On the distribution modulo 1 of the sequence $am^2 + bn^2 + cn$

by

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1. Introduction. Let $||$ denote distance to the nearest integer. Let $\varepsilon \neq 0$, and let $a, b, c$ denote arbitrary real numbers. Recently W. M. Schmidt showed [6] that for $N > c_1(\varepsilon)$ there is a natural number $n \leq N$ having

$||am^2 + bn^2 + cn|| < N^{-1/2 + \varepsilon}$

This generalizes the well known theorem of Heilbronn [3] and sharpens a result of Davenport [2].

Schmidt's method enabled him to prove that for $N > c_1(\varepsilon)$ there is a natural number $n \leq N$ having

$||am^2 + bn^2 + cn|| < N^{-1/4 + \varepsilon}$

For $\gamma = 0$, the exponent $-1/5 + \varepsilon$ could be replaced by $-1/4 + \varepsilon$ [6]. Both results sharpen those of Davenport [2].

In the present paper we shall show that for $N > c_1(\varepsilon)$ there is a natural number $n \leq N$ having

$||am^2 + bn^2 + cn|| < N^{-1/4 + \varepsilon}$

It is no more difficult to prove a more general theorem. We denote by $k$ an integer greater than 1 and write $K = 2^{k-1}$.

Theorem 1. Suppose $k \geq 3$ and $N > c_k(\varepsilon)$. Then there is a natural number $n \leq N$ with

(1) $||am^k + bn^{k-1} + cn|| < N^{-1/k + \varepsilon}$

We also strengthen Schmidt's theorem [6] for an arbitrary polynomial of degree $k \geq 3$ with constant term zero, but only when $k$ is odd.

Theorem 2. Let $k$ be an odd integer, $k \geq 3$, and write $K = 2^{k-1} - 1$. Let $N > c_k(\varepsilon)$. Given a polynomial $P(n)$ of degree $k$ with constant term zero, there is a natural number $n \leq N$ with

(2) $||P(n)|| < N^{-1/k + \varepsilon}$

We shall use ideas normally associated with "minor arcs" in the circle method [4]. Schmidt's method, on the other hand, is a very original development of "minor arc" ideas.
2. The final coefficient lemma. We write \( e(a) = e^{2\pi i a} \), \( e_q(a) = e(a/q) \). In [7], Chapter 4, I. M. Vinogradov showed that, given a large exponential sum

\[
\sum_{n=1}^{N} e_{k}(a_n n^k + a_{k-1} n^{k-1} + \cdots + a_1 n)
\]

and a good simultaneous rational approximation to \( a_k, a_{k-1}, \ldots, a_1 \), one can (under suitable conditions) obtain a good simultaneous approximation to \( a_k, a_{k-1}, \ldots, a_1, a_0 \). Lemma 4 (below) is a refined version of this principle. Other applications of Lemma 4 are given in [1].

We shall need some preliminary lemmas. Lemma 2 is rather like Lemma 11.11 of Elia's book [4].

**Lemma 1.** Let \( G(a) = u_0 a^k + \cdots + u_k a \) be a polynomial with integer coefficients. Let \( q \) be an integer and write \( d \) for the greatest common divisor,

\[
d = (q, u_2, \ldots, u_k). \]

Then when \( 1 \leq m \leq q \), we have

\[
\sum_{a=1}^{m} e_{q}(G(a)) = O(q^{-1/2k} d^{1/2}).
\]

**Proof.** This is Theorem 2 of [4]. The implied constants, here and subsequently, depend at most on \( k \) and \( \varepsilon \).

In the following lemmas the polynomials occurring have real coefficients.

**Lemma 2.** Let \( k \geq 2 \). Let \( f(x) = a_0 a_1 x^{k-1} + \cdots + a_k x \) and suppose that there are integers \( N, q, u_1, u_2, \ldots, u_k \) such that

\[
u_j = u_j - u_{j-1} \leq N^{1-\varepsilon}, \quad (1 \leq j \leq k).
\]

Writing

\[
\beta_j = a_j - u_j/q \quad (j = 1, \ldots, k), \quad v_j = \sum_{j=1}^{k} \beta_j v_j,
\]

\[
G(v) = \sum_{j=1}^{k} u_j v_j, \quad S(q) = \sum_{j=1}^{k} e_{q}(G(v)),
\]

we have

\[
\sum_{n=0}^{N-1} e(f(n)) = q^{-1} S(q) \int_{0}^{N} e(g(y)) dy + O(q^{-1/2k} N^{k}).
\]
It remains to estimate $B$. We begin by observing that the $k$th derivative of $o(\alpha y^k)$ takes the shape

$$D^k(o(\alpha y^k)) = \sum_{\beta \leq r \leq h} o(r, \beta, j) a^{y^{r-h}} o(\alpha y^j).$$

For $1 \leq j \leq h$, $0 \leq y \leq A$, $b_j^{-1} \leq r \leq h$ we have, in view of (4),

$$o(r, \beta, j) a^{y^{r-h}} o(\alpha y^j) < q^j q^{-y^{r-h}} q^{N(r-1-j-o) N^{r-h}} q^{\beta-r} \leq (q N^{1+\epsilon})^j H^{-h} \leq H^{-h}.$$  

It follows that for $1 \leq j \leq h$, $0 \leq y \leq A$,

$$D^k(o(\alpha y^k)) = O(C_1(k) H^{-h})$$

and we easily deduce that

$$(11) \quad D^k(H(y)) = O(C_1(k) H^{-h}) \quad (0 \leq y \leq A).$$

Thus the third term in (10) is

$$O(q N q^{-1} M_j N^{-L}) = O(1).$$

Write $s_n = \sum_{\alpha=1}^q o(\alpha) G(n)$. If $t$ is any real number,

$$\sum_{\alpha=1}^q o(\alpha) G(n) b_{n+t} \left( \frac{n}{\alpha} - t \right) = \sum_{\alpha=1}^q o(\alpha) \left( b_{n+t} \left( \frac{n}{\alpha} - t \right) - b_{n+t} \left( \frac{n+1}{\alpha} - t \right) \right) + o(n b_{n+1} (1-t)),$$

so that for $0 \leq t \leq 1$,

$$\left| \sum_{\alpha=1}^q o(\alpha) G(n) b_{n+t} \left( \frac{n}{\alpha} - t \right) \right| \leq (V_{r+1} + M_{r+1}) \max_{\nu \leq 0} |s_\nu|$$

$$\leq O(q N^{1-\epsilon} H^{2h}) = O(q^{1-\epsilon} N^h)$$

in view of Lemma 1, (3) and (4). Taking (11) into account, it follows that the first and second terms in (10) are

$$O(q^{1-\epsilon} N^h).$$

The same estimate thus applies to $B$, and Lemma 2 is proved.

**Lemma 3.** Let $g(x) = \beta_2 x^2 + \ldots + \beta_k x$. Then

$$\int_0^N o(g(x)) \, dx = O(N^{1+h}),$$

where $Z = \max(1, N |\beta_1|, \ldots, N^k |\beta_k|)$.

**Proof.** This follows at once from Lemma 10.1 of [4].

We may therefore apply Lemma 2. Now in view of (13) and $N > o_5(b, \epsilon)$, the quantity $O(q^{1-\epsilon} N^h)$ is smaller than $\frac{1}{4} H$. It follows that

$$|q^{-1} S(q) \int_0^N o(g(y)) \, dy| > \frac{1}{4} H$$

where $S(q)$ and $g(y)$ are as in Lemma 2.

We now use the estimate

$$q^{-1} S(q) \ll q^{-1/2} N^{1+k}$$

which follows from (3) and Lemma 1. In the notation of Lemma 3, then, we see that

$$H \ll q^{-1/2} N^{1+2k} Z^{-1/2}$$

or

$$qZ = \max(q, N |q a_1|, \ldots, N^k |q a_k|) \ll N^{k+2k} H^{-k}.$$
3. Proofs of the theorems.

Lemma 5. Suppose \(N > a_1(k, \epsilon)\) and \(1 \leq M \leq N^{1/\epsilon}.\) Let
\[
P(x) = ax^{k} + \beta x^{k-1} + \ldots + \alpha x.
\]
Suppose that there is no natural number \(n \leq N\) having
\[
|\mathcal{P}(n)| \leq n^{k-1} \left(\|a\| + n^{k-2} \|\beta\| + \|\gamma\|\right).
\]
Then there exists a natural number \(r\) with
\[
r \leq M N^{1/\epsilon}, \quad \|ar\| \leq M^{k-1} N^{1/\epsilon - k}, \quad \|br\| \leq M^{k-1} N^{1/\epsilon - k} + 1,
\]
and there is a natural number \(m \leq MN^\epsilon\) such that
\[
\sum_{n=1}^{N} e\left(m \mathcal{P}(n)\right) \geq N^{1-\epsilon} M^{-1}.
\]

Proof. As far as (14) goes, this is a special case of Lemma 8A of [8]. The inequality (15) is an easy consequence of the proof of Lemma 8A.

Proof of Theorem 1. Suppose that there is no natural number \(n \leq N\) having (1). Let \(M = N^{1/\epsilon}.\) We apply Lemma 5 with \(\epsilon = \epsilon/\delta\) in place of \(\epsilon\). Thus there is a natural number \(r \leq M N^{1/\epsilon}\) such that
\[
\|ar\| \leq M^{k-1} N^{1/\epsilon - k}, \quad \|br\| \leq M^{k-1} N^{1/\epsilon - k} + 1,
\]
and a natural number \(m \leq MN^{\epsilon}\) such that
\[
\sum_{n=1}^{N} e\left(m \mathcal{P}(n)\right) \geq H = N^{1-\epsilon} M^{-1}.
\]
Write \(f(x) = mP(x) = a_0 x^k + a_{k-1} x^{k-1} + \ldots + a_1 x\). Evidently \(1 \leq r \leq N^{1-\delta}\), \(\|a_0 r\| \leq N^{1-\epsilon - \delta}\), \(2 \leq j \leq k\), and moreover \(r^{-1} N^{(k+1)\delta} \leq M^{k-1} N^{\epsilon \delta} \leq H\).

Applying Lemma 4, with \(\epsilon_1\) in place of \(\epsilon\), there is a natural number \(q\) such that
\[
q \leq N^{k+1+2\delta} H^{-k} \leq M^{k} N^{\delta k},
\]
\[
\|qa_0\| = \|qma_0\| \leq N^{2\delta k} H^{-k} \leq M^{k} N^{\delta k - k} \leq M^{k-1} N^{\delta k - k},
\]
and similarly
\[
\|qab\| \leq M^{k-1} N^{1/\epsilon - k + 1}, \quad \|qam\| \leq M^{k-1} N^{1/\epsilon - k + 1}.
\]

Write \(n = qm\). Then
\[
n \leq M^{k+1} N^{\delta k} \leq M^{k} N^{\delta k} \leq N,
\]
while
\[
\|\mathcal{P}(n)\| \leq n^{k-1} \|a\| + n^{k-2} \|\beta\| + \|\gamma\| + \|\gamma\| \leq M^{k-1} M^{k-1} N^{1/\epsilon - k + 1} + M^{k-1} N^{1/\epsilon - k + 1} + M^{k-1} N^{1/\epsilon - k + 1}
\]
\[
\leq 3 N^{1+4\delta k} N^{1/\epsilon} \leq N^{1/\epsilon - k + 1}.
\]
This is a contradiction, and Theorem 1 is proved.

Proof of Theorem 2. This is true for \(k = 3\) by Theorem 1. We proceed by induction from \(k-2\) to \(k\). Write
\[
P(n) = an^k + \beta n^{k-1} + \ldots + \gamma n = an^k + \beta n^{k-1} + P(n),
\]
and put \(M = N^{1/\epsilon - k}\). Suppose that there is no natural number \(n \leq N\) having (2). Let \(r\) be as in Lemma 5. We apply the induction hypothesis to the polynomial \(P(n)\). Thus there exists a natural number \(s \leq M^{K_1} N^\epsilon\) with
\[
\|\mathcal{P}(rs)\| \leq \frac{1}{2} M^{-1},
\]
where \(K_1 = \frac{1}{2}(2k-2-1)\).

Putting \(n = rs\), we have \(n \leq M^{K_1+K_2} N^\epsilon = M^{K_1} N^\epsilon \leq N\). Moreover,
\[
\|\mathcal{P}(n)\| \leq p^{k-1} \|ar\| + p^{k-2} \|br\| + \|\gamma\| + \|\gamma\| \leq M^{K_1+K_2+1} N^{(k+1)\delta - k} + M^{(k+1)K_1+1} N^{k+1} + \frac{1}{2} M^{-1} \leq M^{-1}.
\]
This is a contradiction, and Theorem 2 is proved.

References


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