have by (2.2.7)

\[
\left| e^{\varepsilon N_1(\phi^*-\phi)} - e^{-\varepsilon N_1(\phi^*-\phi)} \right| < e^{-\varepsilon N_1(\phi^*-\phi)} e^{\varepsilon N_1} \sqrt{\frac{1}{1 + \log^2 T}} + \frac{1}{2} \log^2 T < \frac{1}{2},
\]

This, (6.4.6) and (6.4.1) give

\[
|s|^2 < \frac{1}{4}
\]

and hence (6.4.3) gives the lower estimation

\[
Z > \frac{1}{4} \left( \frac{L_1}{23(m+L_0)} \right)^{\frac{1}{2}} = \frac{1}{4} \left( \frac{e^{\varepsilon N_1(\phi^*-\phi)}}{23(N_1^{-1} + e^{\varepsilon N_1})} \right) = \frac{1}{4} T^{-\varepsilon N_1(\phi^*-\phi)} > \frac{1}{4} T^{-\varepsilon N_1(\phi^*-\phi)}.
\]

Comparing this with (6.2.6) we get for \( T > c_0(\varepsilon) \)

\[
\frac{1}{4} T^{-\varepsilon N_1(\phi^*-\phi)} < c_0(\varepsilon) \log^3 T \left[ 2^{1-a} + \frac{1}{2} + \left( \frac{25}{2} \right) \right],
\]

i.e., since for all sufficiently small \( \varepsilon \)'s

\[
e^{\varepsilon N_1(\phi^*-\phi)} < 1,
\]

then

\[
e^{-\varepsilon N_1(\phi^*-\phi)} < c_0(\varepsilon) \log^3 T \left[ 2^{1-a} + \frac{1}{2} + \left( \frac{25}{2} \right) \right],
\]

and for \( T > c_0(\varepsilon) \)

\[
T^{-\varepsilon N_1(\phi^*-\phi)} < \frac{1}{2} + \frac{1}{2} e^{\varepsilon N_1(\phi^*-\phi)} \log^3 T,
\]

which means that for all sufficiently small \( \varepsilon \)'s

\[
e^{\varepsilon N_1(\phi^*-\phi)} < \frac{1}{2} + \frac{1}{2} e^{\varepsilon N_1(\phi^*-\phi)} \log^3 T,
\]

i.e., lemma VII is proved.

\section{7. Proof of the theorem}

1. According to lemma VII and (5.5.3) we have for the \( \alpha \)'s in (2.2.5) for \( T > c_0(\varepsilon) \) a fortiori

\[
(7.1.1) \quad N(\alpha + 3\varepsilon^2, 2T) - N(\alpha + \varepsilon^2, T) < T^{\frac{1}{2} - \nu}.
\]

Replacing \( T \) by \( T/2, T/2^2, \ldots, T/2^n \), where

\[
\frac{T}{2^n} > c_0(\varepsilon) \geq \frac{T}{2^n-1}
\]

and summing, we get

\[
N(\alpha + 3\varepsilon^2, T) < c_0(\varepsilon) T^{\frac{1}{2} - \nu},
\]

or owing to (2.2.5) for

\[
(7.1.2) \quad \frac{1}{2} + \frac{1}{2} e^{\varepsilon N_1(\phi^*-\phi)} < \alpha \leq 1 - \max \left( 3\varepsilon, \frac{6}{e^{\varepsilon N_1(\phi^*-\phi)}} \right)
\]

the inequality

\[
(7.1.3) \quad N(\alpha, T) < c_0(\varepsilon) T^{\frac{1}{2} - \nu + \varepsilon^2}.
\]

Since for

\[
\alpha \leq 1 - 3\varepsilon
\]

we have

\[
2(1 - \alpha) < 2(1 - \varepsilon) < 2(1 + \varepsilon)(1 - \alpha)
\]

and by (2.2.2) and (2.2.3)

\[
125/e^{\varepsilon N_1} < 11\delta_1 < e^\varepsilon,
\]

we obtain for \( T > c_0(\varepsilon) \) and

\[
(7.1.4) \quad N(\alpha, T) < c_0(\varepsilon) T^{\frac{1}{2} - \nu + \varepsilon(1 - \alpha)}.
\]
The case

\[(7.1.5) \quad 1 - \max \left( 3e, \frac{6}{eN}, \right) \leq a \leq 1\]

is, by (1.1.10) already settled, provided \(e\) is so small that with the \(e\) from (1.1.9)

\[\frac{6}{eN} < a < 3e < a;\]

the exponent of \(T\) in (7.1.4) becomes

\[(7.1.6) \quad 2(1-a)[1+300(1-a)^{3,8}] \leq 2 \left[ 1+300 \left( \max \left( 3e, \frac{6}{eN} \right) \right)^{7/6} \right](1-a).\]

Since the case

\[(7.1.7) \quad 1 < a < 1 + 4a^2\]

is trivial, our theorem is proved.

18. Appendix

1. As has been said, we shall outline for \(\tau > \tau_0(\delta, \beta)\) a proof of the inequality

\[(8.1.1) \quad M(\tau, a_0, \beta) \leq 0.71 \log \frac{\tau}{2}\]

for the number \(M(\tau, a_0, \beta)\) of the zeros of \(\zeta(w)\) in the parallelogram

\[(8.1.2) \quad a_1 - \delta \leq u \leq a_0, \quad |v - \tau| \leq \frac{\delta}{2},\]

when \(\frac{1}{4} < \tau < 1, a_0 \leq a_0 \leq 1, 0 < \delta \leq \frac{1}{4}(x - \frac{1}{2})\) and \(\zeta(w)\) does not vanish in the parallelogram

\[(8.1.3) \quad a_0 \leq u \leq 1, \quad |v - \tau| \leq \frac{\log \tau}{2}.\]

First we need the simple

**Lemma VIII.** If \(\tau > \tau_0(\beta)\), then in the domain

\[(8.1.4) \quad u > a_0 + 48 \cdot \frac{\log \log \tau}{\log \tau}, \quad |v - \tau| \leq \frac{3}{4} \log \tau\]

the inequality

\[(8.1.5) \quad \left| \frac{\zeta' (w)}{\zeta (w)} \right| \leq \frac{\log \tau}{(\log \log \tau)^2}\]

holds.

**Proof.** For the sake of brevity denote

\[(8.1.6) \quad 16 \log \log \tau = A\]

and take \(\tau\) so large that

\[(8.1.7) \quad 3 \log \tau < \log \frac{\tau}{2} - 2.\]

We apply the inequality of Hadamard-Carathéodory \((\S)\) to the circle

\[(8.1.8) \quad |w - \frac{3}{4} - i\tau_6| \leq \frac{3}{4} - a,\]

with

\[|\tau - \tau_6| \leq \left[ \log \frac{\tau}{2} \right] - 1,\]

and to the function

\[\log \frac{\zeta(w)}{\zeta\left( \frac{3}{4} + i\tau_6 \right)},\]

which is certainly regular in our circle. Since in the circle we have roughly for \(\tau > \tau_0\)

\[|\zeta(w)| < \tau,\]

it follows that in the circle

\[(8.1.9) \quad |w - \frac{3}{4} - i\tau_6| \leq \frac{3}{4} - a - A\]

the inequality

\[\left| \log \frac{\zeta(w)}{\zeta\left( \frac{3}{4} + i\tau_6 \right)} \right| \leq c_3 \log \frac{\tau}{A},\]

i.e., also

\[(8.1.10) \quad |\log \zeta(w)| \leq c_3 \log \frac{\tau}{A}\]

\(\S\) According to this theorem if \(f(w)\) is regular for \(|w - \omega_0| \leq R\) and here \(Re f(w) \leq M\), then for \(|w - \omega_0| \leq \tau (\tau < R)\) we have

\[|f(w) - f(\omega_0)| \leq \frac{2\tau}{R - \tau} (M - Re f(\omega_0)).\]
holds. Next we apply the three circle-theorem to log $\zeta'(w)$ and to the circles
\begin{align*}
K_3: & \quad \left| \frac{w - \frac{1}{2} - it_0}{\frac{1}{2} - t_0} \right| \leq 1 - a_3 - 2d, \\
K_4: & \quad \left| \frac{w - \frac{1}{2} - it_0}{\frac{1}{2} - t_0} \right| \leq 1 - a_4 - 2d, \\
K_5: & \quad \left| \frac{w - \frac{3}{2} - it_0}{\frac{3}{2} - t_0} \right| \leq 1.
\end{align*}
This gives by (8.1.10)
\begin{equation}
(3.1.11) \quad \max_{w \in K_5} \log |\zeta'(w)| \leq c_3 \left( \frac{\log \tau}{\log \log \tau} \right)^{1 - \frac{1}{4}}.
\end{equation}
The exponent for $\tau > c_3$ is
\begin{equation}
\frac{(1 - \frac{1}{2} - 4d)}{\log(6 - 4a_3 - 4d)} < 1 - \frac{4d}{(6 - 4a_3 - 4d)(6 - 4a_3 - 4d)} < 1 - \frac{1}{4d},
\end{equation}
\text{i.e., by (3.1.11), for } \tau > c_3
\begin{equation}
\max_{w \in K_5} \log |\zeta'(w)| \leq \left( \log \log \tau \right)^{1 - \frac{1}{2}} \frac{\log \tau}{\log \log \tau} = \log \frac{\log \tau}{\log \log \tau}
\end{equation}
Then we have in the circle
\begin{align*}
K_4: & \quad \left| \frac{w - \frac{1}{2} - it_0}{\frac{1}{2} - t_0} \right| \leq 1 - a_4 - 3d \\
\text{the estimation}
\begin{align*}
\left| \frac{\zeta''}{\zeta} (w) \right| & \leq \frac{1}{A} \log \tau \leq \log \frac{\log \tau}{\log \log \tau}.
\end{align*}
\end{align*}
Since the circles $K_4$ cover $\tau > c_4$, the parallelogram
\begin{align*}
\Delta: & \quad \Delta = [a_4 + 3d, a_4 + 3d] \\
\text{when } \tau, \text{ varies continuously between}
\begin{align*}
\tau \pm \left( \log \frac{\tau}{2} - 1 \right),
\end{align*}
\text{lemma VIII is proved owing to (8.1.7).}

2. Next we turn to the proof of (8.1.1). As is well known (see [3], p. 31),
\begin{equation}
\zeta''(w) = \frac{1}{w-1} + \sum_{\rho} \left( \frac{1}{w-\rho} + \frac{1}{\rho} \right) + \frac{1}{2} \frac{\zeta''(w)}{\log \log \tau} + \frac{1}{2} \frac{f''(w)}{f'(w)^2}.
\end{equation}
where $b$ is a constant. Restricting $w$ to the parallelogram
\begin{equation}
\frac{1}{4} \log \log \tau \leq \tau \leq 1, \quad |\sigma - 1| \leq \frac{3}{4} \log \tau
\end{equation}
and taking real parts we obtain owing to lemma VIII for $\tau > c_3$
\begin{equation}
\left| \sum_{\rho} \left( \frac{u - \rho}{(u - \rho)^2 + (\sigma - \frac{1}{2})^2} - \frac{1}{2} \frac{\log f'(w)}{f'(w)^2} + 1 \right) \right| \leq \frac{2}{\log \log \tau} \log \tau.
\end{equation}
or by Stirling's formula
\begin{equation}
\left| \sum_{\rho} \left( \frac{u - \rho}{(u - \rho)^2 + (\sigma - \frac{1}{2})^2} - \frac{1}{2} \frac{\log f'(w)}{f'(w)^2} + 1 \right) \right| \leq \frac{3}{\log \log \tau} \log \tau.
\end{equation}
The contribution of the $\rho$-zeros with
\begin{equation}
|\sigma - 1| > \left( \log \frac{\tau}{2} \right)
\end{equation}
is by (2.1.3) absolutely
\begin{equation}
\leq c_3 \tau \sum_{\rho \in \rho_{\tau}} \frac{\log \tau + u + 2}{n^2} < c_3,
\end{equation}
\text{i.e., by (8.2.3) and (8.1.3),}
\begin{equation}
\left| \sum_{\rho \in \rho_{\tau}} \left( \frac{u - \rho}{(u - \rho)^2 + (\sigma - \frac{1}{2})^2} - \frac{1}{2} \frac{\log f'(w)}{f'(w)^2} + 1 \right) \right| \leq \frac{4}{\log \log \tau} \log \tau.
\end{equation}
or
\begin{equation}
\left| \sum_{\rho \in \rho_{\tau}} \left( \frac{u - \rho}{(u - \rho)^2 + (\sigma - \frac{1}{2})^2} - \frac{1}{2} \frac{\log f'(w)}{f'(w)^2} + 1 \right) \right| \leq \frac{1}{2} \frac{\log \tau + 4}{\log \log \tau}.
\end{equation}
Owing to (8.2.2) the terms of the sum in (8.2.5) are non-negative. We choose $\tau$ so large that
\begin{equation}
\delta \leq \frac{\log \log \tau}{\log \log \tau}
\end{equation}
and
\begin{equation}
\frac{500 \log \log \tau}{\log \log \tau} \leq \frac{1}{20} \left( \epsilon - \frac{1}{2} \right).
\end{equation}
Keeping on the left of (8.2.5) only the terms whose $q$ belongs to the square (8.1.2) and using (8.2.5) with

$$u = a + \frac{\sqrt{2} - 1}{2} \delta, \quad v = \tau$$

we get

$$\sum_{\substack{q \leq \epsilon \delta \sqrt{2a} \log a \log \log q; \sigma \in \mathbb{C}_0 \cap \mathbb{C}_1; a}} \frac{u - q}{(u - q)^2 + (v - k)^2} \geq M_2(\tau, a, \delta) \min_{a_2, a_3, x \in \mathbb{C}} \frac{a + 2}{a_2 + 2} \frac{\sqrt{2} - 1}{2} \delta - x \frac{1}{A_4} \frac{1}{\delta} = M_2(\tau, a, \delta) \frac{1}{2} \frac{1}{\delta} \frac{1}{A_4}$$

This and (8.2.5) give together

$$M_2(\tau, a, \delta) \leq \frac{\sqrt{2}}{2} \frac{1}{\delta} \frac{4}{\log \tau} \frac{\log \tau}{(\log \log \tau)^2} < 0.71 \frac{1}{2} \frac{1}{\delta}$$

if $\tau > a_0(\delta)$, q. e. d.

**References**


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**On prime numbers in an arithmetical progression**

by

S. Knapowski (Poznań)

1. Let $k \geq 3$, $0 < l < k$, $(i, k) = 1$ be integers ($(i, k)$ denotes the greatest common divisor of $i$, $k$).

Throughout this paper $p$ denotes prime numbers, $\pi(x, k, l)$ denotes the number of primes not exceeding $x$, belonging to the arithmetical progression

$$l, l+k, l+2k, \ldots,$$

$c, a_1, a_2, \ldots$ denote positive numerical constants, $A(n)$ denotes the Dirichlet symbol:

$$A(n) = \begin{cases} \log p & \text{if } n = p^a, a = 1, 2, \ldots, \\ 0 & \text{otherwise}, \end{cases}$$

$\varphi(k) = k - \text{Euler's function}$, $L(s, X)$ denotes Dirichlet $L$-functions.

It is well-known that

$$\pi(x, k, l) = \frac{1}{k} \int_{u=1}^{\frac{x}{l}} \frac{du}{\log u} + O(\exp(-c\sqrt{x\log x}))$$

for any fixed $k$.

Write

$$A(x, k, l) = \pi(x, k, l) = \frac{1}{k} \int_{u=1}^{\frac{x}{l}} \frac{du}{\log u}.$$

We can show by classical methods that if $k$ is fixed and

$$(1.1) \quad A(x, k, l) = O(x^{\varepsilon+\delta}) \quad (l \ll \delta < 1, \varepsilon > 0 \text{ freely fixed, } x \to \infty)$$

then

$$(1.2) \quad A(x, k, l) = O(x^{\varepsilon+\delta}) \quad (x \to \infty)$$

for all $l (0 < l < k, (l, k) = 1)$ and each fixed $\varepsilon > 0$.

These methods cannot, however, reduce the relation (1.1) → (1.2) to an explicit inequality.