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Quadratic forms and radicals of fields*

by

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1. Introduction. In [6] Kaplansky introduced the notion of a radical for a field F as the set $R = \{a \in \dot{F} \mid [a, b] = 1 \text{ for all } b \in \dot{F}\}$ where $[a, b]$ is the Hilbert symbol for a, b. Equivalently $R = \bigcap_{a \in \dot{F}} D(\langle 1, a\rangle)$. Kaplansky showed that a field such that $|\dot{F}/R| = 2$ was a formally real field in which every element was the sum of two squares and R was exactly the set of totally positive elements of F. Since then other authors have been investigating the role of the radical with respect to quadratic form theory. In particular, Cordes [2] found that many results which held in terms of $\dot{F}^2$ could be strengthen to results in terms of R. In [1] Berman gives several methods of constructing field with non-trivial radicals, i.e., fields with $\dot{F}^2 \subsetneq R \subsetneq \dot{F}$.

In this paper we defined a sequence of radicals $R_0(F) \subseteq R_1(F) \subseteq R_2(F) \subseteq \ldots$ in which Kaplansky's radical is $R_1(F)$. It turns out that the entire sequence seems to be as rich as Kaplansky's radical. For example in Section 2 we show that a field F with $|\dot{F}/R_n(F)| = 2$ is a formally real field in which every element is a sum of 2^n squares and $R_n(F)$ consist of exactly the totally positive elements of F. We also show that if a is a totally positive element of F and $K = F(\sqrt{a})$ then $R_n(F) \subseteq R_n(K)$. This generalizes the going up theorem of Elman and Lam ([4], Theorem 4.5).

In Section 3 we give a generalization of most of the work done in [2] and in Section 4 we show that [10], Theorem 4.2, and [5], Theorem 2.4, can be strengthened by replacing $\dot{F}^2$ with $R(F)$.

Throughout this paper F will be a field of characteristic not 2 and the notation and terminology of [7] will be used.

2. The sequence $R_n(F)$. The sequence of radicals for a field F is defined by $R_n(F) = \bigcap D(\varphi)$ where φ ranges over all n-fold Pfister forms.

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Remark 2.1. (i) $R_0(F) = \dot{F}^2$, $R_1(F) = R(F)$ and $R_0(F) \subseteq R_1(F) \subseteq \ldots \ldots \subseteq R_n(F) \subseteq \ldots$

(ii) $R_n(F)$ is a subgroup of $D(2^n)$.

(iii) If $D(2) \subseteq R_n(F)$ then $R_n(F) = D(\infty)$ (see [7], Theorem 11.1.6).

PROPOSITION 2.2. *For $r \in \dot{F}$ the following statements are equivalent.*

(1) $r \in R_n(F)$.

(2) $\langle\langle -r, a_2, \ldots, a_{n+1}\rangle\rangle$ *is hyperbolic for all* $a_i \in \dot{F}$.

(3) $\langle\langle -r, a_2, \ldots, a_n\rangle\rangle$ *is universal for all* $a_i \in \dot{F}$.

Proof. (1)$\Rightarrow$(2). Suppose $r \in R_n(F)$ and $\varphi = \langle\langle a_2, \ldots, a_{n+1}\rangle\rangle$. $r\varphi \simeq \varphi$ since $r \in D(\varphi)$ thus $\langle 1, -r\rangle\varphi$ is hyperbolic. Consequently, $\langle\langle -r, a_2, \ldots \ldots, a_{n+1}\rangle\rangle$ is hyperbolic.

(2)$\Rightarrow$(3). Suppose $d \in \dot{F}$ and let $\psi = \langle\langle -r, a_2, \ldots, a_n\rangle\rangle$. By (2), $\langle 1, -d\rangle\psi = \langle\langle -r, a_2, \ldots, a_n, -d\rangle\rangle$ is hyperbolic. It follows that $\psi \simeq d\psi$ and ψ is universal.

(3)$\Rightarrow$(1). Let $\varphi = \langle\langle a_1, \ldots, a_n\rangle\rangle$. By (3), $\langle\langle -r, a_1, \ldots, a_{n-1}\rangle\rangle$ is universal hence $\langle\langle -r, a_1, \ldots, a_n\rangle\rangle$ is hyperbolic. Consequently $r\varphi \simeq \varphi$ and thus $r \in D(\varphi)$.

LEMMA 2.3. *Let $r \in R_n(F)$ and suppose $\varphi = \langle\langle a_1, \ldots, a_{n+1}\rangle\rangle$. Then $\varphi \simeq \langle\langle a_1 r, a_2, \ldots, a_{n+1}\rangle\rangle$.*

Proof. $\varphi - \langle\langle a_1 r, a_2, \ldots, a_{n+1}\rangle\rangle = \langle a_1, -a_1 r\rangle\langle\langle a_2, \ldots, a_{n+1}\rangle\rangle = \langle a_1\rangle\langle\langle -r, a_2, \ldots, a_{n+1}\rangle\rangle$ which is hyperbolic by Proposition 2.2 hence $\varphi \simeq \langle\langle a_1 r, a_2, \ldots, a_{n+1}\rangle\rangle$.

THEOREM 2.4. *Let $r_1, \ldots, r_m \in R_n(F)$, $m \geqslant n+1$, and suppose $a_1, a_2, \ldots \ldots, a_n, \ldots, a_m \in \dot{F}$.*

(1) $\langle\langle a_1, \ldots, a_m\rangle\rangle \simeq \langle\langle a_1 r_1, \ldots, a_m r_m\rangle\rangle$,

(2) $D(\langle\langle a_1, \ldots, a_n\rangle\rangle) = D(\langle\langle a_1 r_1, \ldots, a_n r_n\rangle\rangle)$.

Proof. (1) We induct on m. If $m = n+1$ just apply Lemma 2.3 $(n+1)$ times. Suppose $m > n+1$. $\langle\langle a_1, \ldots, a_m\rangle\rangle \simeq \langle\langle a_1, \ldots, a_{m-1}\rangle\rangle\langle\langle a_m\rangle\rangle \simeq \langle\langle a_1 r_1, \ldots, a_{m-1} r_{m-1}\rangle\rangle\langle\langle a_m\rangle\rangle$ by induction. But again by induction $\langle\langle a_2 r_2, \ldots, a_{m-1} r_{m-1}, a_m\rangle\rangle \simeq \langle\langle a_2 r_2, \ldots, a_m r_m\rangle\rangle$ hence $\langle\langle a_1, \ldots, a_m\rangle\rangle \simeq \langle\langle a_1 r_1, \ldots, a_m r_m\rangle\rangle$.

(2) It suffices to show that $D(\langle\langle a_1, \ldots, a_n\rangle\rangle) \subseteq D(\langle\langle a_1 r_1, \ldots, a_n r_n\rangle\rangle)$. If $d \in D(\langle\langle a_1, \ldots, a_n\rangle\rangle)$ then $\langle\langle a_1, \ldots, a_n, -d\rangle\rangle$ is hyperbolic. By (1), $\langle\langle a_1 r_1, \ldots, a_n r_n, -d\rangle\rangle \simeq \langle\langle a_1, \ldots, a_n, -d\rangle\rangle$. Consequently $d \in D(\langle\langle a_1 r_1, \ldots, a_n r_n\rangle\rangle)$.

THEOREM 2.5. *There are no anisotropic $(n+1)$-fold Pfister forms killed by 2 in $W(F)$ if and only if*

$$D(\infty) = D(2^n) = R_n(F) = R_{n+1}(F) = \ldots$$

Proof. Suppose there are no anisotropic $(n+1)$-fold Pfister forms killed by 2 in $W(F)$. In view of Remark 2.1 (iii) it suffices to show $D(2) \subseteq R_n(F)$. Let $a \in D(2)$ and suppose φ is an n-fold Pfister form. $\langle 1, -a\rangle\varphi$ is an $(n+1)$-fold Pfister form killed by 2 hence $\langle 1, -a\rangle\varphi$ must be hyperbolic. Consequently $a \in D(\varphi)$ and $a \in R_n(F)$.

To prove the converse suppose φ is an $(n+1)$-fold Pfister form killed by 2. By [3], Corollary 1, we can write $\varphi \simeq \langle\langle -w, a_2, \ldots, a_{n+1}\rangle\rangle$ where $w \in D(2)$. Consequently $w \in R_n(F)$ and φ is hyperbolic by Proposition 2.2(2).

COROLLARY 2.6. *$I^3(F)$ is torsion free if and only if $R_2(F) = D(4)$ and we have*

$$R_n(F) = D(4) = D(\infty) \quad \textit{for all } n \geqslant 2.$$

Proof. This follows from [3], Corollary 3, and Theorem 2.5.

Using the technique of [4], Theorem 4.5, we prove the following:

THEOREM 2.7. *If $a \in D(\infty)$ and $K = F(\sqrt{a})$ then $R_n(F) \subseteq R_n(K)$.*

Proof. Let $s\colon K \to F$ be the linear functional defined by $s(1) = 0$, $s(\sqrt{a}) = 1$. Let $r \in R_n(F)$ and suppose φ is an n-fold Pfister form over K. Write $\psi = \langle\langle -r\rangle\rangle\varphi$. We first show that the transfer $s_*(\psi)$ is hyperbolic. By [3], Lemma 2, we can write $\varphi = \langle\langle z\rangle\rangle\varphi_1$ where $z \in K$ and φ_1 is an $(n-1)$-fold Pfister form over F.

$$s_*(\psi) = s_*(\langle\langle -r\rangle\rangle\varphi_1\langle\langle z\rangle\rangle) = \langle\langle -r\rangle\rangle\varphi_1 s_*(\langle\langle z\rangle\rangle)$$

since s_* is $W(F)$-linear. Now, $s_*(\langle\langle z\rangle\rangle) \in I(F)$ hence we can write $s_*(\langle\langle z\rangle\rangle) = a_1\psi_1 + \ldots + a_n\psi_n$ for suitable $a_i \in \dot{F}$ and 1-fold Pfister forms ψ_i over F. But $s_*(\psi) = \langle\langle -r\rangle\rangle\varphi_1(a_1\psi_1 + \ldots + a_n\psi_n) = 0$ in $W(F)$ by Proposition 2.2(2). By [4], Lemma 2.1, $\psi \simeq (\langle\langle -r\rangle\rangle\varphi_1\langle\langle b\rangle\rangle)_K\gamma_1$ where $b \in \dot{F}$ and γ_1 is a Pfister form over K. Now $\langle\langle -r\rangle\rangle\varphi_1\langle\langle b\rangle\rangle = 0$ in $W(F)$ by Proposition 2.2(2) hence $r\varphi \simeq \varphi$ and $r \in R_n(K)$.

Notice that the going-up theorem ([4], Theorem 4.5), is just the special case of Theorem 2.7 when $R_n(F) = D(\infty)$.

LEMMA 2.8. *Let $a, b, b_1, \ldots, b_n \in \dot{F}$. In $W(F)$ we have*

$$\langle\langle 1, -a, bb_1, bb_2, \ldots, bb_n\rangle\rangle - \langle\langle -a, -b, -b_1, -b_2, \ldots, -b_n\rangle\rangle = \langle\langle -a, -ab, -ab_1, \ldots, -ab_n\rangle\rangle.$$

Proof. We induct on n. If $n = 0$, $\langle\langle 1, -a\rangle\rangle - \langle\langle -a, -b\rangle\rangle = \langle 1, 1, -a, -a, -1, a, b, -ab\rangle = \langle 1, -a, b, -ab\rangle = \langle\langle -a, -ab\rangle\rangle$. In general,

$$\langle\langle 1, -a, bb_1, \ldots, bb_n\rangle\rangle - \langle\langle -a, -b, -b_1, \ldots, -b_n\rangle\rangle$$
$$= \langle\langle 1, -a, bb_1, \ldots, bb_n\rangle\rangle - \langle\langle -a, -b, -b_1, \ldots, -b_{n-1}, bb_n\rangle\rangle$$
$$= (\langle\langle 1, -a, bb_1, \ldots, bb_{n-1}\rangle\rangle - \langle\langle -a, -b, -b_1, \ldots, -b_{n-1}\rangle\rangle)\langle 1, bb_n\rangle$$
$$= \langle\langle -a, -ab, -ab_1, \ldots, -ab_{n-1}\rangle\rangle\langle 1, bb_n\rangle \quad \text{(by induction)}$$
$$= \langle\langle -a, -ab_1, \ldots, -ab_{n-1}\rangle\rangle\langle\langle -ab, bb_n\rangle\rangle$$
$$= \langle\langle -a, -ab_1, \ldots, -ab_{n-1}\rangle\rangle\langle\langle -ab, -ab_n\rangle\rangle$$
$$= \langle\langle -a, -ab, -ab_1, \ldots, -ab_{n-1}, -ab_n\rangle\rangle$$

as desired.

PROPOSITION 2.9. *If $[\dot{F}: R_n(F)] = 2$, $n \geqslant 1$, then $D(2) \subseteq R_n(F)$.*

Proof. Let $a \in D(2)$ and assume $a \notin R_n(F)$. By Proposition 2.2(2) it suffices to show that the form $\varphi = \langle\langle -a, a_2, \ldots, a_{n+1}\rangle\rangle$ is hyperbolic for all $a_i \in \dot{F}$. Since $[\dot{F}: R_n(F)] = 2$, any element in $\dot{F}$ is of the form $-r$ or $-ar$ for some $r \in R_n(F)$. If $a_i = -r \in -R_n(F)$ for some i we are done by Proposition 2.2(2) so we may assume that for all i, $a_i = -ar_i$ for some $r_i \in R_n(F)$. By Lemma 2.8,

$$\varphi = \langle\langle -a, -ar_2, \ldots, -ar_{n+1}\rangle\rangle$$
$$= \langle\langle 1, -a, r_2r_3, r_2r_4, \ldots, r_2r_{n+1}\rangle\rangle - \langle\langle -a, -r_2, -r_3, \ldots, -r_{n+1}\rangle\rangle.$$

Since $a \in D(2)$ and $r_2 \in R_n(F)$,

$$\langle\langle 1, -a, r_2r_3, \ldots, r_2r_{n+1}\rangle\rangle = \langle\langle -a, -r_2, -r_3, \ldots, -r_{n+1}\rangle\rangle = 0$$

in $W(F)$. Consequently $\varphi = 0$ in $W(F)$ as desired.

THEOREM 2.10. *Suppose $[\dot{F}: R_n(F)] = 2$, $n \geqslant 1$. Then F is a formally real field with $R_n(F) = D(2^n) = D(\infty)$ and there are no anisotropic $(n+1)$-fold torsion Pfister forms in $W(F)$.*

Proof. By Proposition 2.9, $D(2) \subseteq R_n(F)$ thus $R_n(F) = D(2^n) = D(\infty)$ by Remark 2.1 (iii). By Theorem 2.5, there are no anisotropic torsion $(n+1)$-fold Pfister forms in $W(F)$. To show F is formally real suppose $-1 \in D(\infty)$. Then $D(\infty) = \dot{F}$ hence $\dot{F} = R_n(F)$. This contradicts the fact that $[\dot{F}: R_n(F)] = 2$.

We conclude this section with examples of fields satisfying Theorem 2.10. Let $u = u(F)$ denote the maximum dimension of all anisotropic torsion forms over F.

EXAMPLE 2.11. For any $n \in N$ there exist a field K_n and $m > n$ such that $[\dot{K}_n: R_m(K_n)] = 2$.

Proof. Prestel ([8], Theorem 2.1) gave a construction of a sequence of fields $K_0, K_1, K_2, \ldots$ each having a unique ordering such that $P(K_n) = 2^n$ ($P(F)$ is the smallest integer $k < \infty$ such that every sum of squares in F equals a sum of at most k squares). Using [8], Lemma 1.5(d), it is easy to see that for each K_n, $2^n \leqslant u(K_n) < \infty$. If $2^m > u(K_n)$ then $I^m(K_n)$ is torsion free. By Theorem 2.5 we have $R_{m-1}(K_n) = D(\infty)$. But since K_n has a unique ordering we must have $[\dot{K}_n: R_{m-1}(K_n)] = 2$ as desired.

3. Radicals of non-real fields. Throughout this section F will be a non-real field of characteristic not 2. We will need the following lemma of Kneser.

LEMMA 3.1 (Kneser's Lemma). *If $\varphi + \langle a\rangle$ is anisotropic, then $D(\varphi) \subsetneqq D(\varphi + \langle a\rangle)$.*

Let φ be an s-fold Pfister form ($s \geqslant n$). Since $D(\varphi)$ is a group, φ represents coset of $R_n(F)$. Note also that if $\varphi \in I^s(F)$ and φ has a simple decomposition (i.e. $\varphi \simeq a_1\varphi_1 + \ldots + a_r\varphi_r$ where φ_i are s-fold Pfister forms) then φ also represents cosets of $R_n(F)$.

LEMMA 3.2. *Suppose $\varphi = \langle a_1, \ldots, a_m\rangle$, $m \geqslant 2$ and let ψ be an $(n-1)$-fold Pfister form. Then $\varphi\psi$ represents cosets of $R_n(F)$.*

Proof. Suppose $x \in D(\varphi\psi)$ and $r \in R_n(F)$. If m is even then $\varphi\psi$ has a simple decomposition in $I^n(F)$ thus we may assume m is odd. Let $\varrho = \langle a_1, \ldots, a_{m-1}\rangle$. Then $\varrho\psi$ represents cosets of $R_n(F)$. There exist $y \in D(\varrho\psi)$ and $z \in D(a_m\psi)$ such that $x \in D(\langle y, z\rangle)$. $rx \in D(\langle ry, rz\rangle)$ hence $rx \in D(\varrho\psi + ra_m\psi) = D(a_1\psi + \ldots + a_{m-2}\psi + a_{m-1}\langle 1, ra_m a_{m-1}\rangle\psi) = D(\varphi\psi)$ by Theorem 2.4(2).

If E is a subset of $\dot{F}$ consisting of cosets of $\dot{F}^2$ then we will denote the number of those cosets by $V(E)$.

THEOREM 3.3. *Suppose φ is an m-dimensional form and ψ is an $(n-1)$-fold Pfister form with $\varphi\psi$ anisotropic. If $R_n(F) \neq \dot{F}$ then for $m \geqslant 2$,*

$$V(\varphi\psi) \geqslant mV(R_n(F)).$$

Proof. It suffices to show that $\varphi\psi$ represents at least m cosets of $\dot{F}/R_n(F)$. We induct on m. Suppose $m = 2$. Here we may assume that $\varphi = \langle 1, a\rangle$. If $\varphi\psi$ represents only one coset of $\dot{F}/R_n(F)$ then $D(\varphi\psi) = R_n(F)$. But then by Theorem 2.4(2) we have $D(2) \subseteq D(\varphi\psi) = R_n(F)$ hence by Remark 2.1 (iii) $R_n(F) = \dot{F}$, a contradiction. Now suppose $\varphi = \langle a_1, \ldots, a_m\rangle$ with $m > 2$ and with $\varphi\psi$ anisotropic. By Lemma 3.2 both $\langle a_1, \ldots, a_{m-1}\rangle\psi$ and $\varphi\psi$ represent cosets of $\dot{F}/R_n(F)$. The result follows now from Kneser's Lemma and induction.

If we denote $|\dot{F}/\dot{F}^2|$ by q we have

COROLLARY 3.4. *Suppose φ is an m-dimensional form over F, $m \geqslant 2$, and let ψ be an $(n-1)$-fold Pfister form with $\varphi\psi$ being an anisotropic u-dimensional form. If $q < \infty$ and $R_n(F) \neq \dot{F}$, then*

$$2^n \leqslant u \leqslant 2^{n-1}q/V(R_n(F)).$$

Scharlau ([9], p. 72) showed that if $-a \in R(F)$ and if s_a is the smallest number of squares of which a is the sum, then s_a is a 2-power.

PROPOSITION 3.5. *Suppose* $-a \in R_n(F)$ *and* $2^n \leqslant 2^k \leqslant s_a < 2^{k+1}$, *then* $s_a = 2^k$.

Proof. If $-a \in R_n(F)$, then $-a \in D(2^k)$ thus $2^k\langle 1\rangle \simeq \langle -a\rangle(2^k\langle 1\rangle) \simeq 2^k\langle -a\rangle$. Consequently, $2^{k+1}\langle 1\rangle \simeq 2^k\langle 1\rangle + 2^k\langle -a\rangle$. Since $s_a < 2^{k+1}$, $2^{k+1}\langle 1\rangle \simeq \langle 1, a, \ldots\rangle$ hence $2^{k+1}\langle 1\rangle \simeq \langle\langle a, \ldots\rangle\rangle$. By Proposition 2.2(2), $2^{k+1}\langle 1\rangle$ is hyperbolic. It follows that $2^k\langle 1\rangle \simeq 2^k\langle a\rangle$ thus $s_a = 2^k$.

A proof similar to [2], lemma following Theorem 1, gives us

PROPOSITION 3.6. *Suppose* $-a \in R_n(F)$ *and* $s_a \geqslant 2^n$. *Then* $s_a = s$, *the level of* F.

From [2], Theorem 2, we see that if $R(F) \neq \dot F$ then $V(D(\langle 1, 1\rangle)) \geqslant sV(R(F))$. A similar argument yields the following generalization.

THEOREM 3.7. *If* $R_n(F) \neq \dot F$, $n \geqslant 1$, *then* $V(D(2^n)) \geqslant (s/2^{n-1})\, V(R_n(F))$.

Proof. If $s < 2^n$ then $V(D(2^n)) = q$ and clearly $q \geqslant V(R_n(F)) \geqslant (s/2^{n-1})\, V(R_n(F))$. If $s = 2^n$ then $2^n\langle 1\rangle$ represents both 1 and -1. Since $2^n\langle 1\rangle$ represents cosets of $\dot F/R_n(F)$ we see that $V(D(2^n)) \geqslant 2V(R_n(F))$. Suppose $s > 2^n$. Write $-1 = \sum_{i=1}^{s} x_i^2$ and let

$$y_j = \sum_{i=(j-1)2^n+1}^{j2^n} x_i^2.$$

Note that $y_1, \ldots, y_{s/2^n} \in D(2^n)$. Assume $y_1y_2 \in R_n(F)$. Then,

$$-y_1y_2 = -y_1 + y_1 - y_1y_2 = \sum_{i=1}^{s} x_i^2 y_1 + y_1 - y_1y_2$$
$$= \sum_{i=1}^{s/2^n} y_i y_1 + y_1 - y_1y_2 = y_1^2 + \sum_{i=3}^{s/2^n} y_i y_1 + y_1.$$

Consequently, $-y_1y_2 \in D([1+((s/2^n)-2)2^n+2^n])$ thus $s_{-y_1y_2} < s$. But $s_{-y_1y_2} \geqslant 2^n$ (if not $-1 = (y_1y_2)(-y_1y_2) \in D(2^n)$). By Proposition 3.6 $s_{-y_1y_2} = s$, a contradiction. Consequently $y_1y_2 \notin R_n(F)$. In this manner we see that $y_1R_n(F), \ldots, y_{s/2^n}R_n(F)$ are distinct cosets of $D(2^n)/R_n(F)$. If $y_j \in R_n(F)$, then

$$-y_j = y_j \sum_{i=1}^{s} x_i = y_j^2 + \sum_{i\neq j} y_j y_i \in D([1+((s/2^n)-1)2^n]).$$

But $s-2^n+1 < s$ thus $s_{-y_j} < s$ and $s_{-y_j} \geqslant 2^n$ contradicting Proposition 3.6. It follows that there are at least $(s/2^n)+1$ distinct cosets of $D(2^n)/R_n(F)$. Since $|D(2^n)/R_n(F)|$ is a 2-power, $|D(2^n)/R_n(F)| \geqslant s/2^{n-1}$. Consequently, $V(D(2^n)) \geqslant (s/2^{n-1})\, V(R_n(F))$.

COROLLARY 3.8. *Suppose* $R_n(F) \neq \dot F$ *and* $s = 2^m$, $m \geqslant n$. *Then*
$$q \geqslant 2^{(m-n+1)(m-n+2)/2}\, V(R_n(F)).$$

Proof. From the proof of Kaplansky's Lemma (see [7], p. 325) we have $[D(2^{j+1}):D(2^j)] \geqslant 2^{m-j}$. It follows that

$$[D(2^m):D(2^n)] = [D(2^m):D(2^{m-1})] \ldots [D(2^{n+1}):D(2^n)]$$
$$= 2^1 \cdot 2^2 \cdot \ldots \cdot 2^{n-m} = 2^{(m-n)(m-n+1)/2}.$$

From Theorem 3.7 we obtain

$$V(D(2^m)) = |D(2^m)/D(2^n)|\,|D(2^n)/\dot F^2| = 2^{(m-n)(m-n+1)/2} \cdot 2^{m-n+1}\, V(R_n(F)).$$

Consequently,

$$q \geqslant V(D(2^m)) \geqslant 2^{(m-n+1)(m-n+2)/2}\, V(R_n(F)).$$

We conclude this section with examples of non-real fields having a finite number of square classes and non-trivial radicals.

EXAMPLE 3.9. For any $n \geqslant 3$ and $m < n-1$ there exist a non-real field K with $|\dot K/\dot K^2| = 2^n$ and $\dot K^2 \neq R_1(K) \subseteq \ldots \subseteq R_{n-m-1}(K) \subsetneq \dot K$.

Proof. Let $F_1 = \mathbf{R}((x_1)) \ldots ((x_t))$, $t = n-m$, and let F_2 be a Pythagorean SAP field with $|\dot F_2/\dot F_2^2| = 2^m$. Using the method of Berman, Bröcker and Craven (see [1], Corollary 6.15) there exist a field F such that $I(F) \cong I(F_1) \times I(F_2)$ and such that $K = F(\sqrt{-1})$ has a non-trivial Kaplansky radical. To show that $R_{t-1}(K)$ is non-trivial it suffices to show $I^t(K) \neq 0$. For any field L let H_L^s denote the subgroup of $I^s(L)$ defined by

$$H_L^s = \{\varphi \in I^s(L) \mid \varphi \in 2I(L)\}.$$

Clearly $I^t(F) \cong I^t(F_1) \times I^t(F_2)$ and moreover

$$\frac{I^t(F)}{H_F^t} \cong \frac{I^t(F_1)}{H_{F_1}^t} \times \frac{I^t(F_2)}{H_{F_2}^t}.$$

Now the form $\langle\langle x_1, \ldots, x_t\rangle\rangle \notin H_{F_1}^t$. Consequently $H_F^t \subsetneq I^t(F)$. It follows that the kernel of the surjection $r^*\colon I^t(F) \to I^t(K)$ is not all of $I^t(F)$ hence $I^t(K) \neq 0$.

4. Radicals of formally real fields. The purpose of this section is to strengthen [10], Theorem 4.2 and [5], Theorem 2.4, by replacing $\dot F^2$ by $R(F)$.

Let $X(F)$ be the set of signatures on F. For $a \in \dot F$, let $W(a) = \{\sigma \in X(F) \mid \sigma(a) = -1\}$. In [10] Ware introduced the notion of effective diagonalization of quadratic forms. A form $\varphi = \langle a_1, \ldots, a_n\rangle$ over a formally real field F is effectively diagonalized if $W(a_i) \subset W(a_{i+1})$,

$i = 1, 2, \ldots, n-1$. F is said to satisfy ED if every form over F has an effective diagonalization.

Throughout this section F will be a formally real field. A form φ over F is said to be *totally positive* if $D(\varphi)$ consists of totally positive elements.

PROPOSITION 4.1. *Suppose φ is a totally positive form over F of dimension $n \geqslant 2$. If $D(\varphi) \neq D(\infty)$ and if $R(F) \neq D(\infty)$ then φ represents at least n cosets of $D(\infty)/R(F)$.*

Proof. We induct on n. Note that $\langle a, b\rangle$ represents the same number of cosets of $D(\infty)/R(F)$ as does $\langle 1, ab\rangle$ so to prove the $n = 2$ case it suffices to show that the form $\langle 1, a\rangle$ represents at least 2 cosets of $D(\infty)/R(F)$. Suppose not, then $D(\langle 1, a\rangle) = R(F)$. By [2], Proposition 1, $D(\langle 1, a\rangle) = D(\langle 1, 1\rangle)$ since $a \in R(F)$. But then $D(\langle 1, 1\rangle) = R(F)$. By Remark 2.1 (iii) $D(\infty) \subseteq R(F)$, a contradiction. Now suppose $\varphi = \langle a_1, \ldots, a_n\rangle$. By induction $\langle a_1, \ldots, a_{n-1}\rangle$ represents at least $n-1$ coset of $D(\infty)/R(F)$. By [10], Lemma 4.1, $D(\langle a_1, \ldots, a_{n-1}\rangle) \neq D(\langle a_1, \ldots, a_n\rangle)$ hence φ represents at least n cosets of $D(\infty)/R(F)$.

COROLLARY 4.2. *Suppose φ is a totally positive form over F of dimension $n \geqslant 2$. If $D(\varphi) \neq D(\infty)$ and if $R(F) \neq D(\infty)$ then $V(D(\varphi)) \geqslant nV(R(F))$.*

THEOREM 4.3. *Suppose F satisfies* ED *and $u < \infty$. If $R(F) \neq D(\infty)$ then*

$$u \leqslant |D(\infty)/R(F)|.$$

Proof. Let φ be a u-dimensional torsion form over F. By [10], Lemma 2.1, $\varphi \simeq \varphi_1 - \varphi_2$ where φ_1 and φ_2 are totally positive forms of dimension $u/2$. $D(\varphi_1) \cap D(\varphi_2)$ is empty else φ is isotropic hence $D(\varphi_1) \neq D(\infty)$ and $D(\varphi_2) \neq D(\infty)$. By Corollary 4.2, both φ_1 and φ_2 represent at least $(u/2)V(R(F))$ square classes of $D(\infty)/\dot{F}^2$. Consequently

$$V(D(\infty)) \geqslant 2(u/2)V(R(F)) = uV(R(F))$$

and

$$u \leqslant V(D(\infty))/V(R(F)) = |D(\infty)/R(F)|.$$

COROLLARY 4.4. *Suppose F satisfies* ED *and $R(F) \neq D(\infty)$. If $q < \infty$ then $u \leqslant (1/2^s)|\dot{F}/R(F)|$ where s is the number of distinct orderings of F.*

Proof. By [10], Corollary 4.3, $q = 2^s V(D(\infty))$ and by Theorem 4.3,

$$u \leqslant V(D(\infty))/V(R(F)) = (q/2^s)/V(R(F)) = (1/2^s)|\dot{F}/R(F)|.$$

If E is a subset of F consisting of cosets of $\dot{F}/R(F)$ we denote the number of these cosets by $\overline{V}(E)$. Utilizing the β-decomposition of a form as in [5], Theorem 2.4, we obtain

THEOREM 4.5. *Suppose $q < \infty$ and $|\dot{F}/R(F)| \geqslant 4$. Then*

$$u \leqslant |\dot{F}/R(F)| - 2.$$

Proof. First note that if $u \leqslant 2$, there is nothing to prove. In particular we may assume the torsion subgroup $W_t(F) \neq \{0\}$. Let φ be an anisotropic torsion form of dimension $u > 2$. It suffices to show $\overline{V}(D(\varphi)) \geqslant u+2$. By the proof of [5], Theorem 2.4, we can write $\varphi \simeq \beta_1 + \ldots + \beta_r + \varphi_0$ where $\dim \beta_i = 2$, $2\beta_i = 0$, $4r \geqslant u$ and where $D(\beta_1), \ldots, D(\beta_r), D(\varphi_0)$ are mutually disjoint subsets of $D(\varphi)$. Write $\beta_i = \langle x_i, y_i\rangle$. If $x_i R(F) = y_i R(F)$ then $x_i y_i \in R(F)$. But $0 = 2\beta_i = \langle x_i\rangle\langle 1, 1, x_i y_i, x_i y_i\rangle$ thus $-x_i y_i \in D(2)$, a contradiction. If $-x_i R(F) = y_i R(F)$ then $D(\langle 1, x_i y_i\rangle) = \dot{F}$ hence $\varphi = \beta_i$ and $u \leqslant 2$. Consequently we may assume that for each $i = 1, \ldots, r$, β_i represents at least 4 distinct cosets of $\dot{F}/R(F)$, $x_i R(F)$, $y_i R(F)$, $-x_i R(F)$ and $-y_i R(F)$. If $u = 2r$ then

$$\overline{V}(D(\varphi)) \geqslant \sum_{i=1}^{r} \overline{V}(D(\beta_i)) \geqslant 4r \geqslant u+2.$$

Suppose $u > 2r$. Here φ_0 is a non-zero element of $W_t(F)$ and hence $\overline{V}(D(\varphi_0)) \geqslant 2$. Consequently,

$$\overline{V}(D(\varphi)) \geqslant \sum_{i=1}^{r} \overline{V}(D(\beta_i)) + \overline{V}(D(\varphi_0)) \geqslant 4r+2 \geqslant u+2.$$

COROLLARY 4.6. *Suppose $q < \infty$ and $|\dot{F}/R(F)| \geqslant 4$. Then*

$$u < q/V(R(F)).$$

It has been conjectured by Elman and Lam that if $q < \infty$ then $u \leqslant q/2$. Notice that the condition that $|\dot{F}/R(F)| \geqslant 4$ is needed only for the $u \leqslant 2$ case of Theorem 4.5. Consequently $|\dot{F}/R(F)| \geqslant 4$ may be replaced by the condition $u > 2$ in both Theorem 4.5 and Corollary 4.6. If $u \leqslant 2$ clearly $u \leqslant q/2$. We now have the following:

COROLLARY 4.7. *Suppose $q < \infty$. If $R(F) \neq \dot{F}^2$ then $u \leqslant q/2$.*

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Komposition und Klassenzahlen binärer quadratischer Formen

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Die klassische Kompositionstheorie quadratischer Formen in zwei Variablen von Gauss ([1], Artikel 234–261, 286–287) und Dedekind ([2], X. Supplement) zur Bestimmung der Anzahl der Geschlechter fester Diskriminante kann mittels der Idealtheorie quadratischer Zahlkörper begründet werden (etwa [4], S. 261–292), wobei sich ein Zusammenhang zwischen Klassenzahlen quadratischer Formen und Ringklassenzahlen des Zahlkörpers ergibt. Eine direkte Übertragung auf andere Grundringe als **Z** scheint nur möglich zu sein, wenn man sich auf Ideale beschränkt, die eine Modulbasis haben (vergl. [11] und [12]). Betrachtet man jedoch quadratische Formen auf projektiven Moduln A vom Rang zwei mit endlich vielen, aber nicht notwendig nur zwei Erzeugenden über einem Dedekind-Ring $\mathfrak{o}$, so ergibt sich aus der *Komposition von Moduln*, wenn der Quotientenkörper K von $\mathfrak{o}$ ein algebraischer Zahlkörper ist, ebenfalls eine *Beziehung zwischen verschiedenen Klassenzahlen*, die hier bewiesen werden soll.

Die auf den K-Vektorraum $V = k \otimes_{\mathfrak{o}} A$ fortgesetzte quadratische Form q kann durch einen Skalarfaktor, der Klassenzahlen nicht ändert, so normiert werden, daß es Elemente e in V mit $q(e) = 1$ gibt. Da $\operatorname{char} k \neq 2$ vorausgesetzt wird, ist damit V bis auf Isometrie durch seine Diskriminante dV bestimmt. V sei keine hyperbolische Ebene, also $\delta \underset{2}{=} -dV$ kein Quadrat in k; dann ist $K = k(\sqrt{\delta})$ eine quadratische Erweiterung, und die Norm $N = N_{K|k}$ definiert auf dem zweidimensionalen k-Vektorraum K eine quadratische Form, welche 1 darstellt. Die zugehörige symmetrische Bilinearform ist

$$(x, y) = N(x+y) - N(x) - N(y) = S(x\bar{y})$$

mit der Spur $S = S_{K|k}$, ihre Diskriminante $dK \underset{2}{=} -\delta$. Man darf also $V = K$ und $q = N$ annehmen.