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On the class number of $Q(\sqrt{-p})$ modulo 16, for $p \equiv 1 \pmod 8$ a prime

by

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1. Introduction. Throughout this paper p denotes a prime congruent to 1 modulo 8, and we set $p = 8l+1$. For such primes, the class number $h(-p)$ of the imaginary quadratic field $Q(\sqrt{-p})$ satisfies

$$h(-p) \equiv 0 \pmod 4, \tag{1.1}$$

see for example [1], p. 413, and the class number $h(p)$ of the real quadratic field $Q(\sqrt{p})$ satisfies

$$h(p) \equiv 1 \pmod 2, \tag{1.2}$$

see for example [2], p. 100. The fundamental unit ε_p (>1) of the real quadratic field $Q(\sqrt{p})$ has norm -1 and can be written in the form

$$\varepsilon_p = T + U\sqrt{p}, \tag{1.3}$$

where T and U are positive integers such that

$$T \equiv 0 \pmod 4, \qquad U \equiv 1 \pmod 4. \tag{1.4}$$

Recently Lehmer ([8], p. 48), Cohn and Cooke ([3], p. 368) and Kaplan ([6], p. 240) have proved that

$$h(-p) \equiv T \pmod 8. \tag{1.5}$$

It is our purpose to determine $h(-p)$ modulo 16.

We prove

THEOREM. *If $p \equiv 1 \pmod 8$ is a prime, then*

(1.6)

$$\begin{cases} h(-p) \equiv T+(p-1) \pmod{16}, & \text{if} \quad h(-p) \equiv 0 \pmod 8, \\ h(-p) \equiv T+(p-1)+4\big(h(p)-1\big) \pmod{16}, & \text{if} \quad h(-p) \equiv 4 \pmod 8. \end{cases}$$

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We set $\varrho = \exp(2\pi i/p)$. The cyclotomic polynomial $F(z)$ of index p in the complex variable z is given by

$$(1.7)\qquad F(z) = \frac{z^p-1}{z-1} = \prod_{j=1}^{p-1}(z-\varrho^j) = z^{p-1}+\ldots+z+1.$$

We have

$$(1.8)\qquad F(z) = F_+(z)F_-(z),$$

where $F_+(z)$ and $F_-(z)$ are polynomials of degree $\frac{1}{2}(p-1)$ given by

$$(1.9)\qquad F_+(z) = \prod_{\substack{j=1\\ \left(\frac{j}{p}\right)=+1}}^{p-1}(z-\varrho^j), \qquad F_-(z) = \prod_{\substack{j=1\\ \left(\frac{j}{p}\right)=-1}}^{p-1}(z-\varrho^j).$$

The method used to prove the theorem is completely elementary. We sketch the ideas involved. In §§ 2–4 Dirichlet's class number formulae for $h(p)$ and $h(-p)$ are used to evaluate $F_\pm(1)$ (Lemma 1), $F_\pm(-1)$ (Lemma 2) and $F_\pm(i)$ (Lemma 3). From these evaluations certain linear congruences and equations are obtained (Corollaries 1, 2, 3) for the coefficients a_n and b_n of the polynomials $Y(z) = F_-(z)+F_+(z)$ and $Z(z) = \frac{1}{\sqrt{p}}\big(F_-(z)-F_+(z)\big)$. In § 5 these congruences and equations are combined to give further congruences (Lemma 4) which are required in § 6. In § 6 the quantities $Y(\omega)$, $Z(\omega)$, $Y'(\omega)$, $Z'(\omega)$ $(\omega = 1+i/\sqrt{2})$, are given in terms of the a_n and b_n, and certain equations derived (Lemmas 5 and 6). Finally in § 7 using Dirichlet's class number formulae for $h(-p)$ and $h(-2p)$ and an identity of Liouville, $h(-p)$ is expressed in terms of $Y(\pm\omega)$, $Z(\pm\omega)$, $Y'(\pm\omega)$, $Z'(\pm\omega)$, and the theorem follows by appealing to Lemmas 5 and 6.

2. Evaluation of $F_+(1)$ and $F_-(1)$. Using Dirichlet's class number formula for $h(p)$, we prove

LEMMA 1. *If $p \equiv 1 \pmod 8$ is prime, then*

$$F_+(1) = -\sqrt{p}(T-U\sqrt{p})^{h(p)}, \qquad F_-(1) = \sqrt{p}(T+U\sqrt{p})^{h(p)}.$$

Proof. By Dirichlet's class number formula for $h(p)$ (see for example [7], p. 227), we have

$$(2.1)\qquad \varepsilon_p^{2h(p)} = \prod_{\substack{j=1\\ \left(\frac{j}{p}\right)=-1}}^{p-1}\sin\frac{\pi j}{p}\Big/\prod_{\substack{j=1\\ \left(\frac{j}{p}\right)=+1}}^{p-1}\sin\frac{\pi j}{p}.$$

It is well-known (see for example [11], p. 173) that

$$(2.2)\qquad 2^{p-1}\prod_{\substack{j=1\\ \left(\frac{j}{p}\right)=-1}}^{p-1}\sin\frac{\pi}{p}\prod_{\substack{j=1\\ \left(\frac{j}{p}\right)=+1}}^{1}\sin\frac{\pi j}{p} = \prod_{j=1}^{p-1}2\sin\frac{\pi j}{p} = p.$$

Multiplying (2.1) and (2.2) together we obtain

$$(2.3)\qquad p\varepsilon_p^{2h(p)} = 2^{p-1}\Big\{\prod_{j=1}^{p-1}{}'\sin\frac{\pi j}{p}\Big\}^2,$$

where, here and throughout the rest of the paper, we use a prime (′) to indicate that the product or summation variable is restricted to quadratic non-residues (mod p). Since $\varepsilon_p > 1$ and each $\sin(\pi j/p) > 0$ $(j = 1, \ldots, p-1)$ we have

$$(2.4)\qquad \sqrt{p}\varepsilon_p^{h(p)} = 2^{(p-1)/2}\prod_{j=1}^{p-1}{}'\sin\frac{\pi j}{p} = \prod_{j=1}^{p-1}{}'2\sin\frac{\pi j}{p}.$$

Now, for $j = 1, \ldots, p-1$, we have

$$2\sin\frac{\pi j}{p} = i\varrho^{-j/2}(1-\varrho^j),$$

so, as

$$\sum_{j=1}^{p-1}{}' j = p(p-1)/4,$$

(2.4) gives $F_-(1) = \sqrt{p}\,\varepsilon_p^{h(p)} = \sqrt{p}(T+U\sqrt{p})^{h(p)}$ as required.

Finally, as $h(p) \equiv 1 \pmod 2$ and the norm of ε_p is -1, we have

$$F_+(1) = \frac{F(1)}{F_-(1)} = \frac{p}{\sqrt{p}(T+U\sqrt{p})^{h(p)}} = -\sqrt{p}(T-U\sqrt{p})^{h(p)}.$$

This completes the proof of Lemma 1.

It is clear from (1.9) that $F_+(z)$ and $F_-(z)$ are polynomials in z of degree $\frac{1}{2}(p-1)$ with coefficients in the ring of integers of $Q(\sqrt{p})$ (see for example [10], p. 215). Hence we can write

$$(2.5)\qquad F_+(z) = \tfrac{1}{2}\big(Y(z)-Z(z)\sqrt{p}\big), \qquad F_-(z) = \tfrac{1}{2}\big(Y(z)+Z(z)\sqrt{p}\big),$$

where $Y(z)$ and $Z(z)$ are polynomials of degree at most $\frac{1}{2}(p-1)$ with rational integral coefficients. From (2.5) we have

$$(2.6)\qquad Y(z) = F_-(z)+F_+(z), \qquad Z(z) = \frac{1}{\sqrt{p}}\big(F_-(z)-F_+(z)\big).$$

It is easily verified from (1.9) that for $z \neq 0$

$$z^{(p-1)/2} F_{\pm}\left(\frac{1}{z}\right) = F_{\pm}(z),$$

so that by (2.6) we have

$$z^{(p-1)/2} Y\left(\frac{1}{z}\right) = Y(z), \quad z^{(p-1)/2} Z\left(\frac{1}{z}\right) = Z(z).$$

Hence the coefficient of z^n ($n = 0, 1, 2, \ldots, (p-5)/4$) in $Y(z)$ (resp. $Z(z)$) is the same as that of $z^{(p-1)/2-n}$ in $Y(z)$ (resp. $Z(z)$). Moreover, by (2.6) and Lemma 1, $Y(1)$ and $Z(1)$ are both even, so the middle coefficients of $Y(z)$ and $Z(z)$ are both even. Hence we can set

$$\text{(2.7)} \qquad Y(z) = \sum_{n=0}^{2l} a_n(z^n + z^{4l-n}), \qquad Z(z) = \sum_{n=0}^{2l} b_n(z^n + z^{4l-n}),$$

where the a_n and b_n are integers. It is known (see for example [12], pp. 210–212) that

$$a_0 = 2, \ a_1 = 1, \ a_2 = \tfrac{1}{4}(p+3), \ldots,$$

$$b_0 = 0, \ b_1 = 1, \ b_2 = 1, \ldots$$

Appealing to Lemma 1 we obtain

COROLLARY 1. *If $p = 8l+1$ is a prime, then*

$$\sum_{n=0}^{2l} a_n \equiv 1-4l \pmod{16}, \quad \sum_{n=0}^{2l} b_n \equiv T \pmod{16}, \quad \textit{if} \quad h(-p) \equiv 0 \pmod 8,$$

and

$$\sum_{n=0}^{2l} a_n \equiv 9-4l \pmod{16}, \quad \sum_{n=0}^{2l} b_n \equiv h(p)T \pmod{16}, \quad \textit{if} \quad h(-p) \equiv 4 \pmod 8.$$

Proof. If $h(-p) \equiv 0 \pmod 8$, by (1.5) we have $T \equiv 0 \pmod 8$. Then, as $T^2 - pU^2 = -1$ and $U \equiv 1 \pmod 4$, we have

$$\text{(2.8)} \qquad U \equiv 4l+1 \pmod{16}.$$

Hence, working modulo 16, we have

$$\begin{aligned}
\sum_{n=0}^{2l} a_n &= \tfrac{1}{2} Y(1) && \text{(by (2.7))} \\
&= \tfrac{1}{2}\big(F_-(1) + F_+(1)\big) && \text{(by (2.6))} \\
&= \frac{\sqrt{p}}{2}\{(T+U\sqrt{p})^{h(p)} - (T-U\sqrt{p})^{h(p)}\} && \text{(by Lemma 1)} \\
&\equiv U^{h(p)} p^{(h(p)+1)/2} && (\text{as } h(p) \equiv 1 \pmod 2,\ T \equiv 0 \pmod 4) \\
&\equiv (4l+1)^{h(p)}(8l+1)^{(h(p)+1)/2} && \text{(by (2.8))} \\
&\equiv (4l+1)(8l+1)^{h(p)} \\
&\equiv (4l+1)(8l+1) \\
&\equiv 1-4l,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{2l} b_n &= \tfrac{1}{2} Z(1) && \text{(by (2.7))} \\
&= \frac{1}{2\sqrt{p}}\big(F_-(1) - F_+(1)\big) && \text{(by (2.6))} \\
&= \tfrac{1}{2}\big((T+U\sqrt{p})^{h(p)} + (T-U\sqrt{p})^{h(p)}\big) && \text{(by Lemma 1)} \\
&\equiv h(p) T U^{h(p)-1} p^{(h(p)-1)/2} && (\text{as } T \equiv 0 \pmod 4) \\
&\equiv h(p) T (4l+1)^{h(p)-1}(8l+1)^{(h(p)-1)/2} && \text{(by (2.8))} \\
&\equiv h(p) T (8l+1)^{h(p)-1} && (\text{as } h(p) \equiv 1 \pmod 2) \\
&\equiv h(p) T && (\text{as } h(p) \equiv 1 \pmod 2) \\
&\equiv T && (\text{as } T \equiv 0 \pmod 8).
\end{aligned}$$

The case $h(-p) \equiv 4 \pmod 8$ can be treated similarly. In this case we have $T \equiv 4 \pmod 8$ and $U \equiv 4l+9 \pmod{16}$.

3. Evaluation of $F_+(-1)$ and $F_-(-1)$. A simple argument proves

LEMMA 2. *If $p \equiv 1 \pmod 8$ is prime, then*

$$F_+(-1) = F_-(-1) = 1.$$

Proof. From (1.9) we have

$$F_-(1)F_-(-1) = \prod_{j=1}^{p-1}{}' (-1+\varrho^{2j}) = \prod_{j=1}^{p-1}{}' (1-\varrho^{2j}).$$

As j runs through the quadratic non-residues modulo p, so does $2j$. Hence we have

$$\prod_{j=1}^{p-1}{}'(1-\varrho^{2j}) = \prod_{j=1}^{p-1}{}'(1-\varrho^{j}) = F_-(1),$$

giving

$$F_-(-1) = 1,$$

as $F_-(1) \neq 0$. Finally we have

$$F_+(-1) = \frac{F(-1)}{F_-(-1)} = 1.$$

This completes the proof of Lemma 2.

Appealing to Lemma 2 we obtain

COROLLARY 2. *If $p = 8l+1$ is prime, then*

$$\sum_{n=0}^{2l}(-1)^n a_n = 1, \qquad \sum_{n=0}^{2l}(-1)^n b_n = 0.$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{2l}(-1)^n a_n &= \tfrac{1}{2}Y(-1) && \text{(by (2.7))}\\ &= \tfrac{1}{2}\big(F_-(-1)+F_+(-1)\big) && \text{(by (2.6))}\\ &= 1 && \text{(by Lemma 2)},\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{2l}(-1)^n b_n &= \tfrac{1}{2}Z(-1) && \text{(by (2.7))}\\ &= \frac{1}{2\sqrt{p}}\big(F_-(-1)-F_+(-1)\big) && \text{(by (2.6))}\\ &= 0 && \text{(by Lemma 2)}.\end{aligned}$$

4. Evaluation of $F_+(i)$ and $F_-(i)$. Using Dirichlet's class number formula for $h(-p)$, we prove

LEMMA 3. *If $p \equiv 1 \pmod 8$ is prime, then*

$$F_+(i) = F_-(i) = (-1)^{h(-p)/4}.$$

Proof. As $p \equiv 1 \pmod 8$, we have

$$F_-(i) = \prod_{j=1}^{p-1}{}'(i-\varrho^j) = \prod_{j=1}^{p-1}{}'(1+i\varrho^j), \tag{4.1}$$

so that

$$\overline{F_-(i)} = \prod_{j=1}^{p-1}{}'(1-i\bar{\varrho}^j) = \prod_{j=1}^{p-1}{}'(1-i\varrho^{-j}),$$

that is

$$\overline{F_-(i)} = \prod_{j=1}^{p-1}{}'(1-i\varrho^j), \tag{4.2}$$

since, as j runs through the quadratic non-residues modulo p so does $-j$. Hence, multiplying (4.1) and (4.2) together, we obtain

$$|F_-(i)|^2 = F_-(i)\overline{F_-(i)} = \prod_{j=1}^{p-1}{}'(1+\varrho^{2j}) = \prod_{j=1}^{p-1}{}'(1+\varrho^{j}),$$

since as j runs through the quadratic non-residues modulo p so does $2j$. Thus, appealing to Lemma 2, we obtain

$$|F_-(i)|^2 = \prod_{j=1}^{p-1}{}'(-1-\varrho^j) = F_-(-1) = 1,$$

that is

$$|F_-(i)| = 1. \tag{4.3}$$

An easy calculation shows that for $j = 1, 2, \ldots, p-1$ we have

$$1+i\varrho^j = 2\cos\left(\frac{\pi}{4}+\frac{\pi j}{p}\right)\exp\left\{\left(\frac{\pi}{4}+\frac{\pi j}{p}\right)i\right\}, \tag{4.4}$$

so that

$$F_-(i) = 2^{(p-1)/2}\prod_{j=1}^{p-1}{}'\cos\left(\frac{\pi}{4}+\frac{\pi j}{p}\right)\exp\left\{\tfrac{3}{8}(p-1)\pi i\right\}. \tag{4.5}$$

Let M_p denote the number of integers j satisfying

$$\frac{p}{4} < j < p, \qquad \left(\frac{j}{p}\right) = -1.$$

As $\cos(\pi/4+\pi j/p) > 0$, for $0 < j < p/4$, and $\cos(\pi/4+\pi j/p) < 0$, for $p/4 < j < p$, we have

$$\arg(F_-(i)) = \begin{cases} 0, & \text{if } M_p \equiv 0 \pmod 2,\ p \equiv 1 \pmod{16}, \text{ or } \\ & \quad M_p \equiv 1 \pmod 2,\ p \equiv 9 \pmod{16}, \\ \pi, & \text{if } M_p \equiv 0 \pmod 2,\ p \equiv 9 \pmod{16}, \text{ or } \\ & \quad M_p \equiv 1 \pmod 2,\ p \equiv 1 \pmod{16}. \end{cases} \tag{4.6}$$

Now a formula of Dirichlet ([4], p. 152) asserts that

$$h(-p) = 2\sum_{0<j<p/4}\left(\frac{j}{p}\right),$$

so that we have

$$M_p = \tfrac{3}{8}(p-1)+\frac{h(-p)}{4}. \tag{4.7}$$

Putting (4.6) and (4.7) together we obtain

$$\arg(F_-(i)) = \begin{cases} 0, & \text{if} \quad h(-p)\equiv 0 \pmod 8,\\ \pi, & \text{if} \quad h(-p)\equiv 4 \pmod 8,\end{cases} \tag{4.8}$$

that is

$$e^{i\arg(F_-(i))} = (-1)^{h(-p)/4},$$

and hence

$$F_-(i) = |F_-(i)|e^{i\arg(F_-(i))} = (-1)^{h(-p)/4},$$

and

$$F_+(i) = \frac{F(i)}{F_-(i)} = (-1)^{h(-p)/4}.$$

This completes the proof of Lemma 3.

From Lemma 3 we obtain

COROLLARY 3. *If $p = 8l+1$ is a prime, then*

$$\sum_{n=0}^{l}(-1)^n a_{2n} = (-1)^{h(-p)/4}, \qquad \sum_{n=0}^{l}(-1)^n b_{2n} = 0.$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{l}(-1)^n a_{2n} &= \tfrac{1}{2}Y(i) && \text{(by (2.7))}\\ &= \tfrac{1}{2}\big(F_-(i)+F_+(i)\big) && \text{(by (2.6))}\\ &= (-1)^{h(-p)/4} && \text{(by Lemma 3)},\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{l}(-1)^n b_{2n} &= \tfrac{1}{2}Z(i) && \text{(by (2.7))}\\ &= \frac{1}{2\sqrt{p}}\big(F_-(i)-F_+(i)\big) && \text{(by (2.6))}\\ &= 0 && \text{(by Lemma 3)}.\end{aligned}$$

5. An important lemma. By adding and subtracting the results of Corollaries 1, 2 and 3 as appropriate, we obtain a number of congruences which we put together as Lemma 4. This lemma is essential to what follows in § 6.

LEMMA 4. *If $p = 8l+1$ is a prime, then*

$$\sum_{n=0}^{l} a_{2n} \equiv \begin{cases} -2l+1 \pmod 8, & \text{if} \quad h(-p)\equiv 0 \pmod 8,\\ -2l+5 \pmod 8, & \text{if} \quad h(-p)\equiv 4 \pmod 8,\end{cases}$$

$$\sum_{n=0}^{l-1} a_{2n+1} \equiv \begin{cases} -2l \pmod 8, & \text{if} \quad h(-p)\equiv 0 \pmod 8,\\ -2l+4 \pmod 8, & \text{if} \quad h(-p)\equiv 4 \pmod 8,\end{cases}$$

$$\sum_{n=0}^{[l/2]} a_{4n} \equiv \begin{cases} -l+1 \pmod 4, & \text{if} \quad h(-p)\equiv 0 \pmod 8,\\ -l+2 \pmod 4, & \text{if} \quad h(-p)\equiv 4 \pmod 8,\end{cases}$$

$$\sum_{n=0}^{[l-1/2]} a_{4n+2} \equiv \begin{cases} -l \pmod 4, & \text{if} \quad h(-p)\equiv 0 \pmod 8,\\ -l+3 \pmod 4, & \text{if} \quad h(-p)\equiv 4 \pmod 8,\end{cases}$$

$$\sum_{n=0}^{l} b_{2n} \equiv \sum_{n=0}^{l-1} b_{2n+1} \equiv \begin{cases} T/2 \pmod 8, & \text{if} \quad h(-p)\equiv 0 \pmod 8,\\ h(p)T/2 \pmod 8, & \text{if} \quad h(-p)\equiv 4 \pmod 8,\end{cases}$$

$$\sum_{n=0}^{[l/2]} b_{4n} \equiv \sum_{n=0}^{[l-1/2]} b_{4n+2} \equiv \begin{cases} T/4 \pmod 4, & \text{if} \quad h(-p)\equiv 0 \pmod 8,\\ h(p)T/4 \pmod 4, & \text{if} \quad h(-p)\equiv 4 \pmod 8.\end{cases}$$

6. Evaluation of $Y(\omega)$, $Z(\omega)$, $Y'(\omega)$, $Z'(\omega)$. If $p = 16k+1$, so that $l = 2k$, we define

$$A_1 = \sum_{m=0}^{k} a_{4m}(-1)^m, \tag{6.1}$$

$$B_1 = \tfrac{1}{2}\sum_{m=0}^{k-1}(a_{4m+1}-a_{4m+3})(-1)^m, \tag{6.2}$$

$$C_1 = \sum_{m=0}^{k} b_{4m}(-1)^m, \tag{6.3}$$

$$D_1 = \tfrac{1}{2}\sum_{m=0}^{k-1}(b_{4m+1}-b_{4m+3})(-1)^m, \tag{6.4}$$

and, if $p = 16k+9$, so that $l = 2k+1$, we define

$$A_9 = \sum_{m=0}^{k} a_{4m+2}(-1)^m, \tag{6.5}$$

$$B_9 = \tfrac{1}{2}\Big\{\sum_{m=0}^{k} a_{4m+1}(-1)^m + \sum_{m=0}^{k-1} a_{4m+3}(-1)^m\Big\}, \tag{6.6}$$

$$C_9 = \sum_{m=0}^{k} b_{4m+2}(-1)^m, \tag{6.7}$$

$$D_9 = \tfrac{1}{2}\Big\{\sum_{m=0}^{k} b_{4m+1}(-1)^m + \sum_{m=0}^{k-1} b_{4m+3}(-1)^m\Big\}. \tag{6.8}$$

A_1, A_9, C_1 and C_9 are clearly integers. B_1, B_9, D_1, D_9 are integers by Lemma 4.

Setting $\omega = \exp(2\pi i/8) = (1+i)/\sqrt{2}$ (so that $\omega^2 = i$, $\omega^4 = -1$, $\omega^8 = 1$, $\omega+\omega^3 = i\sqrt{2}$, $\omega-\omega^3 = \sqrt{2}$), a straightforward calculation shows that, for $p \equiv 1 \pmod{16}$, we have

$$(6.9) \qquad 2A_1+2B_1\sqrt{2} = Y(\omega), \qquad 2C_1+2D_1\sqrt{2} = Z(\omega),$$

and, for $p \equiv 9 \pmod{16}$, we have

$$(6.10) \qquad 2A_9 i+2B_9 i\sqrt{2} = Y(\omega), \qquad 2C_9 i+2D_9 i\sqrt{2} = Z(\omega).$$

Our next lemma makes (6.9) and (6.10) more precise.

LEMMA 5. *Let $p \equiv 1 \pmod 8$ be a prime. Then, for $p \equiv 1 \pmod{16}$, we have*

$$B_1 = C_1 = 0, \quad A_1^2-2pD_1^2 = 1, \quad Y(\omega) = 2A_1, \quad Z(\omega) = 2D_1\sqrt{2}, \quad \text{if} \quad h(-p) \equiv 0 \pmod 8,$$

$$A_1 = D_1 = 0, \quad 2B_1^2-pC_1^2 = 1, \quad Y(\omega) = 2B_1\sqrt{2}, \quad Z(\omega) = 2C_1, \quad \text{if} \quad h(-p) \equiv 4 \pmod 8,$$

and for $p \equiv 9 \pmod{16}$, we have

$$B_9 = C_9 = 0, \quad A_9^2-2pD_9^2 = -1, \quad Y(\omega) = 2A_9 i, \quad Z(\omega) = 2D_9 i\sqrt{2}, \quad \text{if} \quad h(-p) \equiv 0 \pmod 8,$$

$$A_9 = D_9 = 0, \quad 2B_9^2-pC_9^2 = -1, \quad Y(\omega) = 2B_9 i\sqrt{2}, \quad Z(\omega) = 2C_9 i, \quad \text{if} \quad h(-p) \equiv 4 \pmod 8.$$

Proof. From (1.7), (1.8) and (2.5) we have

$$(6.11) \qquad Y(z)^2-pZ(z)^2 = 4F_+(z)F_-(z) = 4\frac{(z^p-1)}{(z-1)}.$$

Taking $z = \omega$ in (6.11) we obtain

$$(6.12) \qquad Y(\omega)^2-pZ(\omega)^2 = 4.$$

Using (6.9), (6.10) in (6.12) we obtain, for $p = 16k+1$,

$$(6.13) \qquad \begin{cases} A_1^2+2B_1^2-pC_1^2-2pD_1^2 = 1, \\ A_1B_1-pC_1D_1 = 0, \end{cases}$$

and, for $p = 16k+9$,

$$(6.14) \qquad \begin{cases} A_9^2+2B_9^2-pC_9^2-2pD_9^2 = -1, \\ A_9B_9-pC_9D_9 = 0. \end{cases}$$

Now, from (1.9), we have

$$F_-(\omega)F_-(-\omega) = F_-(i).$$

Hence, by (2.5), (6.9), (6.10) and Lemma 3, we have, for $p = 16k+1$,

$$(6.15) \qquad \begin{cases} A_1^2-2B_1^2+pC_1^2-2pD_1^2 = (-1)^{h(-p)/4}, \\ A_1C_1-2B_1D_1 = 0, \end{cases}$$

and, for $p = 16k+9$,

$$(6.16) \qquad \begin{cases} A_9^2-2B_9^2+pC_9^2-2pD_9^2 = -(-1)^{h(-p)/4}, \\ A_9C_9-2B_9D_9 = 0. \end{cases}$$

The result now follows from (6.13) and (6.15), if $p \equiv 1 \pmod{16}$, and from (6.14) and (6.16), if $p \equiv 9 \pmod{16}$. This completes the proof of Lemma 5.

Next, for $p = 16k+1$, we define

$$(6.17) \qquad E_1 = \tfrac{1}{2}\sum_{m=0}^{k-1}\big(a_{4m+1}(4m+1)+a_{4m+3}(4m+3-8k)\big)(-1)^m,$$

$$(6.18) \qquad F_1 = \sum_{m=0}^{k-1} a_{4m+2}(2m-2k+1)(-1)^m,$$

$$(6.19) \qquad G_1 = \tfrac{1}{2}\sum_{m=0}^{k-1}\big(a_{4m+1}(4m-8k+1)+a_{4m+3}(4m+3)\big)(-1)^m,$$

$$(6.20) \qquad H_1 = k\sum_{m=0}^{k} a_{4m}(-1)^{m+1}.$$

The numbers obtained by replacing each a_n by b_n in (6.17)–(6.20) are denoted by L_1, M_1, N_1, P_1 respectively (eqns. (6.21)–(6.24)). Clearly F_1, H_1, M_1 and P_1 are integers. E_1, G_1, L_1 and N_1 are integers by Lemma 4. By (6.1), (6.3), (6.20), (6.24) and Lemma 5, we have

$$(6.25) \qquad H_1 = -kA_1, \qquad P_1 = -kC_1.$$

Moreover, from (6.2), (6.4), (6.17), (6.19), (6.21), (6.23) and Lemma 5 we have

$$(6.26) \qquad \begin{cases} E_1-G_1 = 4k\sum_{m=0}^{k-1}(a_{4m+1}-a_{4m+3})(-1)^m = 8kB_1, \\ L_1-N_1 = 4k\sum_{m=0}^{k-1}(b_{4m+1}-b_{4m+3})(-1)^m = 8kD_1, \end{cases}$$

so that

$$\begin{cases} E_1 = G_1, \; P_1 = 0, & \text{if} \quad h(-p) \equiv 0 \pmod 8, \\ H_1 = 0, \; L_1 = N_1, & \text{if} \quad h(-p) \equiv 4 \pmod 8). \end{cases}$$

Also, working modulo 4, we have, from (6.18) and Lemma 4,

$$F_1 = \sum_{m=0}^{k-1} a_{4m+2}(2m+1)(-1)^m - 2k \sum_{m=0}^{k-1} a_{4m+2}(-1)^m \equiv \sum_{m=0}^{k-1} a_{4m+2} + 2k \sum_{m=0}^{k-1} a_{4m+2},$$

that is

$$(6.27)(a) \qquad F_1 \equiv \begin{cases} 2k \pmod 4, & \text{if} \quad h(-p) \equiv 0 \pmod 8, \\ 3 \pmod 4, & \text{if} \quad h(-p) \equiv 4 \pmod 8. \end{cases}$$

Similarly we have

$$(6.27)(b) \qquad M_1 \equiv \begin{cases} T/4 \pmod 4, & \text{if} \quad h(-p) \equiv 0 \pmod 8, \\ (2k+1)h(p)T/4 \pmod 4, & \text{if} \quad h(-p) \equiv 4 \pmod 8. \end{cases}$$

Next we note that

$$B_1 + E_1 = \sum_{m=0}^{k-1} a_{4m+1}(2m+1)(-1)^m + \sum_{m=0}^{k-1} a_{4m+3}(2m+1-4k)(-1)^m \equiv \sum_{m=0}^{k-1} a_{4m+1} + \sum_{m=0}^{k-1} a_{4m+3} \pmod 4 \equiv \sum_{m=0}^{2k-1} a_{2m+1} \pmod 4,$$

that is, by Lemma 4,

$$B_1 + E_1 \equiv 0 \pmod 4,$$

and so, in particular, we have by Lemma 5

$$E_1 \equiv 0 \pmod 4, \quad \text{if} \quad h(-p) \equiv 0 \pmod 8.$$

Similarly we obtain

$$D_1 + L_1 \equiv T/2 \pmod 4,$$

so

$$L_1 \equiv T/2 \equiv 2 \pmod 4, \quad \text{if} \quad h(-p) \equiv 4 \pmod 8.$$

Finally an easy calculation shows that

$$(6.28) \qquad \begin{cases} 2E_1 + 4F_1\omega + 2G_1\omega^2 + 8H_1\omega^3 = Y'(\omega), \\ 2L_1 + 4M_1\omega + 2N_1\omega^2 + 8P_1\omega^3 = Z'(\omega). \end{cases}$$

For $p = 16k+9$, we define

$$(6.29) \qquad E_9 = \tfrac{1}{2}\Big\{\sum_{m=0}^{k} a_{4m+1}(4m+1)(-1)^m + \sum_{m=0}^{k-1} a_{4m+3}(8k+1-4m)(-1)^m\Big\},$$

$$(6.30) \qquad F_9 = (2k+1)\sum_{m=0}^{k} a_{4m+2}(-1)^m,$$

$$(6.31) \qquad G_9 = \tfrac{1}{2}\Big\{\sum_{m=0}^{k} a_{4m+1}(8k+3-4m)(-1)^m + \sum_{m=0}^{k-1} a_{4m+3}(4m+3)(-1)^m\Big\},$$

$$(6.32) \qquad H_9 = \sum_{m=0}^{k} a_{4m}(2k-2m+1)(-1)^m.$$

The numbers obtained by replacing each a_n by b_n in (6.29)–(6.32) are denoted by L_9, M_9, N_9, P_9 respectively (eqns. (6.33)–(6.36)). Clearly F_9, H_9, M_9 and P_9 are integers. E_9, G_9, L_9 and N_9 are integers by Lemma 4. By (6.5), (6.7), (6.30), (6.34) and Lemma 5, we have

$$(6.37) \qquad F_9 = (2k+1)A_9, \quad M_9 = (2k+1)C_9.$$

Moreover, from (6.5), (6.7), (6.29), (6.31), (6.33), (6.35) and Lemma 5, we have

(6.38)

$$\begin{cases} E_9 + G_9 = (4k+2)\Big\{\sum_{m=0}^{k} a_{4m+1}(-1)^m + \sum_{m=0}^{k-1} a_{4m+3}(-1)^m\Big\} = (8k+4)B_9, \\ L_9 + N_9 = (4k+2)\Big\{\sum_{m=0}^{k} b_{4m+1}(-1)^m + \sum_{m=0}^{k-1} b_{4m+3}(-1)^m\Big\} = (8k+4)D_9, \end{cases}$$

so that

$$\begin{cases} E_9 = -G_9, \quad M_9 = 0, & \text{if} \quad h(-p) \equiv 0 \pmod 8, \\ F_9 = 0, \quad L_9 = -N_9, & \text{if} \quad h(-p) \equiv 4 \pmod 8). \end{cases}$$

Also, working modulo 4, we have, as before,

$$(6.39)(a) \qquad H_9 \equiv \begin{cases} 2k \pmod 4, & \text{if} \quad h(-p) \equiv 0 \pmod 8, \\ 1 \pmod 4, & \text{if} \quad h(-p) \equiv 4 \pmod 8, \end{cases}$$

$$(6.39)(b) \qquad P_9 \equiv \begin{cases} T/4 \pmod 4, & \text{if} \quad h(-p) \equiv 0 \pmod 8, \\ (2k+1)h(p)T/4 \pmod 4, & \text{if} \quad h(-p) \equiv 4 \pmod 8, \end{cases}$$

and

$$B_9 + E_9 \equiv 2 \pmod 4,$$

$$D_9 + L_9 \equiv T/2 \pmod 4,$$

so that by Lemma 5 we have

$$E_9 \equiv 2 \pmod 4, \quad \text{if} \quad h(-p) \equiv 0 \pmod 8,$$
$$L_9 \equiv T/2 \equiv 2 \pmod 4, \quad \text{if} \quad h(-p) \equiv 4 \pmod 8.$$

Finally an easy calculation shows that

$$(6.40)\qquad \begin{cases} 2E_9+4F_9\omega+2G_9\omega^2+4H_9\omega^3 = Y'(\omega), \\ 2L_9+4M_9\omega+2N_9\omega^2+4P_9\omega^3 = Z'(\omega). \end{cases}$$

Differentiating (6.11) and setting $z = \omega$, we obtain

$$(6.41)\qquad Y(\omega)Y'(\omega)-pZ(\omega)Z'(\omega) = -8l(1+\omega+\omega^2+\omega^3).$$

Using (6.25), (6.26), (6.28), (6.37), (6.38), (6.40) and appealing to Lemma 5, (6.41) gives

LEMMA 6. *Let $p = 8l+1$ be a prime. Then*

$$\begin{cases} A_1E_1-2pD_1M_1 = -4k, & \text{if} \quad p \equiv 1 \pmod{16},\ h(-p) \equiv 0 \pmod 8, \\ A_1F_1-pD_1N_1 = 2k(A_1^2-2), \end{cases}$$

$$\begin{cases} 2B_1F_1-pC_1L_1 = -4k, & \text{if} \quad p \equiv 1 \pmod{16},\ h(-p) \equiv 4 \pmod 8, \\ B_1E_1-pC_1M_1 = 2kpC_1^2, \end{cases}$$

$$\begin{cases} A_9E_9+2pD_9P_9 = -4k-2, & \text{if}\ p \equiv 9 \pmod{16},\ h(-p) \equiv 0 \pmod 8, \\ A_9H_9+pD_9L_9 = (2k+1)(A_9^2+2), \end{cases}$$

$$\begin{cases} -2B_9H_9+pC_9N_9 = -4k-2, & \text{if} \quad p \equiv 9 \pmod{16}, \\ B_9E_9+pC_9P_9 = (2k+1)(pC_9^2-2), & h(-p) \equiv 4 \pmod 8. \end{cases}$$

7. Proof of theorem. For $p = 8l+1$ a prime, we define for $j = 0, 1, \ldots, 7$

$$(7.1)\qquad S_j = \sum_{jp/8<s<(j+1)p/8} \left(\frac{s}{p}\right) = \sum_{s=jl+1}^{(j+1)l} \left(\frac{s}{p}\right),$$

so

$$(7.2)\qquad \sum_{j=0}^{7} S_j = \sum_{s=1}^{p-1} \left(\frac{s}{p}\right) = 0.$$

Setting $s = jl+t$ $(t = 1, \ldots, l)$ in (7.1) we have, as $(2/p) = 1$,

$$S_j = \sum_{t=1}^{l} \left(\frac{jl+t}{p}\right) = \sum_{t=1}^{l} \left(\frac{8jl+8t}{p}\right) = \sum_{t=1}^{l} \left(\frac{j(p-1)+8t}{p}\right),$$

that is

$$(7.3)\qquad S_j = \sum_{t=1}^{l} \left(\frac{8t-j}{p}\right).$$

Mapping $t \to l+1-t$ in the right-hand side of (7.3), we obtain (as $(-1/p) = +1$)

$$(7.4)\qquad S_j = S_{7-j} \quad (j = 0, 1, \ldots, 7).$$

From [4], p. 152, and [5], p. 120, we have

$$(7.5)\qquad h(-p) = 2(S_0+S_1), \quad h(-2p) = 2(S_0-S_3), \quad S_1 = S_3.$$

Putting (7.2), (7.4) and (7.5) together, we obtain

$$(7.6)\qquad \begin{cases} S_0 = S_7 = \frac{1}{4}\big(h(-p)+h(-2p)\big), \\ S_1 = S_3 = S_4 = S_6 = \frac{1}{4}\big(h(-p)-h(-2p)\big), \\ S_2 = S_5 = \frac{1}{4}\big(-3h(-p)+h(-2p)\big). \end{cases}$$

Next, for any complex number z, we define

$$(7.7)\qquad K(z) = \sum_{s=1}^{p-1} \left(\frac{s}{p}\right) z^{p-1-s}.$$

Taking $z = \omega_r$ $(r = 0, 1, \ldots, 7)$ in (7.7), and using (7.3), we obtain

$$(7.8)\qquad K(\omega^r) = \sum_{j=0}^{7} \omega^{rj} S_j.$$

Choosing $r = 1, 5$ in (7.8), and appealing to (7.6), we get

$$(7.9)\qquad \begin{cases} K(\omega) = h(-p)(\omega-\omega^2)+\dfrac{h(-2p)}{2}(1-\omega+\omega^2-\omega^3), \\ K(-\omega) = h(-p)(-\omega-\omega^2)+\dfrac{h(-2p)}{2}(1+\omega+\omega^2+\omega^3), \end{cases}$$

from which we obtain

$$(7.10)\qquad 4h(-p) = K(\omega)(1+\omega+\omega^2-\omega^3)+K(-\omega)(1-\omega+\omega^2+\omega^3).$$

Now Liouville ([9], p. 415) has shown that

$$(7.11)\qquad \frac{2}{1-z}K(z) = Y(z)Z'(z)-Y'(z)Z(z).$$

Taking $z = \pm\omega$ in (7.11) we obtain

$$(7.12)\quad \begin{cases} 2K(\omega) = (1-\omega)\{Y(\omega)Z'(\omega) - Y'(\omega)Z(\omega)\}, \\ 2K(-\omega) = (1+\omega)\{Y(-\omega)Z'(-\omega) - Y'(-\omega)Z(-\omega)\}. \end{cases}$$

Substituting (7.12) into (7.10) we obtain

$$(7.13)\quad 4h(-p) = \omega^3\{Y'(\omega)Z(\omega) - Y(\omega)Z'(\omega) + Y(-\omega)Z'(-\omega) - Y'(-\omega)Z(-\omega)\}.$$

Now suppose that $h(-p) \equiv 0 \pmod 8$. By (6.25), (6.26), (6.28), (6.37), (6.38), (6.40), (7.13) and Lemma 5, we have

$$h(-p) = \begin{cases} 4A_1M_1 - 4D_1E_1, & \text{if } p \equiv 1 \pmod{16}, \\ -4A_9P_9 - 4D_9E_9, & \text{if } p \equiv 9 \pmod{16}. \end{cases}$$

Hence, as $E_1 \equiv 0 \pmod 4$, $E_9 \equiv 2 \pmod 4$, $D_9 \equiv 1 \pmod 2$, we have

$$h(-p) \equiv \begin{cases} 4A_1M_1 \pmod{16}, & \text{if } p \equiv 1 \pmod{16}, \\ -4A_9P_9 + 8 \pmod{16}, & \text{if } p \equiv 9 \pmod{16}. \end{cases}$$

Appealing to (6.27)(b) and (6.39)(b), we obtain

$$h(-p) \equiv \begin{cases} A_1T \pmod{16}, & \text{if } p \equiv 1 \pmod{16}, \\ -A_9T + 8 \pmod{16}, & \text{if } p \equiv 9 \pmod{16}. \end{cases}$$

As $T \equiv 0 \pmod 8$ and $A_1 \equiv A_9 \equiv 1 \pmod 2$, we have

$$h(-p) \equiv \begin{cases} T \pmod{16}, & \text{if } p \equiv 1 \pmod{16}, \\ T + 8 \pmod{16}, & \text{if } p \equiv 9 \pmod{16}, \end{cases}$$

that is

$$h(-p) \equiv T + p - 1 \pmod{16},$$

as required.

Finally we suppose that $h(-p) \equiv 4 \pmod 8$. As above we have

$$h(-p) = \begin{cases} 4B_1L_1 - 4C_1F_1, & \text{if } p \equiv 1 \pmod{16}, \\ 4B_9L_9 + 4C_9H_9, & \text{if } p \equiv 9 \pmod{16}. \end{cases}$$

Hence, as $B_1 \equiv C_1 \equiv 1 \pmod 2$, $L_1 \equiv 2 \pmod 4$, $F_1 \equiv 3 \pmod 4$, $B_9 \equiv 0 \pmod 2$, $C_9 \equiv 1 \pmod 2$, $L_9 \equiv 2 \pmod 4$, $H_9 \equiv 1 \pmod 4$, we have

$$h(-p) \equiv \begin{cases} 8 + 4C_1 \pmod{16}, & \text{if } p \equiv 1 \pmod{16}, \\ 4C_9 \pmod{16}, & \text{if } p \equiv 9 \pmod{16}. \end{cases}$$

Now if $p \equiv 1 \pmod{16}$ we have from Lemma 6

$$pC_1M_1 = B_1E_1 - 2kpC_1^2.$$

Multiplying by $M_1 \equiv 1 \pmod 2$, we get

$$\begin{aligned} C_1 &\equiv B_1E_1M_1 - 2kM_1 \pmod 4 \\ &\equiv -B_1^2M_1 - 2kM_1 \pmod 4 \\ &\equiv -(1+2k)M_1 \pmod 4 \\ &\equiv -h(p)T/4 \pmod 4, \end{aligned}$$

so that

$$h(-p) \equiv 8 - h(p)T \equiv T + (p-1) + 4\big(h(p)-1\big) \pmod{16}.$$

On the other hand if $p \equiv 9 \pmod{16}$ we have from Lemma 6

$$pC_9P_9 = (2k+1)(pC_9^2 - 2) - B_9E_9.$$

Multiplying by $P_9 \equiv 1 \pmod 2$, we get

$$\begin{aligned} C_9 &\equiv -(2k+1)P_9 - B_9E_9P_9 \pmod 4 \\ &\equiv -(2k+1)P_9 - B_9(2-B_9)P_9 \pmod 4 \\ &\equiv -(2k+1)P_9 \pmod 4 \\ &\equiv -h(p)T/4 \pmod 4, \end{aligned}$$

so that

$$h(-p) \equiv 8 - h(p)T \equiv T + (p-1) + 4\big(h(p)-1\big) \pmod{16},$$

as required.

This completes the proof of the theorem.

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The ideas of this paper have been extended to determine $h(-2p) \pmod{16}$, where $p \equiv 1 \pmod 8$ is prime.

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On the distribution modulo 1 of the sequence $\alpha n^3+\beta n^2+\gamma n$

by

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1. Introduction. Let $\|\cdot\|$ denote distance to the nearest integer. Let $\varepsilon > 0$, and let α, β, γ denote arbitrary real numbers. Recently W. M. Schmidt showed [5] that *for $N > c_1(\varepsilon)$ there is a natural number $n \leqslant N$ having*

$$\|\alpha n^2+\beta n\| < N^{-1/2+\varepsilon}.$$

This generalizes the well known theorem of Heilbronn [3] and sharpens a result of Davenport [2].

Schmidt's method enabled him to prove that for $N > c_2(\varepsilon)$ there is a natural number $n \leqslant N$ having

$$\|\alpha n^3+\beta n^2+\gamma n\| < N^{-1/5+\varepsilon}.$$

For $\gamma = 0$, the exponent $-1/5+\varepsilon$ could be replaced by $-1/4+\varepsilon$ [6]. Both results sharpen those of Davenport [2].

In the present paper we shall show that *for $N > c_3(\varepsilon)$ there is a natural number $n \leqslant N$ having*

$$\|\alpha n^3+\beta n^2+\gamma n\| < N^{-1/4+\varepsilon}.$$

yt is no more difficult to prove a more general theorem. We denote by k an integer greater than 1 and write $K = 2^{k-1}$.

THEOREM 1. *Suppose $k \geqslant 3$ and $N > c_1(k, \varepsilon)$. Then there is a natural number $n \leqslant N$ with*

(1) $$\|\alpha n^k+\beta n^{k-1}+\gamma n\| < N^{-1/K+\varepsilon}.$$

We also strengthen Schmidt's theorem [6] for an arbitrary polynomial of degree $k \geqslant 3$ with constant term zero, but only when k is odd.

THEOREM 2. *Let k be an odd integer, $k \geqslant 3$, and write $K_1 = \frac{4}{3}(2^{k-1}-1)$. Let $N > c_2(k, \varepsilon)$. Given a polynomial $F(n)$ of degree k with constant term zero, there is a natural number $n \leqslant N$ with*

(2) $$\|F(n)\| < N^{-1/K_1+\varepsilon}.$$

We shall use ideas normally associated with "major arcs" in the circle method [4]. Schmidt's method, on the other hand, is a very original development of "minor arc" ideas.