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On the theorem of Jarník and Besicovitch

by

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1. Let $\alpha > 0$ be fixed in all that follows and $E(\alpha)$ be the set of real numbers x such that the inequality $\|nx\| \leqslant n^{-1-\alpha}$ has arbitrarily large integer solutions n. (As usual $\|t\|$ is the distance from t to the nearest integer.) We recall the classical theorem of Jarník [3] and Besicovitch [1].

I. $E(\alpha)$ has Hausdorff dimension $2(2+\alpha)^{-1}$.

We shall obtain for $E(\alpha)$ a stronger property:

II. There exists a positive measure μ whose support is a compact subset of $E(\alpha)$, whose Fourier–Stieltjes transform obeys the relation

$$\hat{\mu}(u) = o(\log|u|)\,|u|^{-1/2+\alpha}, \qquad |u| \to +\infty.$$

By a theorem of Beurling [3, III], the closed support of μ (or any Borel set of positive μ-measure) must have dimension at least $2(2+\alpha)^{-1}$; however the property of $E(\alpha)$ is not shared by certain sets of positive Lebesgue measure, so that II could not be deduced from I (see [4], p. 351).

2. In this paragraph we define some auxiliary functions, and obtain an inequality on Fourier coefficients, more or less equivalent to the main result II. First of all we construct the function $F_R(x)$, $0 < R < 1/4$,

$$F_R(x) = \begin{cases} 35(32)^{-1}R^{-7}(R^2-x^2)^3, & |x| \leqslant R, \\ 0, & R \leqslant |x| \leqslant 1/2. \end{cases}$$

Then we extend F_R to a periodic function and expand it in a Fourier series

$$F_R(x) = \sum a_m^{(R)} e^{2\pi i m x},$$

$$a_0^{(R)} = 1, \quad |a_m^{(R)}| \leqslant 1, \quad |a_m^{(R)}| \leqslant m^{-2}R^{-2}.$$

In the construction below M is a large positive integer and $R = (4M)^{-1-\alpha}$; we write $q_m(x) = \sum_p^* F_R(px)$, where $\sum_p^*$ means that the sum is extended over primes p in the interval $M \leqslant p \leqslant 2M$. We also write C_M for the number of these primes, $C_M = \pi(2M) - \pi(M-1) \cong M/\log M$. Then we have the expansion

$$q_M(x) - C_M = \sum_m{}' \sum_p{}^* a_m^{(R)} e^{2\pi i m p x}.$$

From this formula we can derive upper bounds for the Fourier coefficients $\hat{q}_M$. Because $|a_m^{(R)}| \leqslant 1$, $\hat{q}_M(k)$ does not exceed the number of solutions of the equation $k = mp$. Hence

$$\hat{q}_M(k) = 0, \quad 1 \leqslant |k| < M,$$

$$|\hat{q}_M(k)| \leqslant 2\log|k|/\log M, \quad \text{always}.$$

The last bound can be improved when $|k| > 2MR^{-1}$. In this case all the solutions $mp = k$ satisfy $|m| \geqslant |k|/|p| > |k|/2M$, so $|a_m^{(R)}| \leqslant m^{-2}R^{-2} \leqslant 4k^{-2}M^2R^{-2}$ and then

$$|\hat{q}_M(k)| \leqslant 4k^{-2}M^2R^{-2}\log|k|/\log M.$$

Let us now set $g_M(x) = C_M^{-1}q_M(x)$. From the previous inequalities we find

$$\text{(a)} \qquad \begin{aligned} &|\hat{g}_M(k)| \leqslant A\log M/M, \quad 1 \leqslant |k| \leqslant 2MR^{-1}, \\ &|\hat{g}_M(k)| \leqslant Ak^{-2}MR^{-2}\log|k|, \quad |k| > 2MR^{-1}. \end{aligned}$$

Recalling that $R = (4M)^{-1-\alpha}$ we get

$$\text{(b)} \qquad |\hat{g}_M(k)| \leqslant A|k|^{-1/2+\alpha}\log|k|, \quad |k| > 2MR^{-1}.$$

In (a) and (b), A depends only on α.

Let now ψ be a function of class C^2 and compact support. The next lemma expresses in compact form the basic step in the construction of μ.

LEMMA. *For the Fourier transform of* (ψg_M) *we have the inequalities*

$$|(\psi g_M)\hat{}\,(u) - \hat{\psi}(u)| \leqslant B\log(2+|u|)(|u|+1)^{-1/2+\alpha},$$

$$|(\psi g_M)\hat{}\,(u) - \hat{\psi}(u)| \leqslant B\log M/M,$$

where B *depends only on* α *and* ψ.

Proof. The difference in question is a sum $\sum_k' \hat{\psi}(u-k)\hat{g}_M(k)$; because ψ is C^2, we can majorize this by

$$A\sum_k{}' (1+|u-k|)^{-2}|\hat{g}_M(k)|.$$

Clearly this sum is most $A\max|\hat{g}_M(k)|$ (maximum for $k \neq 0$) and this, by (a) and (b) gives the second inequality. The second is stronger than the first when $|u| < 4MR^{-1}$, so we can assume $|u| \geqslant 4MR^{-1}$. In the sum to be estimated, we handle the range $|u-k| > |u|/2$ and its complement. The first sum is easily $O(|u|^{-1})$, and the second sum is at most $A\max|\hat{g}_M(k)|$: $|k| > |u|/2$. Using (b), and the inequality $2+\alpha > 1$, we find the second inequality of the lemma.

3. Before passing to the proof of II, we write a more refined estimate than given in the lemma. We fix ψ as above and $\varepsilon > 0$, and attempt to find a function G (like g_M) so that

$$|(\psi G)\hat{}\,(u) - \hat{\psi}(u)| \leqslant \varepsilon\log(2+|u|)(1+|u|)^{-1/2+\alpha}.$$

For this function G we cannot choose g_M (unless our method of estimation is improved) but we can choose $N^{-1}(g_{M_1} + \ldots + g_{M_N})$, with N large and $M_1 < \ldots < M_N$ an appropriate sequence. To see this we have only to observe that each $g_M\psi$ is C^2, so its Fourier transform is $O(u^{-2})$.

Finally, we remark that if $G(x) > 0$ then $\|px\| < p^{-1-\alpha}$ for a certain prime p in the interval $[M_1, 4M_N]$.

4. Let $\psi_0 \geqslant 0$, $\int\psi_0 = 1$, ψ_0 of class C^2 and compact support. By the method just described we choose $G_1, \ldots, G_k, \ldots$ and $H_k = G_1 \ldots \ldots G_k$ $(H_0 = 1)$, so that

$$|\psi_0 \hat{H_k} - \psi_0 \hat{H_{k+1}}| \leqslant 2^{-k-1}\log(2+|u|)(1+|u|)^{-1/2+\alpha}.$$

Then the w^* = limit of the measures $\psi_0(x)H_k(x)\,dx$, a positive measure μ, certainly fulfills the relation $\hat{\mu}(u) = o(\log|u|)|u|^{-1/2+\alpha}$ as $|u| \to \infty$. For each k, the closed support of μ is contained in the closure F_k of the set $\{\psi_0(x)G_k(x) > 0\}$. Clearly $\bigcap F_k \subset E(\alpha)$ and our main theorem is proved.

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