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On the distribution modulo 1 of the sequence $\alpha n^3+\beta n^2+\gamma n$

by
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1. Introduction. Let $\|\cdot\|$ denote distance to the nearest integer. Let $\varepsilon > 0$, and let α, β, γ denote arbitrary real numbers. Recently W. M. Schmidt showed [5] that *for* $N > c_1(\varepsilon)$ *there is a natural number* $n \leqslant N$ *having*

$$\|\alpha n^2+\beta n\| < N^{-1/2+\varepsilon}.$$

This generalizes the well known theorem of Heilbronn [3] and sharpens a result of Davenport [2].

Schmidt's method enabled him to prove that for $N > c_2(\varepsilon)$ there is a natural number $n \leqslant N$ having

$$\|\alpha n^3+\beta n^2+\gamma n\| < N^{-1/5+\varepsilon}.$$

For $\gamma = 0$, the exponent $-1/5+\varepsilon$ could be replaced by $-1/4+\varepsilon$ [6]. Both results sharpen those of Davenport [2].

In the present paper we shall show that *for* $N > c_3(\varepsilon)$ *there is a natural number* $n \leqslant N$ *having*

$$\|\alpha n^3+\beta n^2+\gamma n\| < N^{-1/4+\varepsilon}.$$

yt is no more difficult to prove a more general theorem. We denote by k an integer greater than 1 and write $K = 2^{k-1}$.

THEOREM 1. *Suppose* $k \geqslant 3$ *and* $N > c_1(k, \varepsilon)$. *Then there is a natural number* $n \leqslant N$ *with*

$$\|\alpha n^k+\beta n^{k-1}+\gamma n\| < N^{-1/K+\varepsilon}. \tag{1}$$

We also strengthen Schmidt's theorem [6] for an arbitrary polynomial of degree $k \geqslant 3$ with constant term zero, but only when k is odd.

THEOREM 2. *Let* k *be an odd integer,* $k \geqslant 3$, *and write* $K_1 = \frac{4}{3}(2^{k-1}-1)$. *Let* $N > c_2(k, \varepsilon)$. *Given a polynomial* $F(n)$ *of degree* k *with constant term zero, there is a natural number* $n \leqslant N$ *with*

$$\|F(n)\| < N^{-1/K_1+\varepsilon}. \tag{2}$$

We shall use ideas normally associated with "major arcs" in the circle method [4]. Schmidt's method, on the other hand, is a very original development of "minor arc" ideas.

2. The final coefficient lemma. We write $e(x) = e^{2\pi i x}$, $e_q(x) = e(x/q)$. In [7], Chapter 4, I. M. Vinogradov showed that, given a large exponential sum

$$\sum_{n=1}^{N} e(a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n)$$

and a good simultaneous rational approximation to $a_k, a_{k-1}, \ldots, a_2$, one can (under suitable conditions) obtain a good simultaneous approximation to $a_k, a_{k-1}, \ldots, a_2, a_1$. Lemma 4 (below) is a refined version of this principle. Other applications of Lemma 4 are given in [1].

We shall need some preliminary lemmas. Lemma 2 is rather like Lemma 7.11 of Hua's book [4].

LEMMA 1. *Let $G(x) = u_k x^k + \ldots + u_1 x$ be a polynomial with integer coefficients. Let q be an integer and write d for the greatest common divisor,*

$$d = (q, u_2, \ldots, u_k).$$

Then when $1 \leqslant m \leqslant q$, we have

$$\sum_{x=1}^{m} e_q(G(x)) = O(q^{1-1/k+\varepsilon} d^{1/k}).$$

Proof. This is Theorem 2 of [4]. The implied constants, here and subsequently, depend at most on k and ε.

In the following lemmas the polynomials occurring have real coefficients.

LEMMA 2. *Let $k \geqslant 2$. Let $f(x) = a_k x^k + \ldots + a_1 x$ and suppose that there are integers $N, q, u_1, u_2, \ldots, u_k$ such that*

(3) $$d = (q, u_2, \ldots, u_k) \leqslant N^{\varepsilon}$$

and

(4) $$1 \leqslant q \leqslant N^{1-\varepsilon}, \quad |q a_j - u_j| \leqslant N^{1-j-\varepsilon} \quad (1 \leqslant j \leqslant k).$$

Writing

$$\beta_j = a_j - u_j/q \ (j = 1, \ldots, k), \quad g(x) = \sum_{j=1}^{k} \beta_j x^j,$$

$$G(v) = \sum_{j=1}^{k} u_j v^j, \quad S(q) = \sum_{v=1}^{q} e_q(G(v)),$$

we have

$$\sum_{n=0}^{N-1} e(f(n)) = q^{-1} S(q) \int_0^N e(g(y))\,dy + O(q^{1-1/k} N^{\varepsilon}).$$

Proof. Write $S = \sum_{n=0}^{N-1} e(f(n))$, then

$$S = \sum_{v=1}^{q} \sum_{(5)} e\left(\sum_{j=1}^{k} \left(\frac{u_j}{q} + \beta_j\right)(mq+v)^j\right)$$

where the inner summation is over integers m satisfying

(5) $$0 \leqslant m + v/q < N/q.$$

Thus

(6) $$S = \sum_{v=1}^{q} e_q(G(v)) \sum_{(5)} e(g(mq+v)).$$

Let $l = [1/\varepsilon]+1$. Write $H(x) = e(g(qx))$ and $A = Nq^{-1}$. By Euler's sum formula ([4], p. 80) we have for all t

(7) $$\sum_{0 \leqslant m+t < A} H(m+t) = \int_0^A H(x)\,dx + \sum_{r=0}^{l-1} \{H^{(r)}(A) b_{r+1}(t-A) - H^{(r)}(0) b_{r+1}(t)\} - \int_0^A H^{(l)}(x) b_l(t-x)\,dx.$$

Here $b_1(x), b_2(x), \ldots$ are functions of period one defined inductively by: $b_1(x) = x - [x] - 1/2$,

$$b_{l+1}(x) = b_{l+1}(0) + \int_0^x b_l(y)\,dy.$$

We write V_r for the total variation of b_r on $[0, 1]$ (evidently $V_r < \infty$) and $M_r = \sup_x |b_r(x)|$.

We note that

(8) $$\int_0^A H(x)\,dx = q^{-1} \int_0^N e(g(y))\,dy.$$

Combining (6), (7) (with $t = v/q$) and (8), we find that

(9) $$S = q^{-1} S(q) \int_0^N e(g(y))\,dy + E,$$

where

(10) $$E = \sum_{r=0}^{l-1} H^{(r)}(A) \sum_{v=1}^{q} e_q(G(v)) b_{r+1}\left(\frac{v}{q} - A\right) - \sum_{r=0}^{l-1} H^{(r)}(0) \sum_{v=1}^{q} e_q(G(v)) b_{r+1}\left(\frac{v}{q}\right) - \sum_{v=1}^{q} e_q(G(v)) \int_0^A H^{(l)}(x) b_l\left(\frac{v}{q} - x\right) dx.$$

It remains to estimate E. We begin by observing that the hth derivative of $e(\alpha y^j)$ takes the shape

$$D^h\big(e(\alpha y^j)\big) = \sum_{hj^{-1}\leqslant r\leqslant h} C(r, h, j)\,\alpha^r y^{jr-h} e(\alpha y^j).$$

For $1\leqslant j\leqslant k$, $0\leqslant y\leqslant A$, $hj^{-1}\leqslant r\leqslant h$ we have, in view of (4),

$$(q^j\beta_j)^r y^{jr-h} \leqslant q^{jr} q^{-r} N^{r(1-j-\varepsilon)} N^{jr-h} q^{h-jr} \leqslant (qN^{-1+\varepsilon})^{h-r} N^{-h\varepsilon} \leqslant N^{-h\varepsilon}.$$

It follows that for $1\leqslant j\leqslant k$, $0\leqslant y\leqslant A$,

$$D^h\big(e(\beta_j q^j y^j)\big) = O\big(C_1(h) N^{-h\varepsilon}\big)$$

and we easily deduce that

$$D^h\big(H(y)\big) = O\big(C_2(h) N^{-h\varepsilon}\big) \quad (0\leqslant y\leqslant A). \tag{11}$$

Thus the third term in (10) is

$$O(qNq^{-1}\cdot M_l N^{-l\varepsilon}) = O(1).$$

Write $s_v = \sum_{w=1}^{v} e_q\big(G(w)\big)$. If t is any real number,

$$\sum_{v=1}^{q} e_q\big(G(v)\big) b_{r+1}\Big(\frac{v}{q}-t\Big)$$

$$= \sum_{v=1}^{q-1} s_v\Big\{b_{r+1}\Big(\frac{v}{q}-t\Big) - b_{r+1}\Big(\frac{v+1}{q}-t\Big)\Big\} + s_q b_{r+1}(1-t),$$

so that for $0\leqslant r<l$,

$$\Big|\sum_{v=1}^{q} e_q\big(G(v)\big) b_{r+1}\Big(\frac{v}{q}-t\Big)\Big| \leqslant (V_{r+1}+M_{r+1})\max_{v\leqslant q}|s_v|$$

$$= O(q^{1-1/k+\varepsilon/2} d^{1/k}) = O(q^{1-1/k} N^{\varepsilon})$$

in view of Lemma 1, (3) and (4). Taking (11) into account, it follows that the first and second terms in (10) are

$$O(q^{1-1/k} N^{\varepsilon}).$$

The same estimate thus applies to E, and Lemma 2 is proved.

LEMMA 3. *Let $g(x) = \beta_k x^k + \ldots + \beta_1 x$. Then*

$$\int_0^N e\big(g(x)\big)\,dx \ll NZ^{-1/k},$$

where $Z = \max(1, N|\beta_1|, \ldots, N^k|\beta_k|)$.

Proof. This follows at once from Lemma 10.1 of [4].

LEMMA 4. *Let $f(x) = \alpha_k x^k + \ldots + \alpha_1 x$ and suppose there are integers $N > c_2(k, \varepsilon)$ and r such that*

$$1\leqslant r\leqslant N^{1-2\varepsilon}, \quad \|\alpha_j r\| \leqslant N^{1-j-2\varepsilon} \quad (2\leqslant j\leqslant k). \tag{12}$$

Suppose further that

$$\Big|\sum_{n=1}^{N} e\big(f(n)\big)\Big| \geqslant H \geqslant r^{1-1/k} N^{2\varepsilon}. \tag{13}$$

Then there is a divisor s of r and a natural number $t\leqslant N^{\varepsilon}$ such that, writing $q = st$,

$$q\leqslant N^{k+3k\varepsilon} H^{-k}, \quad \|\alpha_j q\| \leqslant N^{k-j+3k\varepsilon} H^{-k} \quad (1\leqslant j\leqslant k).$$

Proof. Write

$$\|r\alpha_j\| = |r\alpha_j - v_j| \quad (j = 2, \ldots, k).$$

Let $d = (r, v_2, \ldots, v_k)$ and define $s = rd^{-1}$, $w_j = v_j d^{-1}$ $(j = 2, \ldots, k)$.

By Dirichlet's theorem there is a natural number $t\leqslant N^{\varepsilon}$ such that

$$\|\alpha_1 st\| = |\alpha_1 st - u_1| \leqslant N^{-\varepsilon}.$$

Write $q = st$, $u_j = w_j t$ $(j = 2, \ldots, k)$; then

$$(q, u_2, \ldots, u_k) = t(s, w_2, \ldots, w_k) = t\leqslant N^{\varepsilon};$$

and in view of (12),

$$1\leqslant q\leqslant rN^{\varepsilon}\leqslant N^{1-\varepsilon},$$

$$|q\alpha_j - u_j| = td^{-1}|r\alpha_j - v_j| \leqslant N^{1-j-\varepsilon} \quad (2\leqslant j\leqslant k).$$

We may therefore apply Lemma 2. Now in view of (13) and $N > c_2(k, \varepsilon)$, the quantity $O(q^{1-1/k}N^{\varepsilon})$ is smaller than $\frac{1}{2}H$. It follows that

$$\Big|q^{-1}S(q)\int_0^N e\big(g(y)\big)\,dy\Big| > \tfrac{1}{2}H$$

where $S(q)$ and $g(y)$ are as in Lemma 2.

We now use the estimate

$$q^{-1}S(q) \ll q^{-1/k} N^{2\varepsilon}$$

which follows from (3) and Lemma 1. In the notation of Lemma 3, then, we see that

$$H \ll q^{-1/k} N^{1+2\varepsilon} Z^{-1/k}$$

or

$$qZ = \max(q, N\|q\alpha_1\|, \ldots, N^k\|q\alpha_k\|) \ll N^{k+2k\varepsilon} H^{-k}.$$

Since $N > c_2(k, \varepsilon)$, this proves Lemma 4.

3. Proofs of the theorems.

LEMMA 5. *Suppose* $N > c_3(k, \varepsilon)$ *and* $1 \leqslant M \leqslant N^{1/K-\varepsilon}$. *Let*

$$F(x) = \alpha x^k + \beta x^{k-1} + \ldots + \omega x.$$

Suppose that there is no natural number $n \leqslant N$ *having*

$$\|F(n)\| \leqslant M^{-1}.$$

Then there exists a natural number r *with*

$$r \leqslant M^K N^\varepsilon, \quad \|\alpha r\| \leqslant M^{K-1} N^{\varepsilon-k}, \quad \|\beta r\| \leqslant M^{K-1} N^{\varepsilon-k+1}; \tag{14}$$

and there is a natural number $m \leqslant MN^\varepsilon$ *such that*

$$\Big|\sum_{n=1}^{N} e\big(mF(n)\big)\Big| \geqslant N^{1-\varepsilon} M^{-1}. \tag{15}$$

Proof. As far as (14) goes, this is a special case of Lemma 8A of [6]. The inequality (15) is an easy consequence of the proof of Lemma 8A.

Proof of Theorem 1. Suppose that there is no natural number $n \leqslant N$ having (1). Let $M = N^{1/K-\varepsilon}$. We apply Lemma 5 with $\varepsilon_1 = \varepsilon/5k$ in place of ε. Thus there is a natural number $r \leqslant M^K N^{\varepsilon_1}$ such that

$$\|\alpha r\| \leqslant M^{K-1} N^{\varepsilon_1 - k}, \quad \|\beta r\| \leqslant M^{K-1} N^{\varepsilon_1 - k+1},$$

and a natural number $m \leqslant MN^{\varepsilon_1}$ such that

$$\Big|\sum_{n=1}^{N} e\big(mF(n)\big)\Big| \geqslant H = N^{1-\varepsilon} M^{-1}.$$

Write $f(x) = mF(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + \ldots + \alpha_1 x$. Evidently

$$1 \leqslant r \leqslant N^{1-2\varepsilon_1}, \quad \|\alpha_j r\| \leqslant N^{1-j-2\varepsilon_1} \quad (2 \leqslant j \leqslant k),$$

and moreover

$$r^{1-1/k} N^{2\varepsilon_1} \leqslant M^{K-1} N^{3\varepsilon_1} \leqslant H.$$

Applying Lemma 4, with ε_1 in place of ε, there is a natural number q such that

$$q \leqslant N^{k+3k\varepsilon_1} H^{-k} \leqslant M^k N^{4k\varepsilon_1},$$

$$\|q\alpha_k\| = \|qm\alpha\| \leqslant N^{3k\varepsilon_1} H^{-k} \leqslant M^k N^{4k\varepsilon_1 - k} \leqslant M^{K-1} N^{4k\varepsilon_1 - k},$$

and similarly

$$\|qm\beta\| \leqslant M^{K-1} N^{4k\varepsilon_1 - k+1}, \quad \|qm\gamma\| \leqslant M^{K-1} N^{4k\varepsilon_1 - 1}.$$

Write $n = qm$. Then

$$n \leqslant M^{k+1} N^{5k\varepsilon_1} \leqslant M^K N^{5k\varepsilon_1} \leqslant N,$$

while

$$\begin{aligned}\|F(n)\| &\leqslant n^{k-1}\|\alpha n\| + n^{k-2}\|\beta n\| + \|\gamma n\| \\ &\leqslant N^{k-1} M^{K-1} N^{4k\varepsilon_1 - k} + N^{k-2} M^{K-1} N^{4k\varepsilon_1 - k+1} + M^{K-1} N^{4k\varepsilon_1 - 1} \\ &\leqslant 3N^{-1+4k\varepsilon_1} N^{1-1/K} \leqslant N^{-1/K+\varepsilon}.\end{aligned}$$

This is a contradiction, and Theorem 1 is proved.

Proof of Theorem 2. This is true for $k = 3$ by Theorem 1. We proceed by induction from $k-2$ to k. Write

$$F(n) = \alpha n^k + \beta n^{k-1} + \ldots + \omega n = \alpha n^k + \beta n^{k-1} + P(n),$$

and put $M = N^{1/K_1 - \varepsilon}$. Suppose that there is no natural number $n \leqslant N$ having (2). Let r be as in Lemma 5. We apply the induction hypothesis to the polynomial $P(rn)$. Thus there exists a natural number $s \leqslant M^{K_2} N^\varepsilon$ with

$$\|P(rs)\| < \tfrac{1}{2} M^{-1},$$

where $K_2 = \frac{4}{3}(2^{k-3} - 1)$.

Putting $n = rs$, we have $n \leqslant M^{K+K_2} N^{2\varepsilon} = M^{K_1} N^{2\varepsilon} \leqslant N$. Moreover,

$$\begin{aligned}\|F(n)\| &\leqslant s^k r^{k-1}\|\alpha r\| + s^{k-1} r^{k-2}\|\beta r\| + \|P(rs)\| \\ &\leqslant M^{k(K_2+K)-1} N^{(k+1)\varepsilon - k} + M^{(k-1)(K_2+K)-1} N^{k\varepsilon - k+1} + \tfrac{1}{2} M^{-1} \\ &< M^{-1}.\end{aligned}$$

This is a contradiction, and Theorem 2 is proved.

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