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On Linnik's constant

by

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Let q be a large positive integer, $(a, q) = 1$, and $p(q, a)$ the least prime $p \equiv a \pmod{q}$. The celebrated theorem of Linnik ([12], [13]) states that there exists an absolute constant C such that $p(q, a) < q^C$ for q sufficiently large. The first to obtain an explicit value for C was Pan [16], who proved that $C \leqslant 5448$. This was subsequently improved to 770 ([2]), 550 ([10]), 168 ([3]), 80 ([11]), and 36 ([5]). In this paper, we show that one may take $C = 20$.

Theorem 1. *If q is sufficiently large and $(a, q) = 1$, then there is a prime $p \equiv a \pmod{q}$ such that $p < q^{20}$.*

Our proof depends on several results concerning zeros of L-functions. Let $\varrho = \beta + i\gamma$ denote a generic zero of $L(s, \chi)$, where χ is a character mod q. Miech [14] has shown that $\prod_{\chi \bmod q} L(s, \chi)$ has at most one zero in the region

$$1 - \frac{.05}{\log q(|\gamma|+2)} \leqslant \beta < 1. \tag{1}$$

Schoenfeld has informed me that the constant .05 may be replaced by .10367. However, the following two theorems are superior for our purposes.

Theorem 2. *For $\nu = 1, 2$, let*

$$\varrho_\nu = 1 - \frac{\xi_\nu}{\log qT} + i\gamma_\nu$$

be a zero of $L(s, \chi_\nu)$, where χ_ν is a character mod q, *$|\gamma_\nu| \leqslant T$, and $T \geqslant 1$. Suppose that if $\chi_1 = \chi_2$ then $\varrho_1 \neq \varrho_2$, or if $\chi_1 = \overline{\chi}_2$ then $\varrho_1 \neq \overline{\varrho}_2$. If q is sufficiently large, then*

$$\xi_2 \geqslant .752 - \left(\frac{\sqrt{\xi_1^2 + 8\xi_1} - \xi_1}{2}\right) \tag{2}$$

and

$$\max(\xi_1, \xi_2) \geqslant 6/29. \tag{3}$$

THEOREM 3. *For q sufficiently large and $T \geqslant 1$, the product $\prod_{\chi \bmod q} L(s, \chi)$ has at most four zeros satisfying*

$$1-\frac{18}{65\log qT} < \beta < 1, \quad |\gamma| \leqslant T. \tag{4}$$

In (3), 6/29 may be replaced by

$$\frac{10(3-2\sqrt{2})}{3(5-\sqrt{5})}-\varepsilon$$

for any $\varepsilon > 0$, provided $q > q_0(\varepsilon)$. Under the same circumstances, the constant 18/65 in (4) may be replaced by

$$\frac{20(2-\sqrt{3})}{7(5-\sqrt{5})}-\varepsilon.$$

Weaker forms of (3) and (4) were proved in the author's thesis. The improvement comes from using a device of Stechkin [18].

We also require a form of Linnik's density theorem. Let $N(\chi, \alpha, T)$ be the number of zeros of $L(s, \chi)$ in the rectangle

$$\alpha \leqslant \beta < 1, \quad |\gamma| \leqslant T.$$

Selberg [17] introduced the device of "pseudo-characters" to prove that

$$\sum_{\chi \bmod q} N(\chi, \alpha, T) \ll_{\varepsilon} (qT)^{(3+\varepsilon)(1-\alpha)} \tag{5}$$

for every $\varepsilon > 0$. The important feature of (5) is that it is sharp for α close to 1. Selberg's proof was further refined by Motohashi [15] and Jutila [11].

To prove Linnik's theorem, it suffices to bound $N(\lambda)$, where $N(\lambda)$ denotes the number of characters $\bmod q$ such that $L(s, \chi)$ has a zero in the region

$$1-\frac{\lambda}{\log q} \leqslant \beta < 1, \quad |\gamma| \leqslant \log q. \tag{6}$$

Jutila, for his proof of $C = 80$ ([11]), proved that if $\varepsilon > 0$, $a > a_0(\varepsilon)$, $b > b_0(\varepsilon)$, $q > q_0(\varepsilon, a, b)$, $\lambda < \log\log\log q$, then

$$N(\lambda) < (1+\varepsilon)\frac{\pi^2(1/2+a+b)^2}{6ab}e^{(2+6a+2b)\lambda}.$$

If we set $c = 6a+2b$ and choose a to minimize the right-hand side we obtain

$$N(\lambda) < (1+\varepsilon)\frac{\pi^2}{6}\left(4+\frac{16}{c}+\frac{12}{c^2}\right)e^{(2+c)\lambda}$$

for $c > c_0(\varepsilon)$. In Section 6, we prove the following sharper result.

THEOREM 4. *If $0 < \varepsilon \leqslant 1$, $c \geqslant 48\varepsilon$, $\lambda < \log\log\log q$, and $q > q_0(\varepsilon, c)$, then*

$$N(\lambda) < (1+\varepsilon)\left(4+\frac{12}{c}+\frac{27}{4c^2}\right)e^{(3/2+c)\lambda}.$$

Jutila's proof uses Halász's method, pseudo-characters and an asymptotic formula due to the author [6]. We follow the general outline of Jutila's proof, but we have made a number of minor modifications. We use the coefficients θ_d (introduced in Section 5) instead of pseudo-characters; this is technically simpler and allows us to save a factor of $\pi^2/6$. We also use Burgess' estimates for L-functions; this allows us to replace $2+c$ by $3/2+c$ in the exponent.

We also require a quantitative form of the Deuring–Heilbronn phenomenon. Jutila, in his proof of $C = 80$ ([11]), proved a very strong form of this. His result is sufficiently strong for our purposes; we quote it as Lemma 4.

Finally, we remark that we will use Selberg's result (5) in an auxiliary capacity.

2. Notation. The letter q denotes a positive integer, which will henceforth be assumed large enough for the purpose at hand. $\mathcal{L}$ denotes $\log q$, χ is a Dirichlet character $\bmod q$, and χ_0 is the principal character $\bmod q$. As noted before, $\varrho = \beta+i\gamma$ is a generic zero of $L(s, \chi)$. We use $s = \sigma+it$ to denote a complex variable, and we let $\tau = |t|+2$. The letter $\varkappa$ denotes the constant $(5-\sqrt{5})/10$. The letter ε denotes a positive constant less than 1, and c is a positive constant such that $c \geqslant 48\varepsilon$. The constants implied by "$\ll$" and "O" symbols depend at most on c and ε. In Section 7, we choose specific values of c and ε, so the implied constants there are absolute.

3. Lemmata for Theorems 1, 2, and 3. The first two lemmata will be used in the proofs of Theorems 2 and 3. Lemmata 3 and 4 will be used in the proof of Theorem 1.

LEMMA 1. *Suppose that χ is non-principal, and let $\varrho_1, \varrho_2, \ldots, \varrho_k$ be zeros of $L(s, \chi)$ such that $\operatorname{Re}\varrho_i \geqslant 1/2$. If $1 < \sigma \leqslant 2$, then*

$$-\operatorname{Re}\frac{L'}{L}(s, \chi) \leqslant \varkappa\log q\tau - \sum_{i=1}^{k}\operatorname{Re}\frac{1}{s-\varrho_i} + O(k+\log\log q), \tag{7}$$

where $\varkappa = (5-\sqrt{5})/10$ and $\tau = |t|+2$.

If $\sigma > 1$ and χ is principal, then

$$-\operatorname{Re}\frac{L'}{L}(s, \chi) \leqslant \operatorname{Re}\frac{1}{s-1} + O\left(\frac{\log q\tau}{\log\log q}\right). \tag{8}$$

Proof. The inequality (8) is due to Miech [14], Lemma 5.

To prove (7), we first suppose that χ is primitive. By a classical partial fraction formula ([4], p. 85),

$$\frac{L'}{L}(s,\chi) = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + B(\chi) + \sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right),$$

where the summation is over all zeros of $L(s,\chi)$ with $\beta > 0$,

$$\operatorname{Re}B(\chi) = -\sum_{\varrho}\operatorname{Re}\frac{1}{\varrho},$$

and $a = 1/2(1-\chi(-1))$. Since

$$\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) = \log\tau + O_x(1)$$

for $1 < \sigma \leqslant x$ it follows that

$$-\operatorname{Re}\frac{L'}{L}(s,\chi) = \frac{1}{2}\log q\tau - \sum_{\varrho}\operatorname{Re}\frac{1}{s-\varrho} + O_x(1).$$

If ϱ is a zero of $L(s,\chi)$, then $1-\bar{\varrho}$ is also a zero. Therefore

$$\sum_{\varrho}\operatorname{Re}\frac{1}{s-\varrho} = \sum_{\beta\geqslant 1/2}{}'\operatorname{Re}\left(\frac{1}{s-\varrho}+\frac{1}{s-1+\bar{\varrho}}\right),$$

where the dash indicates that those terms with $\beta = 1/2$ are counted with weight 1/2. Let $\sigma_1 = \sqrt{1+4\sigma^2}$ and $s_1 = \sigma_1 + it$. Define, for complex s and z,

$$F(s,z) = \operatorname{Re}\{(s-z)^{-1} + (s-1+\bar{z})^{-1}\}.$$

If $1 < \sigma \leqslant 2$, then

$$-\operatorname{Re}\frac{L'}{L}(s,\chi) + \frac{1}{\sqrt{5}}\operatorname{Re}\frac{L'}{L}(s_1,\chi)$$
$$= \varkappa\log q\tau - \sum_{\beta\geqslant 1/2}{}'\{F(s,\varrho) - \sqrt{1/5}\,F(s_1,\varrho)\} + O(1).$$

By a result of Stechkin ([18], Lemma 2)

$$F(s,z) - \sqrt{1/5}\,F(s_1,z) \geqslant 0$$

whenever $\sigma > 1$ and $1/2 \leqslant \operatorname{Re}z \leqslant 1$. Since $\sigma_1 \geqslant \sqrt{5}$, it follows that

$$\frac{L'}{L}(s_1,\chi) \ll 1$$

and

$$F(s,\varrho_i) = \operatorname{Re}(s-\varrho_i)^{-1} + O(1).$$

This proves (7) for primitive χ.

If χ is induced by a primitive character χ_1 mod q_1 and if $\sigma > 1$, then

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\chi_1) + \sum_{p|q}\frac{\chi(p)\log p}{p^s-1} = \frac{L'}{L}(s,\chi_1) + O\left(\sum_{p|q}\frac{\log p}{p}\right).$$

Since we have already proved (7) for primitive χ and since

$$\sum_{p|q}\frac{\log p}{p} \ll \log\log q,$$

this completes the proof.

Lemma 2. *Suppose a, b, d are positive real numbers such that $bd \leqslant a^2$. Then for all real x,*

$$\frac{a}{a^2+x^2} - \frac{a+b}{(a+b)^2+x^2} - \frac{a+d}{(a+d)^2+x^2} \leqslant 0. \tag{9}$$

Proof. If we multiply the left-hand side of (9) by

$$(a^2+x^2)\big((a+b)^2+x^2\big)\big((a+d)^2+x^2\big),$$

it becomes

$$-(a+b+d)x^4 -$$
$$-\{(a+b)(a^2+ad+bd)+(a+d)(a^2+ad+bd)\}x^2 +$$
$$+(bd-a^2)a(a+b)(a+d),$$

and this is clearly $\leqslant 0$ for all x.

Lemma 3. *Let χ be non-principal, and let $\mathscr{R}(\chi)$ be the set of zeros of $L(s,\chi)$ satisfying*

$$1 - \frac{\log\log\mathscr{L}}{\mathscr{L}} < \beta < 1, \quad |\gamma| \leqslant \mathscr{L}. \tag{10}$$

Then

$$\sum_{\varrho\in\mathscr{R}(\chi)}\min\left(1, \frac{4}{|\varrho-1|^2\mathscr{L}^2}\right) \leqslant (1+\sqrt{2\varkappa})^2 + O\big((\log\mathscr{L})^4\mathscr{L}^{-1}\big).$$

Proof. Let $\sigma = 1 + h\mathscr{L}^{-1}$, where $h > 0$ will be chosen later. If $h \geqslant 2$, then

$$\min\left(1, \frac{4}{|\varrho-1|^2\mathscr{L}^2}\right) \leqslant \frac{h+2}{\mathscr{L}}\operatorname{Re}\frac{1}{\sigma-\varrho}.$$

By Lemma 3,

$$\sum_{\varrho} \operatorname{Re}\frac{1}{\sigma-\varrho} \leqslant \varkappa\mathscr{L} + \left|\frac{L'}{L}(\sigma,\chi)\right| + O\big(\log\mathscr{L} + |\mathscr{R}(\chi)|\big).$$

By (5), $|\mathscr{R}(\chi)| \ll (\log\mathscr{L})^4$. Furthermore,

$$\left|\frac{L'}{L}(\sigma,\chi)\right| \leqslant \sum_{n=1}^{\infty} \Lambda(n)n^{-\sigma} = \frac{1}{\sigma-1} + O(1) = \frac{\mathscr{L}}{h} + O(1).$$

Thus

$$\sum_{\varrho} \min\left(1, \frac{4}{|\varrho-1|^2\mathscr{L}^2}\right) \leqslant (h+2)(\varkappa+1/h) + O\big((\log\mathscr{L})^4\mathscr{L}^{-1}\big).$$

To minimize the right-hand side, we take $h = \sqrt{2/\varkappa}$; this completes the proof.

The next lemma is the Deuring–Heilbronn phenomenon, and it is a direct consequence of a theorem of Jutila ([11], Theorem 2) and Siegel's theorem ([4], Chap. 21).

Lemma 4. *Suppose that χ and χ_1 are (not necessarily distinct) characters* mod q, *and that χ_1 is real and non-principal. Suppose that $\beta_1 = 1-\xi_1\mathscr{L}^{-1}$ is a real zero of $L(s,\chi_1)$ such that $\xi_1 < .05$, and that $\varrho = 1-\xi\mathscr{L}^{-1}+i\gamma$ is a zero of $L(s,\chi)$ such that $|\gamma| \leqslant \mathscr{L}$. If $\varepsilon > 0$ and $q > q_0(\varepsilon)$, then*

(11) $$\xi \geqslant (1/2-\varepsilon)\log\left(\frac{1}{8\xi_1}\right).$$

We remark that Jutila does not appeal to Siegel's theorem, but he obtains a somewhat more complicated inequality for ξ. Furthermore, the inequality (11) can be slightly strengthened. The author [5] has shown that

$$\xi \geqslant (2/3-\varepsilon)\log\left(\frac{2-\varepsilon}{3\xi_1}\right)$$

but we will not need this result here.

4. Proofs of Theorems 2 and 3. We first prove (2). Since $\zeta(s)$ has no zeros in the region

$$\beta \geqslant 1-\frac{A}{(\log T\log\log T)^{3/4}}, \quad |\gamma| \leqslant T$$

([19], eq. 6.15.1] we may assume that χ_1 and χ_2 are non-principal. Let

$\sigma = 1+a(\log qT)^{-1}$, where a will be chosen later. Then

$$0 \leqslant 2\sum_{n=1}^{\infty}\{1+\operatorname{Re}\chi_1(n)n^{-i\gamma_1}\}\{1+\operatorname{Re}\chi_2(n)n^{-i\gamma_2}\}\Lambda(n)n^{-\sigma}$$

$$= -\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(\sigma)+\frac{L'}{L}(\sigma+i\gamma_1,\chi_1)+\frac{L'}{L}(\sigma+i\gamma_2,\chi_2)+\right.$$
$$\left.+\frac{L'}{L}(\sigma+i\gamma_1+i\gamma_2,\chi_1\chi_2)\right\} -$$
$$-\operatorname{Re}\left\{\frac{\zeta'}{\zeta}(\sigma)+\frac{L'}{L}(\sigma+i\gamma_1,\chi_1)+\frac{L'}{L}(\sigma-i\gamma_2,\overline{\chi}_2)+\frac{L'}{L}(\sigma+i\gamma_1-i\gamma_2,\chi_1\overline{\chi}_2)\right\}$$
$$= S(\chi_1,\chi_2;\varrho_1,\varrho_2)+S(\chi_1,\overline{\chi}_2;\varrho_1,\varrho_2),$$

say. If $\chi_1\chi_2$ is not principal, then

(12) $$S(\chi_1,\chi_2;\varrho_1,\varrho_2) \leqslant 3\varkappa\log qT+\frac{1}{\sigma-1}-\frac{1}{\sigma-\beta_1}-\frac{1}{\sigma-\beta_2}+O(\log\mathscr{L})$$
$$= \log qT\left\{3\varkappa+\frac{1}{a}-\frac{1}{a+\xi_1}-\frac{1}{a+\xi_2}+O\big((\log\mathscr{L})^{-1}\big)\right\}$$

by Lemma 1. If $\chi_1\overline{\chi}_2$ is also non-principal, then

$$0 \leqslant 3\varkappa+\frac{1}{a}-\frac{1}{a+\xi_1}-\frac{1}{a+\xi_2}+O\big((\log\mathscr{L})^{-1}\big).$$

If we take

(13) $$a = 1/2(\sqrt{\xi_1^2+8\xi_1}-\xi_1)$$

then

$$\frac{1}{a}-\frac{1}{a+\xi_1} = \frac{1}{2}$$

and

$$\xi_2 > \frac{2}{6\varkappa+1}-a+O\big((\log\mathscr{L})^{-1}\big).$$

This proves (2), provided neither $\chi_1 = \chi_2$ nor $\chi_1 = \overline{\chi}_2$.

To prove (2) when $\chi_1 = \chi_2$ or $\chi_1 = \overline{\chi}_2$, it clearly suffices to prove (12) when $\chi_1 = \overline{\chi}_2$ and a is given by (13). Moreover, we may assume that

(14) $$\xi_2 \leqslant 2-a,$$

since the negation of (14) is stronger than (2).

If $\chi_1 = \bar{\chi}_2$, then $\bar{\varrho}_1$ is a zero of $L(s, \chi_2)$ and $\bar{\varrho}_2$ is a zero of $L(s, \chi_1)$. Let $x = (\gamma_1+\gamma_2)\log qT$. By Lemma 1,

$$S(\chi_1, \chi_2; \varrho_1, \varrho_2) \leqslant 2\varkappa\log qT + \frac{1}{\sigma-1} - \frac{1}{\sigma-\beta_1} - \frac{1}{\sigma-\beta_2} + $$
$$+\mathrm{Re}\,\frac{1}{(\sigma-1)+i(\gamma_1+\gamma_2)} - \mathrm{Re}\,\frac{1}{(\sigma-\beta_1)+i(\gamma_1+\gamma_2)} -$$
$$-\mathrm{Re}\,\frac{1}{(\sigma-\beta_2)+i(\gamma_1+\gamma_2)} + O(\log \mathscr{L})$$
$$= \log qT\left\{2\varkappa + \frac{1}{a} - \frac{1}{a+\xi_1} - \frac{1}{a+\xi_2} + O\big((\log\mathscr{L})^{-1}\big)\right\} +$$
$$+\log qT\left\{\frac{a}{a^2+x^2} - \frac{a+\xi_1}{(a+\xi_1)^2+x^2} - \frac{a+\xi_2}{(a+\xi_2)^2+x^2}\right\}.$$

We apply Lemma 2 to the expression in the second bracket. The hypothesis of Lemma 2 is satisfied by (14), so this expression is non-positive. This proves (12) when $\chi_1 = \bar{\chi}_2$, and thus completes the proof of (2).

To prove (3), we let $\xi = \max(\xi_1, \xi_2)$. By essentially the same argument as above, we see that

$$0 \leqslant 3\varkappa + \frac{1}{a} - \frac{2}{a+\xi} + O\big((\log\mathscr{L})^{-1}\big)$$

when

$$a = \frac{10(\sqrt{2}-1)}{3(5-\sqrt{5})}.$$

Consequently

$$\xi > \frac{2a}{3\varkappa a+1} - a + O\left(\frac{1}{\log\mathscr{L}}\right) > \frac{6}{29}.$$

We make one further remark, which we will use in the proof of Theorem 3. If $\chi_1 = \bar{\chi}_1 = \chi_2$, then we obtain

$$0 \leqslant 2\varkappa + \frac{1}{a} - \frac{2}{a+\xi} + O\left(\frac{1}{\log\mathscr{L}}\right)$$

for

$$a = \frac{5(\sqrt{2}-1)}{(5-\sqrt{5})}.$$

Therefore

$$\max(\xi_1, \xi_2) \geqslant \frac{5(\sqrt{2}-1)^2}{5-\sqrt{5}} + O\left(\frac{1}{\log\mathscr{L}}\right) > \frac{3}{10} \tag{15}$$

when $\chi_1 = \bar{\chi}_1 = \chi_2$.

We now prove Theorem 3. Suppose $\prod_{\chi \bmod q} L(s, \chi)$ has five zeros in the region

$$1 - \frac{\xi}{\log qT} < \beta < 1, \quad |\gamma| \leqslant T.$$

Choose three zeros in this region and denote them $\varrho_1, \varrho_2, \varrho_3$. Suppose that they are zeros of $L(s, \chi_1)$, $L(s, \chi_2)$, and $L(s, \chi_3)$ respectively. It is easy to see that we can choose these zeros in such a manner that if $\chi_j = \chi_k$ $(j \neq k)$, then $\varrho_j \neq \varrho_k$, and if $\chi_j = \bar{\chi}_k$ $(j \neq k)$ then $\varrho_j \neq \bar{\varrho}_k$. It suffices to show that $\xi \geqslant 18/65$. As in the proof of Theorem 2, we may assume that no χ_j is principal.

Let $\sigma = 1 + a(\log qT)^{-1}$, where a will be chosen later. Define

$$S(\chi_1, \chi_2, \chi_3; \varrho_1, \varrho_2, \varrho_3)$$
$$= -\mathrm{Re}\left\{\frac{\zeta'}{\zeta}(\sigma) + \frac{L'}{L}(\sigma+i\gamma_1, \chi_1) + \frac{L'}{L}(\sigma+i\gamma_2, \chi_2) + \right.$$
$$+\frac{L'}{L}(\sigma+i\gamma_3, \chi_3) + \frac{L'}{L}(\sigma+i\gamma_1+i\gamma_2, \chi_1\chi_2) +$$
$$+\frac{L'}{L}(\sigma+i\gamma_1+i\gamma_3, \chi_1\chi_3) + \frac{L'}{L}(\sigma+i\gamma_2+i\gamma_3, \chi_2\chi_3) +$$
$$\left.+\frac{L'}{L}(\sigma+i\gamma_1+i\gamma_2+i\gamma_3, \chi_1\chi_2\chi_3)\right\}.$$

Then

$$0 \leqslant 4\sum_{n=1}^{\infty} \Lambda(n)n^{-\sigma}\prod_{j=1}^{3}\big(1+\mathrm{Re}\,\chi_j(n)n^{-i\gamma_j}\big) \tag{16}$$
$$= S(\chi_1, \chi_2, \chi_3; \varrho_1, \varrho_2, \varrho_3) + S(\chi_1, \chi_2, \bar{\chi}_3; \varrho_1, \varrho_2, \bar{\varrho}_3) +$$
$$+ S(\chi_1, \bar{\chi}_2, \chi_3; \varrho_1, \bar{\varrho}_2, \varrho_3) + S(\chi_1, \bar{\chi}_2, \bar{\chi}_3; \varrho_1, \bar{\varrho}_2, \bar{\varrho}_3).$$

If any of the products $\chi_j\chi_k$ $(j \neq k)$ is principal, then $\chi_1\chi_2\chi_3$ is not principal. Thus the proof of Theorem 3 breaks down into five cases: (i) none of $\chi_1\chi_2$, $\chi_1\chi_3$, $\chi_2\chi_3$, $\chi_1\chi_2\chi_3$ principal, (ii) exactly one the products $\chi_j\chi_k$ principal, (iii) exactly two of the products $\chi_j\chi_k$ principal, (iv) all three of the products $\chi_j\chi_k$ principal, (v) $\chi_1\chi_2\chi_3$ principal.

In case (iv), we have $\chi_1\chi_2 = \chi_1\chi_3 = \chi_2\chi_3 = \chi_0$, and therefore $\chi_1 = \chi_2 = \chi_3 = \bar{\chi}_1$. By (15), $\xi \geqslant 3/10 > 18/65$.

In the other cases, it suffices to show that

$$S(\chi_1, \chi_2, \chi_3; \varrho_1, \varrho_2, \varrho_3) \leqslant \log qT\left\{7\varkappa + \frac{1}{a} - \frac{3}{a+\xi} + O\left(\frac{1}{\log\mathscr{L}}\right)\right\} \tag{17}$$

when

$$a = \frac{10}{7}\frac{(\sqrt{3}-1)}{5-\sqrt{5}} \quad \text{and} \quad \xi \leqslant \sqrt{a} = .6151\ldots$$

For (16) and (17) combine to give

$$0 \leqslant 7\varkappa + \frac{1}{a} - \frac{3}{a+\xi} + O\left(\frac{1}{\log\mathscr{L}}\right).$$

In case (i), (17) follows directly from Lemma 1. In the other cases, (17) follows from a routine applications of Lemmata 1 and 2; we leave the details to the reader.

5. Definitions and lemmata for Theorem 4. Let ε and c be as in the statement of Theorem 4; i.e. $0 < \varepsilon \leqslant 1$, and $c \geqslant 48\varepsilon$. Let a be a real positive number satisfying

(18) $$4\varepsilon \leqslant a \leqslant (c-12\varepsilon)/6;$$

the precise value of a depends on c and will be chosen later. Let $b = 1/2(c-6a)$ and $\delta = \varepsilon^2/8$.

Define

$$R = q^{a-6\delta},$$
$$z_1 = q^{3/8+2a},$$
$$z_2 = q^{3/8+2a+b-6\delta},$$
$$y = q^{3/4+3a+b},$$
$$G = \sum_{\substack{r\leqslant R\\(r,q)=1}} \frac{\mu^2(r)}{\varphi(r)},$$
$$\lambda_d = \begin{cases} \mu(d) & \text{if} \quad d \leqslant z_1,\\ \mu(d)\dfrac{\log(z_2/d)}{\log(z_2/z_1)} & \text{if} \quad z_1 < d \leqslant z_2,\\ 0 & \text{if} \quad d > z_2.\end{cases}$$

If d is squarefree, $(d, q) = 1$ and $d \leqslant R$, we define

$$\theta_d = \frac{\mu(d)d}{G\varphi(d)} \sum_{\substack{r\leqslant R/d\\(r,dq)=1}} \frac{\mu^2(r)}{\varphi(r)}.$$

Otherwise, set $\theta_d = 0$. Finally, we define

$$B(s,\chi) = \sum_{z_1<n\leqslant y}\Big(\sum_{d|n}\theta_d\Big)^2\chi(n)n^{-s-1},$$
$$G(s,\chi) = \sum_{d,e}\theta_d\theta_e\chi([d,e])[d,e]^{-s},$$
$$M(s,\chi) = \sum_{d,e}\theta_d\lambda_e\chi([d,e])[d,e]^{-s}.$$

LEMMA 5. *If* $1/2 \leqslant \sigma \leqslant 1+2(\log q\tau)^{-1}$, *then*

$$L(s,\chi) \ll (q^{3/16+\delta}\tau^2)^{1-\sigma}\log q\tau,$$
$$G(s,\chi) \ll R^{2(1-\sigma)}\mathscr{L}^3,$$
$$M(s,\chi) \ll (Rz_2)^{1-\sigma}\mathscr{L}^3.$$

Proof. The first inequality is due to Burgess [1], except that his result was not uniform in t. An examination of his proof, however, yields the above result. The implied constant depends on δ, and thus on ε.

The second and third inequalities follow directly from $|\theta_d| \leqslant 1$ ([7], equation (3.1.8)), $|\lambda_d| \leqslant 1$, and partial summation.

LEMMA 6. *Suppose that χ is non-principal and that ϱ is a zero of $L(s,\chi)$ satisfying*

$$1 - \frac{\log\log\mathscr{L}}{\mathscr{L}} < \beta < 1, \qquad |\gamma| \leqslant \mathscr{L}.$$

Then

$$\Big|\sum_{z_1<n\leqslant y}\Big(\sum_{d|n}\lambda_d\Big)\Big(\sum_{e|n}\theta_e\Big)\chi(n)n^{-\varrho}\Big| = 1 + O(\mathscr{L}^{-1}).$$

Proof. Let $T = q^{2\delta}$ and $k = 1-\beta+\mathscr{L}^{-1}$. By the truncated Perron formula ([19], Lemma 3.12)

(19) $$\sum_{n\leqslant y}\Big(\sum_{d|n}\lambda_d\Big)\Big(\sum_{e|n}\theta_e\Big)\chi(n)n^{-\varrho} = \frac{1}{2\pi i}\int_{k-iT}^{k+iT} L(s+\varrho,\chi)\,M(s+\varrho,\chi)\frac{y^s}{s}\,ds + O(\mathscr{L}^{-1}).$$

We move the line of integration to $\operatorname{Re} s = 1/2-\beta$. There are no poles of the integrand. By Lemma 5, the horizontal integrals contribute

$$\ll \frac{q^\delta}{T}\int_{1/2}^{1+\mathscr{L}^{-1}}\left(\frac{q^{3/16+\delta}T^2Rz_2}{y}\right)^{1-\sigma} d\sigma \ll \mathscr{L}^{-1}$$

and the vertical integral is

$$\ll q^{\delta}\int_{-T}^{T}\left(\frac{q^{3/8+\delta}T^2Rz_2}{y}\right)^{1/2}dt \ll \mathscr{L}^{-1}.$$

Therefore the right-hand side of (19) is $\ll \mathscr{L}^{-1}$. Since

$$\sum_{d|n}\lambda_d = \begin{cases}1 & \text{if} \quad n=1,\\ 0 & \text{if} \quad 1<n\leqslant z_1,\end{cases}$$

this completes the proof.

Lemma 7. *Suppose that*

$$0\leqslant\sigma\leqslant 2(\log\log\mathscr{L})\mathscr{L}^{-1}, \quad |t|\leqslant 2\mathscr{L}.$$

Then

$$B(s,\chi)=\frac{E(\chi)}{G}\frac{\varphi(q)}{q}\int_{\log z_1}^{\log y}e^{-sx}dx+O(\mathscr{L}^{-1}),$$

where $E(\chi)=1$ if χ is principal and 0 *otherwise.*

Proof. Let $T=q^{2\delta}$, $k=-\sigma+\mathscr{L}^{-1}$. By the truncated Perron formula ([19], Lemma 3.12)

$$B(s,\chi)=\frac{1}{2\pi i}\int_{k-iT}^{k+iT}L(s+1+w,\chi)G(s+1+w,\chi)\left\{\frac{y^w-z_1^w}{w}\right\}dw+O(\mathscr{L}^{-1}).$$

We pull the contour to the line $\operatorname{Re} w=-1/2-\sigma$. There are no poles of the integrand if χ is non-principal. If χ is principal, there is a pole at $w=-s$ with residue

$$\frac{\varphi(q)}{q}G(1,\chi_0)\int_{\log z_1}^{\log y}e^{-sx}dx.$$

It is easily shown ([7], equation (3.1.7)) that

$$G(1,\chi_0)=\sum_{d,e}\frac{\theta_d\theta_e}{[d,e]}=\frac{1}{G}.$$

By Lemma 5, the horizontal integrals contribute

$$\ll\frac{q^{\delta}}{T}\int_{1/2}^{1+\mathscr{L}^{-1}}\left(\frac{R^2q^{3/8+\delta}T^2}{z_1}\right)^{1-\sigma}d\sigma\ll\mathscr{L}^{-1}$$

and the vertical integrals contribute

$$\ll q^{\delta}\int_{-T}^{T}\left(\frac{R^2q^{3/8+\delta}T^2}{z_1}\right)^{1/2}dt\ll\mathscr{L}^{-1}.$$

Lemma 8. *Let $\mathscr{S}$ be a set of non-principal characters* mod q. *For every $\chi\in\mathscr{S}$, let $s=s(\chi)$ be an arbitrary point satisfying*

$$0\leqslant\sigma\leqslant(\log\log\mathscr{L})\mathscr{L}^{-1}, \quad |t|\leqslant\mathscr{L}.$$

Let f_n $(z_1<n\leqslant y)$ be arbitrary complex numbers, and define

$$F(s,\chi)=\sum_{z_1<n\leqslant y}f_n\Big(\sum_{d|n}\theta_d\Big)\chi(n)n^{-s-1/2}.$$

Then

$$\sum_{\chi\in\mathscr{S}}|F(s,\chi)|^2\leqslant M\sum_{z_1<n\leqslant y}|f_n|^2,$$

where

$$M=\frac{\log(y/z_1)}{\log R}+O(|\mathscr{S}|\mathscr{L}^{-1}).$$

Proof. By a well known duality principle ([8], Theorem 288), it suffices to show that for arbitrary complex numbers $c(\chi)$,

$$\sum_{z_1<n\leqslant y}\Big|\sum_{\chi\in\mathscr{S}}c(\chi)\Big(\sum_{d|n}\theta_d\Big)\chi(n)n^{-s(\chi)-1/2}\Big|^2\leqslant M\sum_{\chi\in\mathscr{S}}|c(\chi)|^2.$$

The left-hand side of the above is

$$\sum_{\chi\in\mathscr{S}}\sum_{\chi'\in\mathscr{S}}c(\chi)\overline{c(\chi')}B\big(s(\chi)+\overline{s(\chi')},\chi\bar{\chi}'\big)$$

$$=\frac{\varphi(q)}{q}\frac{\log(y/z_1)}{G}\sum_{\chi\in\mathscr{S}}|c(\chi)|^2+O\Big(\mathscr{L}^{-1}\sum_{\chi,\chi'\in\mathscr{S}}|c(\chi)|\,|c(\chi')|\Big).$$

It is well known ([7], Lemma 3.1) that $G\geqslant\varphi(q)q^{-1}\log R$.

Furthermore, by Cauchy's inequality,

$$\sum_{\chi,\chi'\in\mathscr{S}}|c(\chi)|\,|c(\chi')|\leqslant\frac{1}{2}\sum_{\chi,\chi'\in\mathscr{S}}\big(|c(\chi)|^2+|c(\chi')|^2\big)=|\mathscr{S}|\sum_{\chi\in\mathscr{S}}|c(\chi)|^2.$$

This completes the proof of Lemma 8.

Lemma 9. *If $1/2<a<1$, then*

$$\sum_{z_1<n\leqslant y}\Big(\sum_{d|n}\lambda_d\Big)^2n^{1-2a}\leqslant\frac{\log(y/z_1)}{\log(z_2/z_1)}y^{2-2a}\{1+O(\mathscr{L}^{-1})\}. \tag{20}$$

Proof. First we quote the result, due to the author [5], that

$$\sum_{z_1<n\leqslant y}\Big(\sum_{d|n}\lambda_d\Big)^2\leqslant\frac{y}{\log(z_2/z_1)}\{1+O(\mathscr{L}^{-1})\}.$$

By partial summation, the left-hand side of (20) is

$$\leqslant\frac{y^{2-2a}-z_1^{2-2a}}{(2-2a)\log(z_2/z_1)}\{1+O(\mathscr{L}^{-1})\}.$$

Since

$$\frac{y^{2-2a}-z_1^{2-2a}}{2-2a}=\int\limits_{\log z_1}^{\log y}e^{(2-2a)x}dx\leqslant(\log y/z_1)y^{2-2a},$$

this completes the proof.

6. Proof of Theorem 4. For convenience, we write $a=1-\lambda\mathscr{L}^{-1}$. Let $\mathscr{S}$ be the set of characters $\chi \bmod q$ such that $L(s,\chi)$ has a zero in the rectangle

(21) $$a\leqslant\beta<1-\frac{18}{65\mathscr{L}},\qquad|\gamma|\leqslant\mathscr{L}.$$

We assume that $\lambda\leqslant\log\log\mathscr{L}$. As we noted in the proof of Theorem 2, all characters in $\mathscr{S}$ are non-principal. For $\chi\in\mathscr{S}$, let $\varrho(\chi)$ be a zero of $L(s,\chi)$ satisfying (21), and define $s=s(\chi)=\varrho(\chi)-a$. Let

$$f_n=\Big(\sum_{d|n}\lambda_d\Big)n^{1/2-a}.$$

By Lemmata 6, 8, and 9,

$$N(\lambda)\leqslant\frac{(3/8+a+b)^2}{(a-6\delta)(b-6\delta)}e^{(3/2+c)\lambda}\{1+O(\mathscr{L}^{-1})\}.$$

Now $a\geqslant 4\varepsilon$ and $\varepsilon\leqslant 1$, so

$$\Big(1-\frac{6\delta}{a}\Big)^{-1}\leqslant\Big(1-\frac{3\delta}{2\varepsilon}\Big)^{-1}\leqslant 1+\frac{3\varepsilon}{13}.$$

Similarly, $b=1/2(c-6a)\geqslant 6\varepsilon$, so

$$\Big(1-\frac{6\delta}{b}\Big)^{-1}\leqslant 1+\frac{\varepsilon}{7}.$$

Therefore

$$\frac{1+O(\mathscr{L}^{-1})}{(a-6\delta)(b-6\delta)}\leqslant\frac{1}{ab}\{1+O(\mathscr{L}^{-1})\}\Big(1+\frac{\varepsilon}{7}\Big)\Big(1+\frac{3\varepsilon}{13}\Big)<\frac{1+\varepsilon}{ab},$$

and

$$N(\lambda)\leqslant(1+\varepsilon)\frac{(3+4c-16a)^2}{32a(c-6a)}e^{(3/2+c)\lambda}.$$

To minimize the right-hand side, we take

$$a=\frac{c(3+4c)}{36+32c};$$

note that (18) follows from the inequality $c\geqslant 48\varepsilon$. This completes the proof.

7. Proof of Theorem 1. Let C be an arbitrary positive number, $B=C-2$, $x=q^{C/2}$, and

$$K(s)=\Big(\frac{x}{q}\Big)^s\Big(\frac{q^s-1}{\log q}\Big).$$

For positive integers n, we define

$$R(n)=\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}K^2(s)n^{-s}ds.$$

It is readily seen that $R(n)=0$ if $n>q^C$ or if $n\leqslant q^{C-2}$. A standard argument (cf. [10], Lemma 3) shows that if $C\geqslant 3$, then

$$\varphi(q)\sum_{\substack{p\leqslant q^C\\ p\equiv a \bmod q}}\frac{R(p)\log p}{p}=1-\sum_{\chi \bmod q}\bar{\chi}(a)\sum_{\varrho}K^2(\varrho-1)+O(q^{-1}),$$

where the inner sum is over all non-trivial zeros of $L(s,\chi)$.

To prove Theorem 1, it suffices to show that for $C\geqslant 20$,

$$\sum_{\chi \bmod q}\sum_{\varrho}|K^2(\varrho-1)|<1-\eta,$$

where $\eta>0$ is independent of q. We note that if $\sigma\leqslant 0$, then

(22) $$|K^2(s)|\leqslant q^{-B\sigma}\min\Big(1,\frac{4}{|s|^2\mathscr{L}^2}\Big).$$

By (1) and (5), we see that if $C\geqslant 7$, then

$$\sum_{\chi}\sum_{\varrho\notin\mathscr{R}(\chi)}|K^2(\varrho-1)|\ll(\log\mathscr{L})^{-1},$$

where $\mathscr{R}(\chi)$ was defined in Lemma 3. Thus to prove Theorem 1, it suffices to show that

(23) $$\sum_{\chi}\sum_{\varrho\in\mathscr{R}(\chi)}|K^2(\varrho-1)|<1-10^{-6}.$$

We take $\varepsilon = 10^{-5}$ and $c = 2$ in Theorem 4, so we obtain

$$N(\lambda) < (1+10^{-5})\frac{167}{18}e^{7/2\lambda}. \tag{24}$$

Let β_1 be the largest ordinate of the zeros of $\prod_{\chi \bmod q} L(s,\chi)$ for which $|\gamma| \leqslant \mathscr{L}$, and let $\xi_1 = (1-\beta_1)\mathscr{L}$. Suppose first that $\xi_1 \geqslant .142$. By Theorems 2 and 3, Lemma 3, (22) and (24), the left-hand side of (23) is

$$\leqslant 2e^{-B\xi_1} + 2e^{-6B/29} + (3.0399)\int_{18/65}^{\infty} e^{-B\lambda}dN(\lambda)$$

$$\leqslant 2e^{-B\xi_1} + 2e^{-6B/29} + \frac{(3.04)187}{16}\,\frac{B}{B-7/2}\,e^{-(B-7/2)18/65} < 1-10^{-6}$$

if $B \geqslant 18$ (i.e. $C \geqslant 20$).

Now suppose that $.05 < \xi_1 \leqslant .142$. Then the left-hand side of (23) is

$$< 2e^{-B\xi_1} + \frac{(3.04)B}{B-7/2}\,\frac{187}{16}\,e^{-(B-7/2)\xi}$$

where

$$\xi = .752 - \left(\frac{\sqrt{8\xi_1+\xi_1^2}-\xi_1}{2}\right),$$

and it is readily seen that (23) is satisfied for $B \geqslant 18$.

Finally, suppose $\xi_1 \leqslant .05$. By (1), there is at most one zero of $\prod_{\chi \bmod q} L(s,\chi)$ satisfying $\beta > 1-.05\mathscr{L}^{-1}$, $|\gamma| \leqslant \mathscr{L}$. By Lemma 4, with $\varepsilon = 10^{-5}$,

$$\xi \geqslant \left(\frac{1}{2}-10^{-5}\right)\log\left(\frac{1}{8\xi_1}\right).$$

Thus the left-hand side of (23) is

$$< e^{-B\xi_1} + \frac{187}{16}(3.04)e^{-(B-7/2)\xi} < e^{-B\xi_1} + \frac{187}{16}(3.04)(8\xi_1)^{(C-4)/2} < 1-10^{-6}$$

if $B \geqslant 18$.

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