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Bounding L-functions by class numbers

by

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1. Introduction. Let $L(s,\chi)$ be the Dirichlet L-function belonging to the real primitive character $\chi \pmod{|D|}$, for a fundamental discriminant $D<0$. The value of $L(s,\chi)$ has attracted much attention, in particular $L(1,\chi)$. The value of $L(1,\chi)$ is given by Dirichlet's class number formula,

$$(1)\qquad L(1,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-1} = \frac{\pi h(D)}{\sqrt{|D|}} \qquad (D<0);$$

where $h(D)$ is the class number of the field $Q(\sqrt{D})$. The size of $L(1,\chi)$ is closely related to the value of $L(s,\chi)$ for $s\in[\frac{1}{2},1)$. For example we mention:

Theorem 1 (Hecke). *Let δ be any fixed real number satisfying $0<\delta<1$, and suppose that there is at least one point α satisfying $\frac{1}{2}\leqslant\alpha<1$ for which*

$$L(\alpha,\chi) > \frac{-7}{6e}\,\frac{\delta}{\Gamma(\alpha)|\zeta(\alpha)|} \qquad (D<-16\pi^2);$$

then

$$L(1,\chi) > (1-\delta)\,\frac{7\pi}{6e}\,2\alpha(1-\alpha)\,\frac{|D|^{(\alpha-1)/2}}{(2\pi)^{\alpha}} \qquad (D<-16\pi^2).$$

This statement of Hecke's theorem requiring a weaker hypothesis and with the constants given explicitly, is implicit in Landau's proof [3].

On the other hand trying to give upper bounds for $L(s,\chi)$ for $s\in[0,1]$ is also a difficult problem. There is a conjecture (see Montgomery [4]) that for $\varepsilon>0$ and $|D|>c_0(\varepsilon)$

$$(2)\qquad L(s,\chi) \ll |D|^{\varepsilon(1-s)} \quad \text{for} \quad s\in[\tfrac{1}{2},1).$$

In connection with giving a lower bound for $L(1,\chi)$ we mention Tatuzawa's [7] near effectivisation of Siegel's theorem [6].

THEOREM 2 (Tatuzawa). *For an explicit function $c(\varepsilon)$ $(0<\varepsilon<1/2)$ there is at most one negative fundamental discriminant $D, |D|>c(\varepsilon)$, for which*

$$h(D)<|D|^{\varepsilon}.$$

We shall prove here that the L-function belonging to this exceptional discriminant satisfies conjecture (2). Alternatively we can state this as:

THEOREM 3. *Let $\chi \pmod{|D|}$, $D<0$, be a real primitive character, and $L(s,\chi)$ be the L-function belonging to χ. Then if $|D|>c(\varepsilon)$ either,*

$$(3)\qquad L(s,\chi)\ll |D|^{\varepsilon(1-s)}\log^2|D| \qquad (\tfrac{1}{2}\leqslant s\leqslant 1)$$

or

$$(4)\qquad h(D)\gg |D|^{\varepsilon} \qquad (0<\varepsilon<\tfrac{1}{2})$$

must hold.

The implied constants above, and in all that follows, will be effectively computable. Probably both (3) and (4) are true but this either/or type of result seems to be a feature of the topic at the moment. For example we mention:

the result of Fluch [2], that

$$\text{either}\quad L'(1,\chi)>1,\quad \text{or}\quad L(1,\chi)\gg(\log|D|)^{-1}.$$

the Deuring phenomenon (see Mordell [5]) that

either *Riemann's hypothesis is true* or *$h(D)\to\infty$ as $D\to-\infty$.*

2. Proofs. A proof of Theorem 3 will follow directly from the following:

LEMMA. *Let χ be a real primitive character $\bmod |D|$, $D<0$. Then,*

$$(5)\qquad |L(s,\chi)|\ll h^{1-s}\log^2(3h)+h^s\log^2(3h)|D|^{\frac{1}{2}-s}+h|D|^{-s/2} \qquad (\tfrac{1}{2}\leqslant s\leqslant 1),$$

where $h=h(D)$ is the class number.

Proof. We use the relations (3.8) of [1], and after summing over all the reduced quadratic forms (a,b,c) get:

$$(6)\qquad \zeta(s)L(s,\chi)$$

$$=\zeta(2s)\sum_a a^{-s}+\left(\frac{|D|}{4}\right)^{\frac{1}{2}-s}\zeta(2s-1)\frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(s)}\sum_a a^{s-1}+O(h|D|^{-\sigma/2}).$$

This equation is valid for $s=\sigma+it$, with $\frac{1}{2}\leqslant\sigma<1$, and bounded t.

The sum occurring in (6) is over all the $h(D)$ reduced binary quadratic forms (a,b,c) such that,

$$\begin{cases}-a<b\leqslant a<c\\ b^2-4ac=D<0.\end{cases}$$

We first need some information about $\sum_a a^{-s}$. We thus let,

$$(7)\qquad \sum_a a^{-s}=\sum_{n=1}^{X}c(n)n^{-s}$$

where $\sum_{n=1}^{X}c(n)=h$. Thus (7) can be written as

$$(8)\qquad \sum_a a^{-s}=s\int_1^X\frac{S(x)}{x^{s+1}}\,dx+\frac{S(X)}{X^s},$$

where,

$$(9)\qquad S(x)=\sum_{n\leqslant x}c(n).$$

But for every a in $\sum_a$, we know there exists an ideal of norm a in $K=Q(\sqrt{D})$. Hence if $\zeta(s)L(s,\chi)=\zeta_K(s)=\sum_{n=1}^{\infty}g(n)n^{-s}$ $(\mathrm{Re}(s)>1)$, then $g(n)=\sum_{t|n}\chi(t)\leqslant d(n)$, where $d(n)$ is the divisor function. Thus we have,

$$(10)\qquad c(n)\leqslant g(n)\leqslant d(n),$$

and using (10) in (9) we get,

$$(11)\qquad S(x)\leqslant\sum_{n\leqslant x}d(n)\ll x\log x.$$

Using (11) in (8) we have,

$$(12)\qquad \sum_a a^{-s}\ll\int_1^X\frac{\log x}{x^s}\,dx+\log X\ll X^{1-s}\log^2(3X) \qquad (0\leqslant s\leqslant 1).$$

From (10) we observe that,

$$(13)\qquad h=\sum_{n=1}^{X}c(n)\leqslant\sum_{n=1}^{X}d(n)\ll X\log X,$$

and hence see that (13) holds if we take $X=ch$, for some positive constant c. If we now take $X=ch$ in (12), then

$$(14)\qquad \sum_a a^{-s}\ll h^{1-s}\log^2(3h) \qquad (0\leqslant s\leqslant 1),$$

with a corresponding expression for $\sum_a a^{-(1-s)}$. Finally using (14) in (6) we get the lemma by noting that for s near 1^- we have

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|1-s|) \qquad (\gamma = 0.577\,\ldots).$$

Proof of Theorem 3. Suppose that (4) is false, i.e. we can assume that $h(D) \ll |D|^{\varepsilon}$ $(0 < \varepsilon < \frac{1}{2})$. Then we have from the Lemma for $\frac{1}{2} \leqslant s \leqslant 1$,

$$|L(s,\chi)| \ll (|D|^{\varepsilon(1-s)} + |D|^{\varepsilon s + \frac{1}{2} - s})\log^2|D| + D^{\varepsilon - s/2}.$$

But clearly,

$$|D|^{\varepsilon(1-s)} \geqslant |D|^{\varepsilon s + \frac{1}{2} - s}$$

and

$$|D|^{\varepsilon(1-s)} \geqslant |D|^{\varepsilon - s/2}, \quad \text{if } 0 < \varepsilon < \tfrac{1}{2} \text{ and } \tfrac{1}{2} \leqslant s \leqslant 1.$$

Hence $|L(s,\chi)| \ll |D|^{\varepsilon(1-s)}\log^2|D|$, thus proving Theorem 3.

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Über die Klassenzahl einfach reeller kubischer Zahlkörper

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1. Einleitung. Die Ausdeutung einer von H. Hasse und C. Meyer stammenden Klassenzahlformel für einfach reelle kubische Zahlkörper ergab in [5], Satz (3.1) einen Zusammenhang zwischen den Klassenzahlen dieser Körper und zyklischen Untergruppen der Ordnung 9 in Ringdivisorenklassengruppen imaginär-quadratischer Körper. Diese Ergebnisse werden durch die Sätze 1 und 2 der vorliegenden Arbeit wesentlich verallgemeinert. Grundlage hierfür ist unter anderem das Lemma 3 aus [8], dessen Verwendung beim Beweis der Sätze 1 und 2 die rein technischen Voraussetzungen (1.20) in [5], Satz (3.1) überflüssig macht.

Sei Σ ein imaginär-quadratischer Zahlkörper der Diskriminante $D < 0$, und für eine natürliche Zahl f bezeichne Ω_f den Ringklassenkörper modulo f über Σ. Ω_f ist im Sinne der Klassenkörpertheorie der Untergruppe H_f^* aller Hauptdivisoren (γ) von Σ zugeordnet, wobei γ alle zu f primen Zahlen aus $\Sigma - \{0\}$ durchläuft, die modulo f zu einer rationalen Zahl kongruent sind (vgl. [7], Abschnitt 3). Jeder einfach reelle kubische Zahlkörper K ist Teilkörper eines geeigneten Ringklassenkörpers, und nach [2] gilt $K \subseteq \Omega_f$ genau dann, wenn die Diskriminante D_K von K die Zerlegung

$$D_K = f_K^2 D \quad \text{mit} \quad f_K \in \boldsymbol{N} \text{ und } f_K | f \tag{1.1}$$

besitzt. Bezeichnet weiterhin $\mathfrak{K}_f^*$ die Ringdivisorenklassengruppe modulo f, d.h. die Faktorgruppe der zu f primen Divisoren von Σ nach der Untergruppe H_f^* und $r_3(f)$ den 3-Rang von $\mathfrak{K}_f^*$, so lautet das Hauptresultat dieser Arbeit:

SATZ 1. *Es sei* $D < -4$, *und* f *besitze einen Primteiler* p *mit* $p^2 \nmid f$, $\left(\frac{D}{p}\right) = -1$ *und* $p+1 \equiv \pm 3 \bmod 9$. *Dann gilt:*

(1) *Für jeden einfach reellen kubischen Zahlkörper* K *der Diskriminante* $D_K = f_K^2 D$ *mit* $p \,|\, f_K | f$ *besteht die Äquivalenz*

$$3 \,|\, h_K c_K(f) \Leftrightarrow 9 \,|\, |\mathfrak{K}_f^*|.$$