1. Introduction. Let $Q_p$ and $Z_p$ denote the locally compact field of $p$-adic numbers and the compact ring of $p$-adic integers respectively, where $p$ is a fixed prime. We suppose that $\mu$ is Haar measure on $Q_p$ normalized so that $\mu(\mathbb{Z}_p) = 1$ and that $| \cdot |_p$ is the $p$-adic absolute value normalized so that $|p|_p = p^{-1}$. For $J = 1, 2, 3, \ldots$ and $j = 0, 1, 2, \ldots, p^J - 1$ we define

$$\varphi(j, J, y) = \begin{cases} 1 & \text{if } |y - j|_p \leq p^{-J}, \\ 0 & \text{if } |y - j|_p > p^{-J}. \end{cases}$$

Thus $\varphi(j, J, y)$ is the characteristic function of the sphere $S^J$ centered at $j$ and having radius $p^{-J}$. A sequence $\{x_n\}$, $n = 1, 2, 3, \ldots$, of $p$-adic integers is said to be uniformly distributed in $Z_p$ if

$$\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} \varphi(j, J, x_n) = p^{-J}$$

for each $J$ and $j$. We define the $p$-adic discrepancy of $\{x_n\}$, $n = 1, 2, \ldots, N$, by

$$A_N = \sup \left| \sum_{n=1}^{N} \varphi(j, J, x_n) - Np^{-J} \right|$$

where the supremum is taken over all $J \geq 1$ and $0 \leq j \leq p^J - 1$. It is well known (see [3] or [4]) that $N^{-1}A_N \to 0$ as $N \to \infty$ if and only if the sequence $\{x_n\}$ is uniformly distributed.

Let $\omega \in Q_p$ and let

$$\omega = \sum_{m=1}^{\infty} a_m p^m = \sum_{m=1}^{1} a_m p^m + \sum_{m=0}^{\infty} a_m p^m$$

** Limit theorems for uniformly distributed $p$-adic sequences**

by

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be its canonical representation. We define the integer part of \( \alpha \) to be the \( p \)-adic integer \( e(\alpha) = \sum_{n=0}^{\infty} a_n p^n \). If \( \{y_n\} \) is a sequence in \( \mathbb{Q}_p \) we may consider the distribution in \( \mathbb{Z}_p \) of the sequence of integer parts \( \{\sigma(y_n)\} \). Now suppose that \( 1 < K_1 < K_2 < \ldots < K_m < \ldots \) is a sequence of positive integers. The purpose of this paper is to investigate the distribution of \( \{\sigma(p^{-K_n} \omega)\} \) for \( \mu \)-almost all \( \omega \) in \( \mathbb{Q}_p \). Our results give \( p \)-adic analogues of theorems proved for real lacunary sequences by L. Gád and S. Gál [2] and by Philipp [5], [6]. In particular, let \( A_N(\alpha) \) be the \( p \)-adic discrepancy of \( \{\sigma(p^{-K_n} \omega)\} \), \( n = 1, 2, \ldots, N, \) for \( \omega \) in \( \mathbb{Q}_p \). It follows easily from ergodic theory that \( \{\sigma(p^{-K_n} \omega)\} \) is uniformly distributed in \( \mathbb{Z}_p \) for \( \mu \)-almost all \( \omega \). In the following theorem we give an almost everywhere bound on the discrepancy.

**Theorem 1.** For \( \mu \)-almost all \( \omega \in \mathbb{Q}_p \),

\[
p^{-1/2} \leq \limsup_{N \to \infty} \frac{A_N(\omega)}{\sqrt{N \log N}} \leq (30)p^{-1/2}.
\]

We say that a function \( f: \mathbb{Z}_p \to \mathbb{R} \) has bounded \( p \)-adic variation if

\[
V^*(f) = \sup_{J \in \mathcal{F}} \left\{ \sum_{n=0}^{N-1} \sup_{x_n \in J} |f(x) - f(y)| \right\}
\]

is finite. Here \( V^*(f) \) is called the total fluctuation of \( f \) (see Tableb15 [5]). Let \( \mathcal{F} \) be the class of functions \( f: \mathbb{Z}_p \to \mathbb{R} \) with total fluctuation not exceeding \( V^* \), satisfying

\[
\int_{\mathbb{Z}_p} f(y) d\mu(y) = 0,
\]

and extended to all of \( \mathbb{Q}_p \) by the requirement that \( f(\omega) = f(e(\omega)) \). Our \( p \)-adic version of Theorem 3 in Philipp [6] is the following result.

**Theorem 2.** For \( \mu \)-almost all \( \omega \in \mathbb{Q}_p \),

\[
\limsup_{N \to \infty} \left( \sum_{n=0}^{N-1} f(p^{-K_n} \omega) \right)^2 / V N \log N \leq (100) V^* p^{1/2}.
\]

We remark that our proof of Theorem 2 is more complicated than the corresponding results for real lacunary sequences. We use an inequality which is similar to Koksma's inequality (see [3]) but one which does not directly involve the \( p \)-adic discrepancy. Let \( \{x_n\} \) be a sequence of \( p \)-adic integers and for each positive integer \( J \) define

\[
A_N(J) = \max_{\varphi \in \mathcal{F}, \varphi(0, J)} \left| \sum_{n=1}^{N-1} \varphi(j, J; x_n) - Np^{-J} \right|,
\]

**Theorem 3.** Let \( f: \mathbb{Z}_p \to \mathbb{R} \) have bounded \( p \)-adic variation. Then for any positive integer \( M \),

\[
\left| \sum_{n=1}^{N} f(x_n) - N \int_{\mathbb{Z}_p} f(y) d\mu(y) \right| \leq V^*(f) \left( Np^{-M} + A_N(M) + p \sum_{J=M}^{\infty} A_N(J) \right).
\]

We now define \( A_N(j, \omega) \) by (1.3) with \( x_n = \sigma(p^{-K_n} \omega) \). Also we suppose that \( M = M(N) = \left\lceil (\log p)^{-1} \log N \right\rceil \), where \( \lceil \cdot \rceil \) is the greatest integer function. Our main result is

**Theorem 4.** For each positive integer \( J \) and for \( \mu \)-almost all \( \omega \in \mathbb{Q}_p \),

\[
\limsup_{N \to \infty} \frac{A_N(J, \omega)}{\sqrt{N \log N \log \log N}} \leq \frac{30}{p^{1/2}}
\]

and

\[
\sum_{J=M}^{\infty} A_N(J, \omega) \leq (100)p^{-1/2}.
\]

In order to deduce Theorem 1 from Theorem 4 we argue as follows. It is clear that

\[
A_N(\omega) = \sup_{1 \leq J < M} A_N(J, \omega).
\]

If \( M = M(N) < J \) then

\[
A_N(J, \omega) \leq \max_{1 \leq J < M} \left| \sum_{n=1}^{N} \varphi(j, J, \sigma(p^{-K_n} \omega)) \right| + Np^{-J}
\]

\[
\leq \max_{1 \leq J < M} \left| \sum_{n=1}^{N} \varphi(n, M, \sigma(p^{-K_n} \omega)) \right| + Np^{-J}
\]

\[
\leq 2Np^{-M} + A_N(M, \omega) + 2pn^{1/2}.
\]

From (1.7) and (1.8) we have

\[
A_N(\omega) \leq \max_{1 \leq J < M} A_N(J, \omega) + 2pn^{1/2}.
\]

Thus the upper bound in Theorem 1 follows from (1.9) and from (1.5) with \( J = 1 \). The lower bound in Theorem 1 is essentially trivial. For it is easily verified that \( \varphi(0, 1, \sigma(p^{-K_n} \omega) - p^{-1}) \), \( n = 1, 2, 3, \ldots \), is a sequence of identically distributed independent random variables on the probability space \( (\mathbb{Z}_p, \mu) \) with mean zero and variance \( p^{-1} - p^{-2} \). By applying the law of the iterated logarithm to this sequence we obtain the lower bound in (1.1) for \( \mu \)-almost all \( \omega \in \mathbb{Z}_p \). But \( \mathbb{Q}_p \) is a countable
disjoint union of translates of $Z_p$ by $p$-adic numbers $\xi$ with $|\xi|_p > p$ and $\sigma(\xi) = 0$. For such an $\xi$ we have $\sigma(p^{-K}(\xi + \omega)) = \sigma(p^{-K}\omega)$ whenever $\omega \in Z_p$. Hence the lower bound in (1.1) must hold for $\mu$-almost all $\omega \in Q_p$.

To prove Theorem 2 we first observe that by Theorem 3, with

$$M = M(N) = [(2\log p)^{-1}\log N],$$

we have

$$\sup_{J \subseteq \mathbb{Z}} \left\{ \sum_{n=1}^{N} f(p^{-K}\omega) \right\} \leq V* \left[ p^{N^{1/2}} + A_N(M, \omega) + p \sum_{J} A_N(J, \omega) \right].$$

If we now use (1.5) with an arbitrarily large value of $L$ and (1.6) with $L = 1$ we obtain (1.2).

Thus it remains only to prove Theorems 3 and 4. We note that it suffices to prove Theorem 4 for $\mu$-almost all $\omega \in Z_p$. Then the same argument used in the proof of the lower bound in Theorem 1 can be applied to extend the result to $\mu$-almost all $\omega \in Q_p$.

2. Preliminary lemmas. Let $T: Z_p \rightarrow Z_p$ be defined by

$$T(a_0 + a_1 p + a_2 p^2 + \ldots) = a_1 + a_2 p + a_3 p^2 + \ldots,$$

so that for each positive integer $K$ the $K$th iterate of $T$ satisfies $T^K(\omega) = \sigma(0)$. The transformation $T$ is $\mu$-measurable on $Z_p$ and it will be convenient to prove our results for the sequence $\{T^K(\omega)\}$. We shall write $\| \|$ for the $L^2$-norm of an integrable real valued function on $Z_p$ with respect to $\mu$-linear measure $\mu$. Also we define $\psi(\omega, J, \omega) = \psi(\omega, J, \omega) - p^{-J}$.

**Lemma 5.** For each integer $L > 0$ and $J > 1$,

$$\left| \sum_{n=1}^{J} \psi(\omega, J, T^n(\omega)) \right| \leq 2p^{-J}.$$

**Proof.** Since $T$ is $\mu$-measurable on $Z_p$ and $\| \|$ for an integrable function $\psi(\omega, J, T^n(\omega))$, we have

$$\left| \sum_{n=1}^{J} \psi(\omega, J, T^n(\omega)) \right| \leq \left\| \psi(\omega, J, T^n(\omega)) \right\| \leq 2p^{-J}.$$

By a simple calculation, the left-hand side of (2.1) is equal to

$$Jp^{-J}(1 - p^{-J}) + 2 \sum_{n=L+1}^{J} \sum_{m=1}^{n} \int \psi(\omega, J, T^n(\omega)) \psi(\omega, J, \omega) \mu(\omega),$$

where we have used the fact that $\int \psi(\omega, J, T^n(\omega)) \mu(\omega) = 0$. Now if $l$ is an integer greater than or equal to $J$ then

$$\int \psi(\omega, J, T^l(\omega)) \psi(\omega, J, \omega) \mu(\omega) = p^{-J} \int \psi(\omega, J, T^{l-J}(\omega)) \mu(\omega) = 0.$$

If $l$ is an integer, $1 \leq l < J$, we may write $j = a + p^l b$ where $a \in \{0, 1, \ldots, p^{l-1} - 1\}$ and $b \in \{0, 1, \ldots, p^{l-1} - 1\}$. Therefore

$$(2.4) \quad \int \psi(\omega, J, T^l(\omega)) \psi(\omega, J, \omega) \mu(\omega) = p^{-J} \int \psi(\omega, J, T^l(\omega) + p^l b) \mu(\omega) = p^{-J} \int \psi(\omega, J, b + p^{-l} y) \mu(\omega).$$

In order to evaluate the integral on the right-hand side of (2.4) we consider two cases. First suppose that $|j - b|_p > p^{-l}$, then

$$p^{-l} < |j - b|_p \leq \max(|j - b - p^{-l} y|_p, |j - b + p^{-l} y|_p) = |j - b - p^{-l} y|_p.$$ 

So in this case

$$(2.5) \quad \int \psi(\omega, J, T^l(\omega)) \psi(\omega, J, \omega) \mu(\omega) = p^{-J} \int (-p^{-l} y) \mu(\omega) = -p^{-2l}.$$

Next suppose that $|j - b|_p \leq p^{-l}$. Then $j = b + p^{-l} y$, and $y \in \{0, 1, \ldots, p^{-l} - 1\}$. Hence the right-hand side of (2.4) is equal to

$$(2.6) \quad p^{-J} \int \psi(\omega, J, b + p^{-l} y) \mu(\omega) = p^{-2l}.$$

If we combine (2.4), (2.5) and (2.6) with the observation that $j = a + p^l b$ implies $b = T^l(j)$ we find that for $1 \leq l < J$,

$$(2.7) \quad \int \psi(\omega, J, T^l(\omega)) \psi(\omega, J, \omega) \mu(\omega) = p^{-l} \psi(\omega, J - l, T^l(\omega)) \leq p^{-l-1}.$$

Returning to (2.2) we obtain

$$(2.8) \quad 2 \sum_{n=L+1}^{J} \sum_{m=1}^{n} \psi(\omega, J, T^n(\omega)) \psi(\omega, J, \omega) \mu(\omega) \leq 2 \sum_{n=L+1}^{J} \sum_{m=1}^{n} p^{-l}(K_m - K_n) \leq 2 \sum_{m=1}^{J} \sum_{r=1}^{r} p^{-l}(m-n) = 2 \sum_{r=1}^{J} (J - r) p^{-l} = 2p^{-l} \left( p^{l} - p^{l-1} - \ldots - p^0 \right) \leq 2p^{-l}.$$
Hence (2.1) follows from (2.2) and (2.8).  

**Lemma 6.** Let $0 < \epsilon < 1/3$ and $\delta < 1$. There exists a positive integer $H_0 = H_0(\epsilon, \delta)$ such that: if $H \geq 1$ and $H, J, \theta$ are integers satisfying $H_0 \leq H$, $1 \leq J \leq (1-\epsilon)^{-1}(\log p)^{-1}\log 2H$ and $0 \leq \theta$, then

$$
\mu \left\{ \omega \in \mathbb{Z}_p^* : \left| \sum_{n=0}^{G+H} \psi(n, J, T^{\mathfrak{L}}(\omega)) \right| \geq (1 + \delta)4\pi p^{R_0}R^{-1} \right\} \leq 6 \exp \left\{ -(1 + \delta)R^{2/3}\log \log H \right\}.
$$

**Proof.** For $r = 0, 1, 2, \ldots$ we define

$$A_r(\omega) = \sum_{n=0}^{G+H} \psi(n, J, T^{\mathfrak{L}}(\omega)).$$

Clearly $\int_{\mathbb{Z}_p} A_r(\omega) d\mu(\omega) = 0$ and by Lemma 5 $\|A_r\|^2 \leq 3Jp^{-r}$. Also the two sequences

(2.9) \hspace{1cm} A_0, A_1, A_2, \ldots, A_r, \ldots

and

(2.10) \hspace{1cm} A_1, A_2, A_2, \ldots, A_{2r+1}, \ldots

are both sequences of independent random variables on $(\mathbb{Z}_p, \mu)$. To see this we write $\omega = a_0(\omega) + a_1(\omega)p + a_2(\omega)p^2 + \ldots$ and note that $a_0(\omega), a_1(\omega), \ldots$ is a sequence of independent random variables. It is easy to check that the functions in the sequence (2.9) depend on certain finite subsets of the set $\{a_0(\omega), a_1(\omega), \ldots\}$ and that these subsets are disjoint. Similarly, the sequence (2.10) consists of independent random variables.

Now, $\exp(\theta) \leq 1 + \epsilon + \frac{1}{2}(1 + \delta)\theta^2$ for $|\theta| \leq \delta$. Thus for any $\lambda$ satisfying $0 < \lambda J \leq \delta$ we have

(2.11) \hspace{1cm} \int_{\mathbb{Z}_p} \exp \left\{ \lambda \sum_{n=0}^{N-1} A_{2r}(\omega) \right\} d\mu(\omega) \leq \prod_{r=1}^{N-1} \int_{\mathbb{Z}_p} \exp \left\{ \lambda A_{2r}(\omega) \right\} d\mu(\omega)

$$\leq \prod_{r=1}^{N-1} \left( 1 + \frac{1}{2}(1 + \delta)3\lambda^2 Jp^{-r} \right)^{\frac{1}{2}} \exp \left\{ \frac{1}{2} (1 + \delta)2^2 Jp^{-r} \right\}. $$

And similarly,

(2.12) \hspace{1cm} \int_{\mathbb{Z}_p} \exp \left\{ \lambda \sum_{n=0}^{N-1} A_{2r+1}(\omega) \right\} d\mu(\omega) \leq \exp \left\{ \frac{1}{2}(1 + \delta)2^2 Jp^{-r} \right\}.

Next we choose a positive integer $Q$ such that $J(Q + 1) \leq H < J(Q + 2)$.

Then

$$
\lambda \left| \sum_{n=0}^{G+H} \psi(n, J, T^{\mathfrak{L}}(\omega)) - \sum_{n=0}^{Q} A_n(\omega) \right| = \lambda \left| \sum_{n=0}^{2Q+H} \psi(n, J, T^{\mathfrak{L}}(\omega)) \right| \leq \lambda J \leq 1.
$$

If $B_1 = \left[ \frac{Q}{2} + 1 \right]$ and $B_2 = \left[ \frac{Q+1}{2} \right]$ then by the Cauchy–Schwarz inequality

$$
\int_{\mathbb{Z}_p} \exp \left\{ \lambda \sum_{n=0}^{2Q+H} \psi(n, J, T^{\mathfrak{L}}(\omega)) \right\} d\mu(\omega)

\leq 3 \int_{\mathbb{Z}_p} \exp \left\{ \lambda \sum_{n=0}^{B_1} A_n(\omega) + \lambda \sum_{n=B_2}^{2Q+H} A_{2n+1}(\omega) \right\} d\mu(\omega)

\leq 3 \left\{ \int_{\mathbb{Z}_p} \exp \left\{ 2\lambda \sum_{n=0}^{B_1} A_n(\omega) \right\} d\mu(\omega) \right\}^{1/2} \left\{ \int_{\mathbb{Z}_p} \exp \left\{ 2\lambda \sum_{n=B_2}^{2Q+H} A_{2n+1}(\omega) \right\} d\mu(\omega) \right\}^{1/2}

\leq 3 \exp \left\{ (1 + \delta)3\lambda^2 (B_1 + B_2) Jp^{-r} \right\} \leq 3 \exp \left\{ (1 + \delta)3\lambda^2 \right\}.
$$

provided that $0 < 2J \leq \delta$, using (2.11) and (2.12). Since an identical calculation holds for $-\psi$, we have

(2.13) \hspace{1cm} \int_{\mathbb{Z}_p} \exp \left\{ \lambda \sum_{n=0}^{2Q+H} \psi(n, J, T^{\mathfrak{L}}(\omega)) \right\} d\mu(\omega) \leq 6 \exp \left\{ (1 + \delta)3\lambda^2 \right\}.

It follows from (2.13) that for $0 < 2J \leq \delta$ and any $W > 0$,

(2.14) \hspace{1cm} \mu \left\{ \omega \in \mathbb{Z}_p : \left| \sum_{n=0}^{G+H} \psi(n, J, T^{\mathfrak{L}}(\omega)) \right| \geq W \right\}

\leq 6 \exp \left\{ (1 + \delta)3\lambda^2 \right\}.

We choose

(2.15) \hspace{1cm} W := (1 + 3p^{-2r}) (1 + \delta) p^{2r-2} (\log \log H)^{1/2}

and

(2.16) \hspace{1cm} \lambda := p^{-2r} (\log \log H)^{-1/2}.

Then using $1 < J < (1-\epsilon)^{-1}(\log p)^{-1}\log 2H$, we find that

(2.17) \hspace{1cm} 2J \leq C \log (2H) (2H)^{-1} (\log \log H)^{-1/2}

for some absolute constant $C > 0$. The right-hand side of (2.17) is $\leq \delta$ provided $H \geq H_0(\epsilon, \delta)$. Thus we may substitute (2.15) and (2.16) into (2.14) and obtain the result.
We will also require the following elementary inequality.

**Lemma 7.** If \( \beta \geq 1 \) and \( s > 1 \) then

\[
(2.18) \quad \sum_{n=1}^{\infty} \exp \{-\beta n^s\} \leq (s-1)^{-1} \exp \{-\beta s\}.
\]

**Proof.** We have \( x^n = (1+(x-1))^{n} \geq 1+n(x-1) \), and so the left-hand side of (2.18) is

\[
\leq \sum_{n=1}^{\infty} \exp \{-\beta n(x-1)\} \leq \exp \{-\beta \sum_{n=1}^{\infty} (n(x-1))\}
\]

\[
= \exp \{-\beta \} (\exp (x-1) - 1)^{-1} \leq (x-1)^{-1} \exp \{-\beta \}.
\]

**3. Proof of Theorem 4.** Throughout this section we assume that \( 0 < \epsilon < 1/5 \), \( 0 < \delta < 1 \) and that \( \eta > 0 \). To simplify some expressions we write

\[
F([G, H]) = \left| \sum_{x \in \mathbb{Z}^{G+H}} \psi(x, J, F^{x}(\omega)) \right|
\]

\[
r(x) = r_{2}(x) = 4(1+\delta)x \log \log x^{1/8}
\]

and

\[
s(x) = s_{2}(x) = [(2 \log p)^{-1} \log 2 + \eta].
\]

For any positive integer \( N \) we define integers \( u \) and \( v \) by \( 2^{u} \leq N < 2^{u+1} \), \( 2^{v-1} \leq 2 \log \log p \leq 2^{v} \). By using Lemma 12 of Erdős and Erdi [1] with only a trivial modification we can determine integers \( m_{l} \) satisfying \( 0 \leq m_{l} < 2^{u+v+1} \) for \( l = u, v+1, \ldots, u+1 \), and an integer \( N^{u} \) satisfying \( 1 \leq N^{u} < 2^{u+1+\log(1-\eta)} \) \( < 2y^{1+\eta} \) such that

\[
F(0, N) \leq F(0, 2^{u}) + \sum_{u} F(2^{u} + m_{l+1} 2^{v-1}, 2^{v}) + F(2^{u} + m_{u} 2^{v}, N^{u}).
\]

Next we will show that if \( u_{0} = u_{0}(\epsilon, \delta, \eta, p) \) is sufficiently large then

\[
\sum_{u_{0}} \sum_{v_{0}} \sum_{j} \sum_{M} \mu \{ \omega : F(0, 2^{u}, J, \omega) \geq p^{(1-\beta)2^{u}} \} \leq \eta
\]

and

\[
\sum_{u_{0}} \sum_{v_{0}} \sum_{j} \sum_{M} \mu \{ \omega : F(2^{u} + m_{2^{v-1} + 1}, 2^{v}, J, \omega) \}
\]

\[
\geq p^{(1-\beta)2^{u}} \rho^{2^{u}} \leq \eta.
\]

Both (3.2) and (3.3) are established by applying Lemmas 6 and 7. We give the details only for (3.3) as (3.2) is similar and in fact easier. First we observe that

\[
(3.4) \quad \mu \{ \omega : F(2^{u} + m2^{v+1}, 2^{v}) \geq p^{(1-\beta)2^{u} \rho^{2^{u}} \eta} \}
\]

\[
\leq \mu \{ \omega : F(2^{u} + m2^{v+1}, 2^{v}) \geq p^{(1-\beta)2^{u} \rho^{2^{u}} \eta} \}.
\]

We use Lemma 6 with \( G = 2^{u} + m2^{v+1} \), \( H = 2^{v} \) and \( R = 2^{u} \log 2 \). Since \( u = 1 + \frac{1}{4}(u(1-\epsilon)) \) and \( 2^{u} \leq H \) we find that

\[
1 \leq u \leq s(\epsilon) \leq (\log p)^{-1} \log(2^{u} \rho^{2^{u}}) \leq (1-\epsilon)^{-1} \log(2^{u} \rho^{2^{u}}).
\]

Thus the required condition on \( J \) is satisfied. Therefore

\[
(3.5) \quad \sum_{J=1}^{u} \sum_{m=0}^{2^{v-1}} \sum_{J=1}^{u} \sum_{j=0}^{2^{v-1}} \mu \{ \omega : F(2^{u} + m2^{v+1}, 2^{v}, J, \omega) \}
\]

\[
\geq p^{(1-\beta)2^{u} \rho^{2^{u}} \eta}.
\]

Now an easy calculation shows that \( \log(\log 2) \geq \log 2 - 2 \). Thus if \( u \) is sufficiently large (depending only on \( \epsilon, \delta, \) and \( p \)) then the right-hand side of (3.5) is

\[
\leq 6 \sum_{u_{0}} \sum_{v_{0}} \sum_{j} \exp \{(u-1+1)\log 2 + \eta \log \log p \}
\]

\[
\leq 6 \sum_{u_{0}} \sum_{v_{0}} \exp \{(u-1+1)\log 2 + \eta \log \log p \}
\]

Applying Lemma 7 twice and (3.6) is

\[
\leq (p-1)^{-1} (1+2^{1-\beta} - 1) \rho^{2^{u}} \leq \epsilon.
\]

This clearly establishes (3.3) if \( u_{0} = u_{0}(\epsilon, \delta, \eta, p) \) is sufficiently large.

Finally we apply (3.1), (3.2) and (3.3) to estimate \( A_{N}(J, \omega) \). For each positive integer \( N \), \( M(N) \leq s(\epsilon) \). Thus for all \( J \) satisfying \( 1 \leq J \leq M(N) \) and all \( N \geq N_{0}(\epsilon, \delta, \eta, p) \) we have

\[
A_{N}(J, \omega) = \max_{0 \leq g \leq p-1} F(0, N, J, g)
\]

\[
\leq (p-1)^{-1} (1+2^{1-\beta} - 1) \rho^{2^{u}} \leq 27 \rho^{2^{u}} \leq 27 \rho^{2^{u}} \eta.
\]

except on a subset of \( Z_{p} \), which has Haar measure less than \( 2\eta \). Since \( \eta > 0 \) was arbitrary,

\[
\lim_{N \to \infty} \frac{\max_{0 \leq g \leq p-1} A_{N}(J, \omega)}{\sqrt{N \log N}} \leq (30)(1+\delta) p^{2^{u}} \leq \eta.
\]
for μ-almost all ω in \( Z_p \). This proves (1.5). Also from (3.7) we have

\[
\sum_{J=0}^{M^N} A_N(J, \omega) 
\leq p^{(2\nu-1)\alpha} (1 - p^{(2\nu-1)\alpha})^{-1} 4(1 + (1 - 2^{-1/4})^{-1}) (1 + \delta) V N \log \log N + 2 N^{(1-\delta) \log N},
\]

from which (1.6) easily follows.

4. Proof of Theorem 3. Let \( f : Z_p \to R \) have bounded \( p \)-adic variation. For \( J = 0, 1, 2, \ldots \) we define \( f_J : Z_p \to R \) by

\[
f_J(\omega) = p^J \int f(y) d\mu(y)
\]

where \( J \) is the unique integer in \( \{0, 1, 2, \ldots, p^J - 1\} \) such that \( \omega \in S^J_{M^J} \). Clearly we may assume without loss of generality that \( f_J(\omega) = 0 \). We also define

\[
v_J(f, J, f) = \sup_{\omega \in S^J_{M^J}} |f(\omega) - f_J(\omega)|.
\]

Then for any positive integer \( M \),

\[
\sum_{n=1}^{N} f_J(x_n) \leq \sum_{n=1}^{N} f_J(x_n) - \sum_{n=1}^{N} f_J(x_n)
\]

Now,

\[
\sum_{n=1}^{N} (f(x_n) - f_M(x_n))
\]

where

\[
\sum_{n=1}^{N} f_M(x_n) - f_M(-1)(x_n)
\]

In view of the identity

\[
\sum_{J=0}^{M^N} \{ f_M(l+ p^J) - f_M(-1)(l) \} = \sum_{J=0}^{M^N} \int f(y) d\mu(y) - \sum_{J=0}^{M^N} \int f(y) d\mu(y) = 0,
\]

the right-hand side of (4.4) is

\[
\sum_{J=0}^{M^N} \left( \sum_{n=1}^{N} f_M(l+ p^J) - f_M(-1)(l) \right) \left( \sum_{n=1}^{N} \nu(l+p^M, M, x_n) - N p^{-M} \right)
\]

It follows from (4.3), (4.4) and (4.5) that

\[
\left| \sum_{n=1}^{N} f_M(x_n) \right| \leq p V^*(f) A_N(M) + p V^*(f) A_N(M) + A_N(M - 1)
\]

and so

\[
\left| \sum_{n=1}^{N} f_M(x_n) \right| \leq p V^*(f) \sum_{J=0}^{M} A_N(J).
\]

Thus (4.1), (4.2) and (4.6) combine to establish the inequality (1.4).
Class number formulas for quaternary quadratic forms

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Introduction. This paper may be regarded as a sequel to [4]. Unless
otherwise indicated, the notation and terminology are taken from [4],
especially §1, §3 and §5.

We recapitulate some of the results on class numbers derived in [3],
[4]. Let \( V \) be a definite quadratic space of dimension four over the field
of rational numbers \( \mathbb{Q} \). Let \( \mathfrak{I} \) be an idealexcept of maximal lattices
on \( V \) (cf. [4], §3). Let \( \Delta \) denote the reduced discriminant of \( \mathfrak{I} \) and \( H \)
the number of proper similitude classes in \( \mathfrak{I} \). In the case where \( V \) has
square discriminant \( \mathfrak{I} \) is uniquely determined, and an explicit formula
for \( H \) was given in [3] (Theorem, p. 297).

If the discriminant \( D(V) \) of \( V \) is not a square, we put \( K = \mathbb{Q}(\sqrt{D(V)}) \),
and denote the discriminant of \( K \) by \( \Delta_K \). It was shown in [4] (Prop. 7)
that

\[ \Delta = \Delta_K(p_1 \cdots p_r)(q_1 \cdots q_f)^2, \]

where \( p_1, \ldots, p_r \) are the anisotropic finite primes of \( V \); \( q_1, \ldots, q_f \) split
in \( K \), and \( p_1, \ldots, p_r \) are distinct rational primes which remain prime in
\( K \). In §6 of [4] explicit formulas were obtained for \( H \) (Theorems 1, 2)
under the following conditions:

(i) \( f = 0 \),

(ii) The fundamental unit of \( K \) has norm \(-1\).

In this paper we obtain such formulas for \( H \) without making either of
these restrictions. As a result, we completely solve the problem of
determining the proper class number of an arbitrary idealexcept of maximal
quaternary lattices (cf. [4], Prop. 11, for the indefinite case). As a special case of these formulas we obtain, in the classical language,
a formula for the number of proper classes of positive definite integral
quaternary forms of discriminant \( \Delta_K \).

By scaling, we may assume that \( \mathfrak{I} \) contains the maximal integral
lattices of \( V \). When \( D(V) \) is a nonsquare, there is a unique quaternion

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