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p-adic T-numbers do exist

by

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1. Introduction. In 1932 K. Mahler [3] gave a classification of the real transcendental numbers, dividing them into three classes: S-numbers, T-numbers and U-numbers. The existence of S- and U-numbers was easily seen, but it was for a long time an open question, whether the class of T-numbers is empty or not. In 1968 W. M. Schmidt [6] was able to prove the existence of real T-numbers using a generalization of K. F. Roth's well known theorem by E. Wirsing.

In 1934 K. Mahler [4] introduced an analogous classification of the transcendental numbers in the field $\boldsymbol{Q}_p$, the completion of $\boldsymbol{Q}$ with respect to a prime p. It is the purpose of this paper to prove

THEOREM 1. *p-adic T-numbers do exist.*

In the proof of Theorem 1 we use a modified version of Schmidt's method [6]. The main difficulty arises at a point, where Schmidt uses the linear ordering of the real numbers, a property, which naturally does not hold in $\boldsymbol{Q}_p$. Another tool in the proof is the application of a theorem about approximations to a p-adic algebraic number by p-adic algebraic numbers of bounded degree, which is a consequence of my paper [5].

2. Mahler's and Koksma's classification. In the following p is a fixed prime of $\boldsymbol{Q}$ and $|\ldots|$ denotes the ordinary absolute value of $\boldsymbol{Q}$, whereas $|\ldots|_p$ denotes the p-adic valuation.

For the convenience of the reader we shall briefly recall Mahler's classification of the p-adic numbers and introduce an analogue for the p-adics to Koksma's well known classification [2] of the reals.

We first deal with Mahler's classfication. Let

$$P(X) = a_n X^n + \ldots + a_0 \in \boldsymbol{Z}[X]$$

and call, as usual,

$$H(P) = \max\{|a_0|, \ldots, |a_n|\}$$

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the *height* of P. For a p-adic number ξ and a natural n put

$$\omega_n(H,\xi) = \min_{\substack{\deg P\leqslant n\\ H(P)\leqslant H\\ P(\xi)\neq 0}} |P(\xi)|_p.$$

Put furthermore

$$\omega_n(\xi) = \overline{\lim_{H\to\infty}}\big(-(\log\omega_n(H,\xi))/\log H\big)$$

and

$$\omega(\xi) = \overline{\lim_{n\to\infty}}\frac{\omega_n(\xi)}{n}.$$

If $\omega_n(\xi) = \infty$ for some n, then let $\mu(\xi)$ be the smallest such n; otherwise put $\mu(\xi) = \infty$. ξ is called an

A-number	if	$\omega(\xi) = 0,$	$\mu(\xi) = \infty,$
S-number	if	$0 < \omega(\xi) < \infty,$	$\mu(\xi) = \infty,$
T-number	if	$\omega(\xi) = \infty,$	$\mu(\xi) = \infty,$
U-number	if	$\omega(\xi) = \infty,$	$\mu(\xi) < \infty.$

We now turn to the analogue of Koksma's classification. For an algebraic number α define the height $H(\alpha)$ as the height of the minimal polynomial of α, say $P(X)\in \boldsymbol{Z}[X]$, where P is supposed to be normalized, such that its coefficients are relatively prime.

For a number $\xi\in \boldsymbol{Q}_p$ and a natural n put

$$\omega_n^*(H,\xi) = \min_{\substack{\deg\alpha\leqslant n\\ H(\alpha)\leqslant H\\ \alpha\neq\xi}} |\xi-\alpha|_p,$$

furthermore

$$\omega_n^*(\xi) = \overline{\lim_{H\to\infty}}\left(\frac{-\log\omega_n^*(H,\xi)}{\log H}\right) \tag{1}$$

and

$$\omega^*(\xi) = \overline{\lim_{n\to\infty}}\frac{\omega_n^*(\xi)}{n}.$$

Define $\mu^*(\xi)$ as being the smallest n, such that $\mu_n^*(\xi) = \infty$, if such an n exists. Otherwise put $\mu^*(\xi) = \infty$. Now call a p-adic number ξ an

S^*-number	if	$0 < \omega^*(\xi) < \infty,$	$\mu^*(\xi) = \infty,$
T^*-number	if	$\omega^*(\xi) = \infty,$	$\mu^*(\xi) = \infty,$
U^*-number	if	$\omega^*(\xi) = \infty,$	$\mu^*(\xi) < \infty.$

Notice that the definition of $\omega_n^*(\xi)$ is slightly different from the usual one in the real case, where one has

$$\omega_n^*(\xi) = \overline{\lim_{H\to\infty}}\left(\frac{-\log\big(H\omega_n^*(H,\xi)\big)}{\log H}\right).$$

The reason, why (1) seems to be the more natural analogue will be explained in the next section.

As in the real case one shows that for both classifications respectively two numbers belonging to different classes are algebraically independent.

Moreover, every S^*-, T^*-, U^*-number is an S-, T-, U-number respectively. Therefore, in order to prove Theorem 1, it suffices to construct T^*-numbers. We shall prove

THEOREM 2. *Let $B_1, B_2, \ldots$ be real numbers with*

$$B_1 > 9,\quad B_i > 3i^2B_{i-1} \text{ for } i>1. \tag{2}$$

There exist numbers $\xi\in\boldsymbol{Q}_p$ with

$$\omega_i^*(\xi) = B_i\quad (i = 1, 2, \ldots). \tag{3}$$

Theorem 1 clearly is a consequence of Theorem 2. The remainder of the paper deals with the proof of Theorem 2.

3. Approximation by algebraic numbers of bounded degree.

THEOREM 3. *Let α be an algebraic number in $\boldsymbol{Q}_p$. Then for any $\varepsilon>0$ there are only finitely many algebraic numbers $\beta\in\boldsymbol{Q}_p$ of degree $\leqslant k$, such that the inequality*

$$|\alpha-\beta|_p < H(\beta)^{-k-1-\varepsilon} \tag{4}$$

is satisfied.

Proof. We shall apply Theorem 1.1 of [5]. Let P be the minimal polynomial of β. Without loss of generality we may suppose that β and α are not conjugates of each other. We then have

$$0\neq P(\alpha) = P(\beta)+(\alpha-\beta)P'(\beta)+(\alpha-\beta)^2\tfrac12 P''(\beta)+\ldots,$$

hence

$$0<|P(\alpha)|_p = |\alpha-\beta|_p|P'(\beta)+(\alpha-\beta)\tfrac12 P''(\beta)+\ldots|_p. \tag{5}$$

Now if $|\alpha-\beta|_p\leqslant 1$, then the second factor on the right-hand side of (5) is $\ll 1$, with a constant which depends only on α. If (4) were satisfied for infinitely many β, then by (5) the inequality

$$0<|P(\alpha)|_p < H(P)^{-k-1-\varepsilon}$$

would be true for infinitely many polynomials $P\in\boldsymbol{Z}[X]$ of degree $\leqslant k$ with relatively prime coefficients. This contradicts Theorem 1.1 of [5].

Remark. If we have a formula similar to (5) in the reals, then t
second factor on the right-hand side can in general only be estima
by $H(\beta)$. That is the reason why in (1) we have the slight difference
compared to the usual Koksma classification.

4. Construction of special algebraic numbers in Q_p. In this secti
we shall construct a sequence $\{\alpha_n\}_{n\in N}$ of algebraic numbers with for e
n, $\alpha_n \in Q_p$, $\deg \alpha_n = n$. Moreover, the sequence will have the prope
that the height $H(\alpha_n)$ is uniformly bounded, where the bound does n
depend on n.

LEMMA 1. (a) *Let p be an odd prime and d be the smallest prime
the arithmetic progression* $1, p+1, 2p+1, \ldots$

(i) *If $n \not\equiv 0 \pmod p$, the polynomial $X^n - d$ is irreducible over
and has a root in Q_p.*

(ii) *If $n \equiv 0 \pmod p$, the polynomial $X^n - dX^{n-1} + dX - d$ is ir
ducible over Q and has a root in Q_p.*

(b) *If $p = 2$, then*

(iii) *If n is odd, the polynomial $X^n - 3$ is irreducible over Q and h
a root in Q_2.*

(iv) *If n is even, the polynomial $X^n + X + 2$ is irreducible over
and has a root in Q_2.*

A proof may be found e.g. in [1] (Korollar, p. 59).

Lemma 1 provides as for any fixed p with a sequence of algebr
numbers α_n in Q_p with the desired property. We may even choose the
in such a way that

$$|\alpha_n|_p = 1 \quad \text{for all } n \in N. \tag{6}$$

5. Outline of the proof of Theorem 2. We shall proceed by an ind
tive method like that of W. M. Schmidt [6], and construct positive consta

$$L_1, L_2, \ldots \tag{7}$$

annuli

$$\begin{aligned} &A_{11} \supset \\ \supset\ &A_{21} \supset A_{22} \supset \\ \supset\ &A_{31} \supset A_{32} \supset A_{33}, \end{aligned} \tag{8}$$

and furthermore p-adic numbers

$$\begin{aligned} &\xi_{11} \\ &\xi_{21}, \xi_{22} \\ &\xi_{31}, \xi_{32}, \xi_{33} \\ &\ldots\ldots\ldots \end{aligned} \tag{9}$$

The numbers ξ_{ki} will be of the form

$$\xi_{ki} = \frac{a_{ki}}{b_{ki}} \alpha_i, \tag{10}$$

where α_i is an element of the sequence constructed in Section 4 and where a_{ki}, b_{ki} are integers with $(a_{ki}, b_{ki}) = 1$ and $p \nmid a_{ki}$, $p \nmid b_{ki}$. Hence by (6) we have $|\xi_{ki}|_p = 1$. The integers a_{ki}, b_{ki} will be such that

$$1 \leqslant a_{ki} \leqslant c b_{ki} \quad (1 \leqslant i \leqslant k,\ k = 1, 2, \ldots) \tag{11}$$

with a constant c, which depends only on p. Finally, the b_{ki} will satisfy the inequalities

$$1 < b_{11} < b_{21} < b_{22} < b_{31} < \ldots$$

For the annuli A_{ki} we shall have: each number $\eta \in A_{ki}$ satisfies

$$|\eta - \xi_{ki}|_p \leqslant b_{ki}^{-iB_i} \tag{12}$$

and

$$|\eta - \beta_j|_p \geqslant \frac{1}{L_j H(\beta_j)^{B_j}} \quad (1 \leqslant j \leqslant k) \tag{13}$$

for all algebraic numbers β_j of degree j with

$$H(\beta_j) \leqslant f_{ki}^{(j)}, \tag{14}$$

where

$$f_{ki}^{(j)} = \begin{cases} \dfrac{1}{8K_{ij}} b_{ki}^{(iB_i/3j)-j} & \text{for } j \leqslant i, \\ b_{ki}^{1/3j} & \text{for } j > i. \end{cases} \tag{15}$$

The K_{ij} occurring in (15) are constants, which will be defined later on more precisely.

Now our sequence of annuli (8) defines a unique point $\xi \in Q_p$, since by (12) the diameters tend to zero. On the other hand we have by Lemma 1 and by (10) and (11) the relation

$$H(\xi_{ki}) \gg \ll b_{ki}^i, \tag{16}$$

where the constant implied by $\ll$ depends only on p. So it follows from (12) and (16), that

$$\omega_i^*(\xi) \geqslant B_i \quad (i = 1, 2, \ldots),$$

whereas (13) implies

$$\omega_i^*(\xi) \leqslant B_i.$$

(For the last assertion notice that for fixed j the upper bound in (14) for the height of the algebraic numbers β_j, which are admitted in (13), tends to infinity.)

Concluding, we obtain the desired result $\omega_i^*(\xi) = B_i$ for $i = 1, 2, \ldots$

6. More properties of the construction. By Theorem 3, there are positive constants $K_{ij} \geqslant 1$ $(i, j = 1, 2, \ldots)$ such that

$$(17) \qquad |a_i - \beta_j|_p \geqslant \frac{1}{K_{ij} H(\beta_j)^{3j}}$$

for all algebraic numbers β_j of degree j with $\beta_j \neq a_i$. These are the constants which we had already in (15). Hence if a and b are relatively prime integers with $\left|\frac{a}{b}\right|_p = 1$ and $1 \leqslant a \leqslant c(p)b$ we have

$$(18) \qquad \left|\frac{a}{b} a_i - \beta_j\right|_p = \left|\frac{a}{b}\right|_p \left|a_i - \frac{b}{a}\beta_j\right|_p \geqslant \frac{1}{K_{ij} \cdot c_1(p) b^{3j^2} H(\beta_j)^{3j}}$$

for all algebraic numbers β_j of degree j with $\beta_j \neq \frac{a}{b} a_i$.

For $k > 1$ let $A_{\overline{ki}}, \xi_{\overline{ki}}, a_{\overline{ki}}, b_{\overline{ki}}$ be respectively the predecessor of $A_{ki}, \xi_{ki}, a_{ki}, b_{ki}$ in the sequences (8)–(10). Write furthermore $f_{\overline{ki}}^{(j)}$ for the predecessor of $f_{ki}^{(j)}$ in the sequence $f_{11}^{(j)}, f_{21}^{(j)}, f_{22}^{(j)}, \ldots$, where j is fixed and $k > 1$.

Given an annulus A with center ξ, defined by $a^{-1} \leqslant |\xi - \eta|_p \leqslant b^{-1}$ with certain naturals a, b, where $p^3 a^{-1} \leqslant b^{-1}$ we put

$$A' = \{\eta \mid a^{-1} \leqslant |\xi - \eta|_p \leqslant p^{-2}(a^{-1} + b^{-1})\}.$$

Now our sequences will have the following further properties:

$$(19) \qquad \xi_{ki} \in A'_{\overline{ki}} \ (k > 1), \qquad \xi_{ki} \notin A_{ki} \ (k \geqslant 1),$$

$$(20) \qquad f_{ki}^{(j)} \geqslant 2 \quad \text{if} \quad 1 \leqslant j \leqslant k,$$

$$(21) \qquad L_j b_{ki} > 8k(8 + 8pK_{ij})^{B_j} \quad \text{if} \quad 1 \leqslant j \leqslant k,$$

$$(22) \qquad (8K_{ij}/L_j)^{1/(B_j - 3j)} b_{ki}^{3j^2/(B_j - 3j)} < L_j^{-1/B_j} b_{ki}^{iB_i/B_j} \quad \text{if} \quad j \leqslant i, j \leqslant k,$$

$$(23) \qquad b_{ki} > 2\mu^{-1}(A_{\overline{ki}}) \quad \text{if} \quad k > 1,$$

where in formula (23) $\mu(A_{\overline{ki}})$ denotes the Haar measure on $\boldsymbol{Q}_p$ as defined e.g. in [7] (p. 66ff.)

Moreover we shall have

$$(24) \qquad |\xi_{ki} - \beta_j|_p \geqslant 1/(L_j H(\beta_j)^{B_j}) \quad (1 \leqslant j \leqslant k)$$

for all numbers β_j of degree j with $\beta_j \neq \xi_{ki}$ and even

$$(25) \qquad |\xi_{ki} - \beta_j|_p \geqslant 2/(L_j H(\beta_j)^{B_j}) \quad (1 \leqslant j \leqslant k)$$

if one of the conditions

$$i = 1, j = k \text{ or } i > 1 \text{ or } j < k$$

is satisfied together with

$$(26) \qquad H(\beta_j) > f_{ki}^{(j)}.$$

On the other hand for numbers β_j of degree j with

$$(27) \qquad H(\beta_j) \leqslant b_{ki}^{1/3j} \quad (1 \leqslant j \leqslant k, \ k > 1)$$

the inequality

$$(28) \qquad |\xi_{ki} - \beta_j|_p \geqslant 1/b_{ki}$$

will hold.

7. Construction of L_i and A_{ki}. We shall construct the sequence (7), (8), (9) in the order

$$(29) \qquad \begin{array}{l} \xi_{11}, L_1, A_{11}, \\ \xi_{21}, L_2, A_{21}, \xi_{22}, A_{22}, \\ \ldots\ldots\ldots\ldots\ldots \end{array}$$

We first choose a_{11}, b_{11} with $(a_{11}, b_{11}) = 1$ and $p \nmid a_{11}, p \nmid b_{11}$, where b_{11} has to be large enough such that (20) is satisfied for $k = i = j = 1$.

As for the construction of L_i we proceed in precisely the same way as W. M. Schmidt [6], p. 18. The same is true for the construction of A_{ki}. Therefore we shall contend ourselves by indicating the defining inequalities for A_{ki}: If $\xi_{ki} = \frac{a_{ki}}{b_{ki}} a_i$, then we put

$$(30) \qquad A_{ki} = \{\eta \mid p^{-3} b_{ki}^{-iB_i} \leqslant |\xi_{ki} - \eta|_p \leqslant b_{ki}^{-iB_i}\}.$$

Then all the computations of Section 9 of [6] apply *mutatis mutandis* also in our case, and therefore we may omit them here. Notice that in [6] the relation $1/4 < a_{ki}/b_{ki} < 1$ was necessary to obtain in the inequality (60) of [6] a lower bound, which depends on b_{ki} but not on a_{ki}. The corresponding inequality in our case is (18), and here all is fine, since we have $\left|\frac{a_{ki}}{b_{ki}}\right|_p = 1$ and $1 \leqslant a_{ki} \leqslant c(p) b_{ki}$.

8. Construction of ξ_{ki}. Assume that $k > 1$ and that all elements of the sequence (29) up to but excluding ξ_{ki} have already been constructed.

For more convenience we shall write

$$\xi_{ki} = \frac{a_{ki}}{b_{ki}} a_i = \frac{a}{b} a_i.$$

We have to ensure that we may find integers a and b such that all the properties of Section 6 hold. If $b > b_{\overline{ki}}$ is sufficiently large, then (20) and (23) hold as well as (21) and (22), if numbers j are concerned, for which L_j is already constructed. To guarantee the condition $\xi_{ki} \in A'_{\overline{ki}}$, i.e. (19), we will choose later on a in a suitable way. Remind, however, that this is a first condition for the integer a.

Now write $A_{\overline{ki}} = A_{k'i'}$. Then if $i > 1$ we have $k' = k$ and $i' = i-1$, whereas for $i = 1$ we have $k' = k-1$ and $i' = k-1$. Since $\xi_{ki} \in A_{\overline{ki}}$ we infer from (13) that (24) holds if $H(\beta_j) \leqslant f_{\overline{ki}}^{(j)}$ and if $1 \leqslant j \leqslant k'$. But $1 \leqslant j \leqslant k$ and $j > k'$ is only possible if $i = 1$ and $j = k$. In this case the truth of (24) is guaranteed by the construction of L_j, since here L_j is constructed *after* $\xi_{ki} = \xi_{k1}$ and even in such a way that the stronger condition (25) is true.

Hence we have only to worry about the question whether (25) is true under the conditions $H(\beta_i) > f_{\overline{ki}}^{(j)}$ with either $i > 1$ or $j < k$. In both cases we have in particular $j \leqslant k'$.

Our aim is to find sufficient conditions for a and b, such that (25) holds. From (18) we infer that

$$\left| a_i \frac{a}{b} - \beta_j \right| \geqslant \left(c_1(p) K_{ij} b^{3j^2} H(\beta_j)\right)^{-1} \geqslant 2/L_j H(\beta_j)^{B_j},$$

and therefore the truth of (25), if $\beta_j \neq a_i \frac{a}{b} = \xi_{ki}$ and

$$2c_1(p) K_{ij} b^{3j^2} H(\beta_j)^{3j} \leqslant L_j H(\beta_j)^{B_j},$$

which means that

$$H(\beta_j) \geqslant \left(2c_1(p) K_{ij}/L_j\right)^{1/(B_j - 3j)} b^{3j^2/(B_j-3j)}.$$

Collecting the results obtained so far, to ensure (25) we have only to study those numbers β_j whose height satisfies

$$f_{\overline{ki}}^{(j)} \leqslant H(\beta_j) \leqslant \left(2c_1(p) K_{ij}/L_j\right)^{1/(B_j-3j)} b^{3j^2/(B_j-3j)} \tag{31}$$

and whose degree satisfies $\deg \beta_j = j \leqslant k'$.

For each number β_j with (31) we define the disk

$$D(\beta_j) = \left\{\eta \in Q_p \,\middle|\, |\eta - \beta_j|_p \leqslant 2/L_j H(\beta_j)^{B_j}\right\}.$$

Write

$$D_j = \bigcup_{\beta_j \text{ satisfies (31)}} D(\beta_j)$$

and

$$D = \bigcup_{1 \leqslant j \leqslant k'} D_j.$$

So far we did not yet take into consideration condition (28), which takes care of numbers β_j with a comparatively small height. Associate with each number β_j of degree j and with $H(\beta_j) \leqslant b^{1/3j}$ the disk

$$E(\beta_j) = \left\{\eta \in Q_p \,\middle|\, |\eta - \beta_j|_p < 1/b\right\},$$

and put

$$E_j = \bigcup_{\beta_j \text{ satisfies (27)}} E(\beta_j),$$

$$E = \bigcup_{1 \leqslant j \leqslant k} E_j,$$

and finally

$$F = D \cup E.$$

9. Estimate of the measure of F. Our claim is that the complement of F in $A'_{\overline{ki}}$ has a measure of an order of size of the measure of $A'_{\overline{ki}}$ if b is large enough.

For j given the sum of the measures of the disks $D(\beta_j)$ is at most

$$\frac{2p}{L_j} \sum_{\beta_j \text{ satisfies (31)}} H(\beta_j)^{-B_j} \leqslant \frac{2p}{L_j} \sum_{H \geqslant f_{\overline{ki}}^{(j)}} H^{-B_j} j (2H+1)^j.$$

Hence we obtain

$$\mu(D_j) \leqslant \frac{2p \cdot j \cdot 3^j}{L_j} \left(f_{\overline{ki}}^{(j)} - 1\right)^{-(B_j - j - 1)} \leqslant \frac{p 6^{B_j}}{L_j} \left(f_{\overline{ki}}^{(j)}\right)^{-(B_j - j - 1)}.$$

Using (2), (15) and (21) we get after some computations

$$\mu(D_j) \leqslant \frac{1}{2k'} \mu(A'_{\overline{ki}}),$$

where we have used moreover $\mu(A'_{\overline{ki}}) \geqslant b_{\overline{ki}}^{-i'B_{i'}}$. So finally we have

$$\mu(D) \leqslant \tfrac{1}{2} \mu(A'_{\overline{ki}}). \tag{32}$$

The measure of the set E is quite easily estimated: For given j there are $\ll b^{2/3}$ numbers β_j with (27). Therefore

$$\mu(E_j) \ll b^{-1/3}$$

with constants, which depend only on p. But then

$$\mu(E) \ll k b^{-1/3}$$

and so if b is large enough, we have

$$\mu(E) \leqslant \tfrac{1}{4}\mu(A'_{\overline{ki}}). \tag{33}$$

Combining (32) and (33), we finally get

$$\mu(F) \leqslant \tfrac{3}{4}\mu(A'_{\overline{ki}}) \tag{34}$$

if b is large enough. Put $G = A'_{\overline{ki}} \setminus F$. Then by (34) and since $\mu(A'_{\overline{ki}}) \geqslant b_{\overline{ki}}^{-i'B_{i'}}$ we obtain

$$\mu(G) \geqslant b_{\overline{ki}}^{-i'B_{i'}}. \tag{35}$$

10. Investigation of the set G. We first show that if we choose b large enough, then there is a point $\gamma \in G$ such that the whole disk $\{\eta \mid |\eta-\gamma|_p \leqslant b^{-1}\}$ lies in G. By (35) we can find c points $\delta_1, \ldots, \delta_c$, where $c \geqslant b b_{\overline{ki}}^{-i'B_{i'}}$ such that $\delta_i \in G$ for $i = 1, \ldots, c$ and $|\delta_i - \delta_j|_p > b^{-1}$ for $i \neq j$ $(1 \leqslant i, j \leqslant b)$. We now consider the disks

$$C_i = \{\eta \mid |\delta_i - \eta|_p \leqslant b^{-1}\} \qquad (1 \leqslant i \leqslant c).$$

By our construction the disks C_i are mutually disjoint. We want to show that there is an i with $1 \leqslant i \leqslant c$, such that

$$C_i \cap D(\beta_j) = \emptyset \quad \text{and} \quad C_i \cap E(\beta_j) = \emptyset \tag{36}$$

for all disks $D(\beta_j)$ and $E(\beta_j)$ which were considered in Section 8.

Assume that for each i $(1 \leqslant i \leqslant c)$ there is a β_j such that $C_i \cap D(\beta_j) \neq \emptyset$ or $C_i \cap E(\beta_j) \neq \emptyset$. Then we have $C_i \subset D(\beta_j)$ or $D(\beta_j) \subset C_i$ or $C_i \subset E(\beta_j)$ or $E(\beta_j) \subset C_i$. The cases $C_i \subset D(\beta_j)$ or $C_i \subset E(\beta_j)$ are excluded, since the center δ_i of C_i lies in G. So we may assume that $D(\beta_j) \subset C_i$ or $E(\beta_j) \subset C_i$. By (31) for given j the number of disks $D(\beta_j)$ is $\ll b^{3j^2(j+1)/(B_j-3j)}$, hence by (2) the total number of disks $D(\beta_j)$ with $1 \leqslant j \leqslant k'$ is $\ll b^{6/7}$. Similarly for given j the number of disks $E(\beta_j)$ is $\ll b^{2/3}$ (recall that in this case we have $H(\beta_j) \leqslant b^{1/3j}$), and therefore the total number of disks $E(\beta_j)$ with $1 \leqslant j \leqslant k$ is $\ll b^{2/3}$.

On the other hand we have $\geqslant b b_{\overline{ki}}^{-i'B_{i'}}$ disks C_i, and these are mutually disjoint. If we choose b large enough, then $b b_{\overline{ki}}^{-i'B_{i'}} > b^{13/14}$. But then there is at least one disk, say C_1, for which (36) is satisfied.

We clearly have $C_1 \subset G$, and so in particular $C_1 \subset A'_{\overline{ki}}$. We now choose b as being a large prime, different from p, and determine the integer a according to the inequality

$$\left| b\frac{\delta_1}{\alpha_i} - a \right|_p < \frac{1}{b}. \tag{37}$$

This is possible since by (6) we have $|\alpha_n|_p = 1$ for all n and hence also $|\delta_1|_p = 1$. Since moreover by our choice of b we have $|b|_p = 1$, it follows

that any integer a satisfying (37) has $|a|_p = 1$. If we write

$$b = \sum_{\nu=0}^{n(b)} b_\nu p^\nu$$

with integers b_ν having $0 \leqslant b_\nu \leqslant p-1$ and $b_{n(b)} \neq 0$, then (37) implies that we may choose a in such a way that

$$a \leqslant (p-1)\sum_{\nu=0}^{n(b)+1} p^\nu \ll p^{n(b)+2};$$

so we may also fulfill the condition

$$1 \leqslant a \leqslant c(p)\cdot b. \tag{38}$$

Now the number $\frac{a}{b}\alpha_i$ satisfies in view of (37)

$$\left|\delta_1 - \frac{a}{b}\alpha_i\right|_p < \frac{1}{b},$$

so $\frac{a}{b}\alpha_i \in C_1$ and therefore $\frac{a}{b}\alpha_i \in A'_{\overline{ki}}$. Moreover, our construction implies that $\frac{a}{b}\alpha_i$ also satisfies (24), (25) and (28). We still have to guarantee the condition $(a, b) = 1$. If this is not true, then we take instead of a the integer $a' = a + p^{n(b)+2}$. a' certainly satisfies again (38), however if $(a, b) \neq 1$ then since b was chosen as a prime we have $(a', b) = 1$.

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