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New formulations of the class number one problem

by

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1. Introduction. Let $h(D)$ be the class number of the quadratic field $Q(\sqrt{D})$ for a fundamental discriminant $D<0$. Around 1800 Gauss [4] conjectured that the only negative fundamental discriminants with $h(D) = 1$ are: -3, -4, -7, -8, -11, -19, -43, -67, -163. Only recently, however, has this conjecture been confirmed by Baker [1] and Stark [8] (1),

We shall give here two results concerning the restrictions placed on D resulting from limiting the class number to be unity. Thus if we could show that these restrictions on D imply that $|D| < A$, for some effectively computable constant A, then we would have a new effective proof of Gauss' conjecture. We remark that both our results can be extended to cover other values of the class number, but we have only stated the results for $h(D) = 1$, since this is the simplest case.

We state our results as:

THEOREM 1. *Let $h(D)$ be the class number of the quadratic field belonging to the fundamental discriminant $D<0$.*

If $h(D) = 1$, then we have

$$\frac{\zeta(2it)\Gamma(it)\sqrt{\pi}}{\zeta(1+2it)\Gamma(\frac{1}{2}+it)} = -\left(\frac{|D|}{4}\right)^{it} + O\big(|D|^{-1/4}\log^7|t|\exp(\pi|t|-\pi\sqrt{|D|})\big), \tag{1}$$

where t satisfies, $\zeta(\frac{1}{2}+it) = 0$.

The equation is remarkable in that the right-hand side is a function of D but the left-hand side is not. However we are still unable to prove (1) false for $|D|$ sufficiently large.

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(1) As pointed by my referee, we should in fairness not forget Heegner [5], who had found a solution (though in a somewhat unintelligible form) in 1952. For further details see also Stark [7]. Yet another solution has also been given by Deuring [3].

THEOREM 2. *With the same notation as Theorem* 1, *if* $h(D)=1$, *then*

$$(2)\qquad 1=(4t^2+1)\sum_{n^2<|D|/4}\int_1^\infty \exp\left(\frac{-2\pi n^2 x}{\sqrt{|D|}}\right)\cos(t\log x)\,\frac{dx}{x}+ O\left(t^2\exp(-\pi\sqrt{|D|}/3)\right),$$

where t *again satisfies* $\zeta(\frac{1}{2}+it)=0$.

Clearly if we could show that the right-hand side of (2) is greater than 1 for $|D|$ sufficiently large, then we could list all $D<0$ for which $h(D)=1$.

2. Proofs. We use the relations (3.8) of [2], and summing over all the reduced quadratic forms (a, b, c) we get for the Dedekind zeta function $\zeta_K(s)$ of the field $K=Q(\sqrt{D})$,

$$(3)\qquad \zeta_K(s)=\zeta(2s)\sum_a a^{-s}+\frac{|D|^{\frac{1}{2}-s}}{2^{1-2s}}\frac{\zeta(2s-1)\Gamma(s-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(s)}\sum_a a^{s-1}+ O\left(\sum_a \exp(-\pi\sqrt{|D|}/a+\pi|t|)\,|D|^{-\sigma/2}|t|^{1-2\sigma}\right);$$

which is valid for $s=\sigma+it$, with $\frac{1}{2}\leqslant\sigma<1$.

We now choose $s=\varrho=\frac{1}{2}+it$, such that

$$(4)\qquad \zeta(\varrho)=0,\qquad \varrho=\tfrac{1}{2}+it,$$

and thus $\zeta_K(\varrho)=0$. Hence we see that

$$|\zeta(2s)|=|\zeta(1+2it)|\gg\log^{-7}|t|;$$

and since $h(D)=1$ we have $a=1$ only, thus Theorem 1 now easily follows from (3).

Proof of Theorem 2. From Theorem 3, p. 260 of [6], we have for $D<-4$,

$$(5)\qquad F(s)=\frac{|D|^{s/2}}{(2\pi)^s}\Gamma(s)\zeta_K(s) =\sum_{n=1}^\infty\int_1^\infty g(n)\exp\left(\frac{-2\pi n x}{\sqrt{|D|}}\right)(x^s+x^{1-s})\frac{dx}{x}-\frac{h(D)}{s(1-s)},\quad \text{for } 1\neq s\geqslant\tfrac{1}{2}.$$

Here $g(n)$ is given by $\zeta_K(s)=\sum_{n=1}^\infty g(n)n^{-s}$ for $s>1$, and we further have,

$$(6)\qquad 0\leqslant g(n)\leqslant d(n),$$

for the divisor function $d(n)$. But if $h(D)=1$, we know (see [9]) that for $n<|D|/4$ we have

$$g(n)=\begin{cases}1, & \text{if } n \text{ is a perfect square},\\ 0, & \text{otherwise}.\end{cases}$$

As in Theorem 1 we choose ϱ to satisfy (4), then

$$x^s+x^{1-s}=2\,\mathrm{Re}(x^s),$$

i.e.

$$(7)\qquad x^s+x^{1-s}=2\sqrt{x}\cos(t\log x);$$

and

$$(8)\qquad \frac{-h(D)}{s(1-s)}=\frac{1}{2\varrho(\varrho-1)}=\frac{-2}{4t^2+1}.$$

Using (7) and (8) in (5) we get,

$$\frac{1}{4t^2+1}=\sum_{n^2<|D|/4}\int_1^\infty\exp\left(\frac{-2\pi n^2x}{\sqrt{|D|}}\right)\cos(t\log x)\frac{dx}{\sqrt{x}}+ \sum_{n\geqslant|D|/4}g(n)\int_1^\infty\exp\left(\frac{-2\pi nx}{\sqrt{|D|}}\right)\cos(t\log x)\frac{dx}{\sqrt{x}}.$$

However from (6) we have $g(n)\leqslant d(n)$ and hence can estimate the expression in (9) for which $n\geqslant|D|/4$ as,

$$\ll\sum_{n\geqslant|D|/4}d(n)\int_1^\infty\exp\left(\frac{-2\pi nx}{\sqrt{|D|}}\right)\frac{dx}{\sqrt{x}}$$

$$\ll\sqrt{|D|}\sum_{n\geqslant|D|/4}\frac{d(n)}{n}\exp\left(\frac{-2\pi n}{\sqrt{|D|}}\right)$$

$$\ll\exp(-\pi\sqrt{|D|}/3);$$

thus proving Theorem 2.

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Bounding L-functions by class numbers

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1. Introduction. Let $L(s, \chi)$ be the Dirichlet L-function belonging to the real primitive character $\chi \pmod{|D|}$, for a fundamental discriminant $D < 0$. The value of $L(s, \chi)$ has attracted much attention, in particular $L(1, \chi)$. The value of $L(1, \chi)$ is given by Dirichlet's class number formula,

$$L(1, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-1} = \frac{\pi h(D)}{\sqrt{|D|}} \quad (D < 0); \tag{1}$$

where $h(D)$ is the class number of the field $Q(\sqrt{D})$. The size of $L(1, \chi)$ is closely related to the value of $L(s, \chi)$ for $s \in [\frac{1}{2}, 1)$. For example we mention:

THEOREM 1 (Hecke). *Let δ be any fixed real number satisfying $0 < \delta < 1$, and suppose that there is at least one point a satisfying $\frac{1}{2} \leqslant a < 1$ for which*

$$L(a, \chi) > \frac{-7}{6e} \frac{\delta}{\Gamma(a)|\zeta(a)|} \quad (D < -16\pi^2);$$

then

$$L(1, \chi) > (1-\delta) \frac{7\pi}{6e} 2a(1-a) \frac{|D|^{(a-1)/2}}{(2\pi)^a} \quad (D < -16\pi^2).$$

This statement of Hecke's theorem requiring a weaker hypothesis and with the constants given explicitly, is implicit in Landau's proof [3].

On the other hand trying to give upper bounds for $L(s, \chi)$ for $s \in [0, 1]$ is also a difficult problem. There is a conjecture (see Montgomery [4]) that for $\varepsilon > 0$ and $|D| > c_0(\varepsilon)$

$$L(s, \chi) \ll |D|^{\varepsilon(1-s)} \quad \text{for} \quad s \in [\tfrac{1}{2}, 1). \tag{2}$$

In connection with giving a lower bound for $L(1, \chi)$ we mention Tatuzawa's [7] near effectivisation of Siegel's theorem [6].