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Limit theorems for uniformly distributed p-adic sequences*

by

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1. Introduction. Let Q_p and Z_p denote the locally compact field of p-adic numbers and the compact ring of p-adic integers respectively, where p is a fixed prime. We suppose that μ is Haar measure on Q_p normalized so that $\mu(Z_p) = 1$ and that $|\ |_p$ is the p-adic absolute value normalized so that $|p|_p = p^{-1}$. For $J = 1, 2, 3, \ldots$ and $j = 0, 1, 2, \ldots, p^J-1$ we define

$$\varphi(j, J, y) = \begin{cases} 1 & \text{if } |y-j|_p \leqslant p^{-J}, \\ 0 & \text{if } |y-j|_p > p^{-J}. \end{cases}$$

Thus $\varphi(j, J, y)$ is the characteristic function of the sphere $S_J^{(j)}$ centered at j and having radius p^{-J}. A sequence $\{x_n\}$, $n = 1, 2, 3, \ldots$, of p-adic integers is said to be *uniformly distributed* in Z_p if

$$\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N}\varphi(j, J, x_n) = p^{-J}$$

for each J and j. We define the *p-adic discrepancy* of $\{x_n\}$, $n = 1, 2, \ldots, N$, by

$$\Delta_N = \sup\Big|\sum_{n=1}^{N}\varphi(j, J, x_n) - Np^{-J}\Big|$$

where the supremum is taken over all $J \geqslant 1$ and j, $0 \leqslant j \leqslant p^J-1$. It is well known (see [3] or [4]) that $N^{-1}\Delta_N \to 0$ as $N\to\infty$ if and only if the sequence $\{x_n\}$ is uniformly distributed.

Let $\omega \in Q_p$ and let

$$\omega = \sum_{m=l}^{\infty} a_m p^m = \sum_{m=l}^{-1} a_m p^m + \sum_{m=0}^{\infty} a_m p^m$$

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be its canonical representation. We define the *integer part* of ω to be the p-adic integer $\sigma(\omega) = \sum_{m=0}^{\infty} a_m p^m$. If $\{y_n\}$ is a sequence in Q_p we may consider the distribution in $\boldsymbol{Z}_p$ of the sequence of integer parts $\{\sigma(y_n)\}$. Now suppose that $1 \leqslant K_1 < K_2 < \ldots < K_n < \ldots$ is a sequence of positive integers. The purpose of this paper is to investigate the distribution of $\{\sigma(p^{-K_n}\omega)\}$ for μ-almost all ω in Q_p. Our results give p-adic analogues of theorems proved for real lacunary sequences by L. Gál and S. Gál [2] and by Philipp [5], [6]. In particular, let $\Delta_N(\omega)$ be the p-adic discrepancy of $\{\sigma(p^{-K_n}\omega)\}$, $n = 1, 2, \ldots, N$, for ω in Q_p. It follows easily from ergodic theory that $\{\sigma(p^{-K_n}\omega)\}$ is uniformly distributed in $\boldsymbol{Z}_p$ for μ-almost all ω. In the following theorem we give an almost everywhere bound on the discrepancy.

THEOREM 1. *For μ-almost all $\omega \in Q_p$,*

$$p^{-1/2} \leqslant \limsup_{N\to\infty} \frac{\Delta_N(\omega)}{\sqrt{N \log\log N}} \leqslant (30)p^{-1/2}. \tag{1.1}$$

We say that a function $f\colon \boldsymbol{Z}_p \to \boldsymbol{R}$ has bounded p-adic variation if

$$V^*(f) = \sup_{0\leqslant J}\Big\{\sum_{j=0}^{p^J-1} \sup_{x,y\in S_J^{(j)}} |f(x)-f(y)|\Big\}$$

is finite. Here $V^*(f)$ is called the *total fluctuation* of f (see Taibleson [7]). Let $\mathscr{F}$ be the class of functions $f\colon \boldsymbol{Z}_p \to \boldsymbol{R}$ with total fluctuation not exceeding V^*, satisfying

$$\int_{\boldsymbol{Z}_p} f(y)\,d\mu(y) = 0,$$

and extended to all of Q_p by the requirement that $f(\omega) = f(\sigma(\omega))$. Our p-adic version of Theorem 3 in Philipp [6] is the following result.

THEOREM 2. *For μ-almost all $\omega \in Q_p$,*

$$\limsup_{N\to\infty}\left\{\frac{\sup_{f\in\mathscr{F}}\Big|\sum_{n=1}^{N} f(p^{-K_n}\omega)\Big|}{\sqrt{N\log\log N}}\right\} \leqslant (100)V^* p^{1/2}. \tag{1.2}$$

We remark that our proof of Theorem 2 is more complicated than the corresponding results for real lacunary sequences. We use an inequality which is similar to Koksma's inequality (see [3]) but one which does not directly involve the p-adic discrepancy. Let $\{x_n\}$ be a sequence of p-adic integers and for each positive integer J define

$$\Delta_N(J) = \max_{0\leqslant j\leqslant p^J-1}\Big|\sum_{n=1}^{N}\varphi(j, J, x_n) - Np^{-J}\Big|. \tag{1.3}$$

THEOREM 3. *Let $f\colon \boldsymbol{Z}_p \to \boldsymbol{R}$ have bounded p-adic variation. Then for any positive integer M,*

$$\Big|\sum_{n=1}^{N} f(x_n) - N\int_{\boldsymbol{Z}_p} f(y)\,d\mu(y)\Big| \leqslant V^*(f)\Big\{Np^{-M} + \Delta_N(M) + p\sum_{J=1}^{M}\Delta_N(J)\Big\}. \tag{1.4}$$

We now define $\Delta_N(J, \omega)$ by (1.3) with $x_n = \sigma(p^{-K_n}\omega)$. Also we suppose that $M = M(N) = [(2\log p)^{-1}\log N]$, where [ ] is the greatest integer function. Our main result is

THEOREM 4. *For each positive integer L and for μ-almost all $\omega \in Q_p$,*

$$\limsup_{N\to\infty}\frac{\max_{L\leqslant J\leqslant M(N)}\Delta_N(J, \omega)}{\sqrt{N\log\log N}} \leqslant (30)p^{-L/2} \tag{1.5}$$

and

$$\limsup_{N\to\infty}\frac{\sum_{J=L}^{M(N)}\Delta_N(J, \omega)}{\sqrt{N\log\log N}} \leqslant (100)p^{-L/2}. \tag{1.6}$$

In order to deduce Theorem 1 from Theorem 4 we argue as follows. It is clear that

$$\Delta_N(\omega) = \sup_{1\leqslant J}\Delta_N(J, \omega). \tag{1.7}$$

If $M = M(N) < J$ then

$$\begin{aligned}\Delta_N(J, \omega) &\leqslant \max_{0\leqslant j\leqslant p^J-1}\Big|\sum_{n=1}^{N}\varphi(j, J, \sigma(p^{-K_n}\omega))\Big| + Np^{-J}\\ &\leqslant \max_{0\leqslant m\leqslant p^M-1}\Big|\sum_{n=1}^{N}\varphi(m, M, \sigma(p^{-K_n}\omega))\Big| + Np^{-J}\\ &\leqslant \Delta_N(M, \omega) + 2Np^{-M} \leqslant \Delta_N(M, \omega) + 2pN^{1/2}.\end{aligned} \tag{1.8}$$

From (1.7) and (1.8) we have

$$\Delta_N(\omega) \leqslant \max_{1\leqslant J\leqslant M(N)}\Delta_N(J, \omega) + 2pN^{1/2}. \tag{1.9}$$

Thus the upper bound in Theorem 1 follows from (1.9) and from (1.5) with $L = 1$. The lower bound in Theorem 1 is essentially trivial. For it is easily verified that $\{\varphi(0, 1, \sigma(p^{-K_n}\omega)) - p^{-1}\}$, $n = 1, 2, 3, \ldots$, is a sequence of identically distributed independent random variables on the probability space $(\boldsymbol{Z}_p, \mu)$ with mean zero and variance $p^{-1} - p^{-2}$. By applying the law of the iterated logarithm to this sequence we obtain the lower bound in (1.1) for μ-almost all $\omega \in \boldsymbol{Z}_p$. But Q_p is a countable disjoint union of translates of Z_p by p-adic numbers ξ with $|\xi|_p \geqslant p$ and $\sigma(\xi) = 0$. For such an ξ we have $\sigma(p^{-K_n}(\xi+\omega)) = \sigma(p^{-K_n}\omega)$ whenever $\omega \in Z_p$. Hence the lower bound in (1.1) must hold for μ-almost all $\omega \in Q_p$.

To prove Theorem 2 we first observe that by Theorem 3, with

$$M = M(N) = [(2\log p)^{-1}\log N],$$

we have

$$\sup_{f\in\mathscr{F}}\Big|\sum_{n=1}^{N} f(p^{-K_n}\omega)\Big| \leqslant V^*\Big\{pN^{1/2} + \Delta_N(M,\omega) + p\sum_{J=1}^{M}\Delta_N(J,\omega)\Big\}.$$

If we now use (1.5) with an arbitrarily large value of L and (1.6) with $L = 1$ we obtain (1.2).

Thus it remains only to prove Theorems 3 and 4. We note that it suffices to prove Theorem 4 for μ-almost all $\omega \in Z_p$. Then the same argument used in the proof of the lower bound in Theorem 1 can be applied to extend the result to μ-almost all $\omega \in Q_p$.

2. Preliminary lemmas. Let $T\colon Z_p \to Z_p$ be defined by

$$T(a_0 + a_1 p + a_2 p^2 + \ldots) = a_1 + a_2 p + a_3 p^2 + \ldots,$$

so that for each positive integer K the Kth iterate of T satisfies $T^K(\omega) = \sigma(p^{-K}\omega)$. The transformation T is μ-measure preserving on Z_p and it will be convenient to prove our results for the sequence $\{T^{K_n}(\omega)\}$. We shall write $\|\ \|$ for the L^2-norm of an integrable real valued function on Z_p with respect to Haar measure μ. Also we define $\psi(j,J,\omega) = \varphi(j,J,\omega) - p^{-J}$.

LEMMA 5. *For each integer $L \geqslant 0$ and $J \geqslant 1$,*

$$\Big\|\sum_{n=L+1}^{L+J}\psi(j,J,T^{K_n}(\omega))\Big\|^2 \leqslant 3Jp^{-J}. \tag{2.1}$$

Proof. Since T is μ-measure preserving,

$$\|\psi(j,J,T^{K_n}(\omega))\|^2 = \|\psi(j,J,\omega)\|^2 = p^{-J}(1-p^{-J})$$

by a simple calculation. Thus the left-hand side of (2.1) is equal to

$$Jp^{-J}(1-p^{-\ }) + {} + 2\sum_{n=L+1}^{L+J-1}\sum_{m=n+1}^{J}\int_{Z_p}\psi(j,J,T^{K_m-K_n}(\omega))\varphi(j,J,\omega)\,d\mu(\omega), \tag{2.2}$$

where we have used the fact that $\int_{Z_p}\psi(j,J,T^{K_m-K_n}(\omega))\,d\mu(\omega) = 0$. Now if l is an integer greater than or equal to J then

$$\int_{Z_p}\psi(j,J,T^l(\omega))\varphi(j,J,\omega)\,d\mu(\omega) \tag{2.3}$$
$$= p^{-J}\int_{Z_p}\psi(j,J,T^l(j+p^J y))\,d\mu(y) = p^{-J}\int_{Z_p}\psi(j,J,T^{l-J}(y))\,d\mu(y) = 0.$$

If l is an integer, $1 \leqslant l < J$, we may write $j = a + p^l b$ where $a \in \{0,1,\ldots,p^l-1\}$ and $b \in \{0,1,\ldots,p^{J-l}-1\}$. Therefore

$$\int_{Z_p}\psi(j,J,T^l(\omega))\varphi(j,J,\omega)\,d\mu(\omega) \tag{2.4}$$
$$= p^{-J}\int_{Z_p}\psi(j,J,T^l(a+p^l b+p^J y))\,d\mu(y)$$
$$= p^{-J}\int_{Z_p}\psi(j,J,b+p^{J-l}y)\,d\mu(y).$$

In order to evaluate the integral on the right-hand side of (2.4) we consider two cases. First suppose that $|j-b|_p > p^{l-J}$, then

$$p^{l-J} < |j-b|_p \leqslant \max\{|j-b-p^{J-l}y|_p,\, p^{l-J}|y|_p\} = |j-b-p^{J-l}y|_p.$$

So in this case

$$\int_{Z_p}\psi(j,J,T^l(\omega))\varphi(j,J,\omega)\,d\mu(\omega) \tag{2.5}$$
$$= p^{-J}\int_{Z_p}(-p^{-J})\,d\mu(y) = -p^{-2J}.$$

Next suppose that $|j-b|_p \leqslant p^{l-J}$. Then $j = b + p^{J-l}c$ for $c \in \{0,1,\ldots,p^l-1\}$. Hence the right-hand side of (2.4) is equal to

$$p^{-J}\int_{Z_p}\varphi(j,J,b+p^{J-l}y)\,d\mu(y) - p^{-2J} \tag{2.6}$$
$$= p^{-J-l}\int_{Z_p}\varphi(j,J,b+p^{J-l}c+p^J\omega)\,d\mu(\omega) - p^{-2J} = p^{-J-l} - p^{-2J}.$$

If we combine (2.4), (2.5) and (2.6) with the observation that $j = a + p^l b$ implies $b = T^l(j)$ we find that for $1 \leqslant l < J$,

$$\int_{Z_p}\psi(j,J,T^l(\omega))\varphi(j,J,\omega)\,d\mu(\omega) \tag{2.7}$$
$$= p^{-J-l}\psi(j,J-l,T^l(j)) \leqslant p^{-J-l}.$$

Returning to (2.2) we obtain

$$2\sum_{n=L+1}^{L+J-1}\sum_{m=n+1}^{J}\int_{Z_p}\psi(j,J,T^{K_m-K_n}(\omega))\varphi(j,J,\omega)\,d\mu(\omega) \tag{2.8}$$
$$\leqslant 2\sum_{n=L+1}^{L+J-1}\sum_{m=n+1}^{J}p^{-J-(K_m-K_n)}$$
$$\leqslant 2\sum_{n=L+1}^{L+J-1}\sum_{m=n+1}^{J}p^{-J-(m-n)} = 2\sum_{r=1}^{J-1}(J-r)p^{-J-r}$$
$$= 2p^{-2J}\{Jp^J - p - p^2 - \ldots - p^J\} \leqslant 2Jp^{-J}.$$

Hence (2.1) follows from (2.2) and (2.8). ■

LEMMA 6. *Let $0 < \varepsilon \leqslant 1/5$ and $0 < \delta \leqslant 1$. There exists a positive integer $H_0 = H_0(\varepsilon, \delta)$ such that: if $R \geqslant 1$ and H, J and G are integers satisfying $H_0 \leqslant H$, $1 \leqslant J \leqslant (1-\varepsilon)^{-1}(\log p)^{-1}\log 2H$ and $0 \leqslant G$, then*

$$\mu\Big\{\omega \in \mathbf{Z}_p : \Big|\sum_{n=G+1}^{G+H} \psi(j, J, T^{K_n}(\omega))\Big| \geqslant (1+\delta)4Rp^{J(2\varepsilon-\frac{1}{2})}(H\log\log H)^{1/2}\Big\} \leqslant 6\exp\{-(1+\delta)Rp^{J\varepsilon}\log\log H\}.$$

Proof. For $\nu = 0, 1, 2, \ldots$ we define

$$A_\nu(\omega) = \sum_{n=G+J\nu+1}^{G+J(\nu+1)} \psi(j, J, T^{K_n}(\omega)).$$

Clearly $\int_{\mathbf{Z}_p} A_\nu(\omega)\,d\mu(\omega) = 0$ and by Lemma 5 $\|A_\nu\|^2 \leqslant 3Jp^{-J}$. Also the two sequences

(2.9) $$A_0, A_2, A_4, \ldots, A_{2\nu}, \ldots$$

and

(2.10) $$A_1, A_3, A_5, \ldots, A_{2\nu+1}, \ldots$$

are both sequences of independent random variables on $(\mathbf{Z}_p, \mu)$. To see this we write $\omega = a_0(\omega) + a_1(\omega)p + a_2(\omega)p^2 + \ldots$ and note that $a_0(\omega)$, $a_1(\omega), \ldots$, is a sequence of independent random variables. It is easy to check that the functions in the sequence (2.9) each depend on certain finite subsets of the set $\{a_0(\omega), a_1(\omega), \ldots\}$ and that these subsets are disjoint. Similarly, the sequence (2.10) consists of independent random variables.

Now, $\exp\{x\} \leqslant 1 + x + \frac{1}{2}(1+\delta)x^2$ for $|x| \leqslant \delta$. Thus for any λ satisfying $0 < \lambda J \leqslant \delta$ we have

(2.11) $$\int_{\mathbf{Z}_p} \exp\Big\{\lambda \sum_{\nu=0}^{B-1} A_{2\nu}(\omega)\Big\}d\mu(\omega) = \prod_{\nu=0}^{B-1} \int_{\mathbf{Z}_p} \exp\{\lambda A_{2\nu}(\omega)\}d\mu(\omega) \leqslant \prod_{\nu=0}^{B-1}\big(1 + \tfrac{1}{2}(1+\delta)3\lambda^2 Jp^{-J}\big) \leqslant \exp\{\tfrac{3}{2}(1+\delta)\lambda^2 JBp^{-J}\}.$$

And similarly,

(2.12) $$\int_{\mathbf{Z}_p} \exp\Big\{\lambda \sum_{\nu=0}^{B-1} A_{2\nu+1}(\omega)\Big\}d\mu(\omega) \leqslant \exp\{\tfrac{3}{2}(1+\delta)\lambda^2 JBp^{-J}\}.$$

Next we choose a positive integer Q such that $J(Q+1) \leqslant H < J(Q+2)$.

Then

$$\lambda\Big|\sum_{n=G+1}^{G+H} \psi(j, J, T^{K_n}(\omega)) - \sum_{\nu=0}^{Q} A_\nu(\omega)\Big| = \lambda\Big|\sum_{n=G+J(Q+1)+1}^{G+H} \varphi(j, J, T^{K_n}(\omega))\Big| \leqslant \lambda J \leqslant 1.$$

If $B_1 = \left[\frac{Q}{2} + 1\right]$ and $B_2 = \left[\frac{Q+1}{2}\right]$ then by the Cauchy–Schwarz inequality

$$\begin{aligned}\int_{\mathbf{Z}_p} \exp\Big\{\lambda \sum_{n=G+1}^{G+H} \psi(j, J, T^{K_n}(\omega))\Big\}\,d\mu(\omega) &\leqslant 3\int_{\mathbf{Z}_p} \exp\Big\{\lambda \sum_{\nu=0}^{B_1-1} A_{2\nu}(\omega) + \lambda \sum_{\nu=0}^{B_2-1} A_{2\nu+1}(\omega)\Big\}\,d\mu(\omega) \\ &\leqslant 3\Big\{\int_{\mathbf{Z}_p} \exp\Big\{2\lambda \sum_{\nu=0}^{B_1-1} A_{2\nu}(\omega)\Big\}\,d\mu(\omega)\Big\}^{1/2}\Big\{\int_{\mathbf{Z}_p} \exp\Big\{2\lambda \sum_{\nu=0}^{B_2-1} A_{2\nu+1}(\omega)\Big\}\,d\mu(\omega)\Big\}^{1/2} \\ &\leqslant 3\exp\{(1+\delta)3\lambda^2(B_1+B_2)Jp^{-J}\} \leqslant 3\exp\{(1+\delta)3\lambda^2 Hp^{-J}\},\end{aligned}$$

provided that $0 < 2\lambda J \leqslant \delta$, using (2.11) and (2.12). Since an identical calculation holds for $-\psi$, we have

(2.13) $$\int_{\mathbf{Z}_p} \exp\Big\{\lambda\Big|\sum_{n=G+1}^{G+H} \psi(j, J, T^{K_n}(\omega))\Big|\Big\}\,d\mu(\omega) \leqslant 6\exp\{(1+\delta)3\lambda^2 Hp^{-J}\}.$$

It follows from (2.13) that for $0 < 2\lambda J \leqslant \delta$ and any $W \geqslant 0$,

(2.14) $$\mu\Big\{\omega \in \mathbf{Z}_p : \Big|\sum_{n=G+1}^{G+H} \psi(j, J, T^{K_n}(\omega))\Big| \geqslant W\Big\} \leqslant 6\exp\{(1+\delta)3\lambda^2 Hp^{-J} - \lambda W\}.$$

We choose

(2.15) $$W = (1 + 3p^{-3\varepsilon J})(1+\delta)Rp^{-J(\frac{1}{2}-2\varepsilon)}(H\log\log H)^{1/2}$$

and

(2.16) $$\lambda = p^{J(\frac{1}{2}-\varepsilon)}H^{-1/2}(\log\log H)^{1/2}.$$

Then using $1 \leqslant J \leqslant (1-\varepsilon)^{-1}(\log p)^{-1}\log 2H$, we find that

(2.17) $$2\lambda J \leqslant C(\log 2H)(2H)^{\frac{1}{2}-\frac{1}{2}\varepsilon}H^{-1/2}(\log\log H)^{1/2}$$

for some absolute constant $C > 0$. The right-hand side of (2.17) is $\leqslant \delta$ provided $H \geqslant H_0(\varepsilon, \delta)$. Thus we may substitute (2.15) and (2.16) into (2.14) and obtain the result.

We will also require the following elementary inequality.

LEMMA 7. *If* $\beta \geqslant 1$ *and* $x > 1$ *then*

$$\sum_{n=1}^{\infty} \exp\{-\beta x^n\} \leqslant (x-1)^{-1}\exp\{-\beta\}. \tag{2.18}$$

Proof. We have $x^n = (1+(x-1))^n \geqslant 1+n(x-1)$, and so the left-hand side of (2.18) is

$$\leqslant \sum_{n=1}^{\infty} \exp\{-\beta - n\beta(x-1)\} \leqslant \exp\{-\beta\} \sum_{n=1}^{\infty} \exp\{-n(x-1)\}$$
$$= \exp\{-\beta\}\{\exp(x-1)-1\}^{-1} \leqslant (x-1)^{-1}\exp\{-\beta\}.$$

3. Proof of Theorem 4. Throughout this section we assume that $0 < \varepsilon \leqslant 1/5$, $0 < \delta \leqslant 1$ and that $\eta > 0$. To simplify some expressions we write

$$F(G, H) = F(G, H, j, J, \omega) = \Big|\sum_{n=G+1}^{G+H} \psi(j, J, T^{K_n}(\omega))\Big|,$$
$$r(x) = r_\delta(x) = 4(1+\delta)(x\log\log x)^{1/2}$$

and

$$s(x) = s_p(x) = [(2\log p)^{-1}\log 2^{x+1}].$$

For any positive integer N we define integers u and v by $2^u \leqslant N < 2^{u+1}$, $2^{v-1} \leqslant 2^{\frac{1}{2}u(1-\varepsilon)} < 2^v$. By using Lemma 12 of Erdös and Gál [1] with only a trivial modification we can determine integers m_l satisfying $0 \leqslant m_l < 2^{u-l+1}$ for $l = v, v+1, \ldots, u+1$, and an integer N^* satisfying $1 \leqslant N^* < 2^{1+\frac{1}{2}u(1-\varepsilon)}$ $< 2N^{\frac{1}{2}(1-\varepsilon)}$ such that

$$F(0, N) \leqslant F(0, 2^u) + \sum_{l=v}^{u} F(2^u + m_{l+1}2^{l+1}, 2^l) + F(2^u + m_v 2^v, N^*). \tag{3.1}$$

Next we will show that if $u_0 = u_0(\varepsilon, \delta, \eta, p)$ is sufficiently large then

$$\sum_{u=u_0}^{\infty} \sum_{J=1}^{s(u)} \sum_{j=0}^{p^J-1} \mu\{\omega\colon F(0, 2^u, j, J, \omega) \geqslant p^{J(2\varepsilon-1/2)} r(2^u)\} < \eta \tag{3.2}$$

and

$$\sum_{u=u_0}^{\infty} \sum_{l=v}^{u} \sum_{m=0}^{2^{u-l+1}-1} \sum_{J=1}^{s(u)} \sum_{j=0}^{p^J-1} \mu\{\omega\colon F(2^u + m2^{l+1}, 2^l, j, J, \omega) \geqslant p^{J(2\varepsilon-1/2)} 2^{\frac{1}{2}(l-u)} r(2^u)\} < \eta. \tag{3.3}$$

Both (3.2) and (3.3) are established by applying Lemmas 6 and 7. We give the details only for (3.3) as (3.2) is similar and in fact easier. First

we observe that

$$\mu\{\omega\colon F(2^u + m2^{l+1}, 2^l) \geqslant p^{J(2\varepsilon-1/2)} 2^{\frac{1}{2}(l-u)} r(2^u)\} \leqslant \mu\{\omega\colon F(2^u + m2^{l+1}, 2^l) \geqslant p^{J(2\varepsilon-1/2)} 2^{\frac{1}{2}(u-l)} r(2^l)\}. \tag{3.4}$$

We use Lemma 6 with $G = 2^u + m2^{l+1}$, $H = 2^l$ and $R = 2^{\frac{1}{2}(u-l)}$. Since $v = 1 + [\frac{1}{2}u(1-\varepsilon)]$ and $2^v \leqslant H$ we find that

$$1 \leqslant J \leqslant s(u) \leqslant (\log p)^{-1}\log(2^{1+\frac{1}{2}u}) \leqslant (1-\varepsilon)^{-1}(\log p)^{-1}\log(2^{1+v}).$$

Thus the required condition on J is satisfied. Therefore

$$\sum_{l=v}^{u} \sum_{m=0}^{2^{u-l+1}-1} \sum_{J=1}^{s(u)} \sum_{j=0}^{p^J-1} \mu\{\omega\colon F(2^u + m2^{l+1}, 2^l, j, J, \omega) \geqslant p^{J(2\varepsilon-1/2)} 2^{\frac{1}{2}(u-l)} r(2^l)\} \tag{3.5}$$
$$\leqslant 6 \sum_{l=v}^{u} \sum_{J=1}^{s(u)} \exp\{(u-l+1)\log 2 + J\log p - (1+\delta)2^{\frac{1}{2}(u-l)} p^{J\varepsilon}\log\log 2^l\}.$$

Now an easy calculation shows that $\log\log 2^l \geqslant \log u - 2$. Thus if u is sufficiently large (depending only on ε, δ, and p) then the right-hand side of (3.5) is

$$\leqslant \sum_{l=v}^{u} \sum_{J=1}^{s(u)} \exp\{-(1+\tfrac{1}{2}\delta)2^{\frac{1}{2}(u-l)} p^{J\varepsilon}\log u\}. \tag{3.6}$$

Applying Lemma 7 twice and (3.6) is

$$\leqslant (p^\varepsilon - 1)^{-1}\big(1 + (2^{1/4}-1)^{-1}\big) u^{-(1+\frac{1}{2}\delta)}.$$

This clearly establishes (3.3) if $u_0 = u_0(\varepsilon, \delta, \eta, p)$ is sufficiently large.

Finally we apply (3.1), (3.2) and (3.3) to estimate $\Delta_N(J, \omega)$. For each positive integer N, $M(N) \leqslant s(u)$. Thus for all J satisfying $1 \leqslant J$ $\leqslant M(N)$ and all $N \geqslant N_0(\varepsilon, \delta, \eta, p)$ we have

$$\Delta_N(J, \omega) = \max_{0 \leqslant j \leqslant p^J - 1} F(0, N, j, J, \omega) \tag{3.7}$$
$$\leqslant p^{J(2\varepsilon-1/2)} r(2^u) + \sum_{l=v}^{u} p^{J(2\varepsilon-1/2)} 2^{\frac{1}{2}(l-u)} r(2^u) + N^*$$
$$\leqslant p^{J(2\varepsilon-1/2)} 4\big(1 + (1-2^{-1/4})^{-1}\big)(1+\delta)\sqrt{N\log\log N} + 2N^{\frac{1}{2}(1-\varepsilon)}$$

except on a subset of $\boldsymbol{Z}_p$ which has Haar measure less than 2η. Since $\eta > 0$ was arbitrary,

$$\limsup_{N\to\infty} \frac{\max_{L \leqslant J \leqslant M(N)} \Delta_N(J, \omega)}{\sqrt{N\log\log N}} \leqslant (30)(1+\delta)p^{L(2\varepsilon-1/2)}$$

for μ-almost all ω in $\boldsymbol{Z}_p$. This proves (1.5). Also from (3.7) we have

$$\sum_{J=L}^{M(N)} \Delta_N(J,\omega) \leqslant p^{L(2\varepsilon-1/2)}(1-p^{2\varepsilon-1/2})^{-1}\,4\big(1+(1-2^{-1/4})^{-1}\big)(1+\delta)\sqrt{N\log\log N}+ 2N^{\frac{1}{2}(1-\varepsilon)}\log N,$$

from which (1.6) easily follows.

4. Proof of Theorem 3. Let $f\colon \boldsymbol{Z}_p\to\boldsymbol{R}$ have bounded p-adic variation. For $J=0,1,2,\ldots$ we define $f_J\colon \boldsymbol{Z}_p\to\boldsymbol{R}$ by

$$f_J(\omega) = p^J \int_{S_J^{(j)}} f(y)\,d\mu(y)$$

where j is the unique integer in $\{0,1,2,\ldots,p^J-1\}$ such that $\omega\in S_J^{(j)}$. Clearly we may assume without loss of generality that $f_0(\omega)=0$. We also define

$$v(j,J,f) = \sup_{x,y\in S_J^{(j)}} |f(x)-f(y)|.$$

Then for any positive integer M,

$$\Big|\sum_{n=1}^{N} f(x_n)\Big| \leqslant \Big|\sum_{n=1}^{N} f_M(x_n)\Big| + \Big|\sum_{n=1}^{N}\{f(x_n)-f_M(x_n)\}\Big|. \tag{4.1}$$

Now,

$$\begin{aligned}\Big|\sum_{n=1}^{N}\{f(x_n)-f_M(x_n)\}\Big| &\leqslant \sum_{m=0}^{p^M-1}\Big|\sum_{n=1}^{N}\varphi(m,M,x_n)\,p^M\int_{S_M^{(m)}}\big(f(x_n)-f(y)\big)\,d\mu(y)\Big|\\ &\leqslant V^*(f)Np^{-M}+\sum_{m=0}^{p^M-1} v(m,M,f)\Big|\sum_{n=1}^{N}\varphi(m,M,x_n)-Np^{-M}\Big|\\ &\leqslant V^*(f)\{Np^{-M}+\Delta_N(M)\}.\end{aligned} \tag{4.2}$$

We estimate the other term on the right of (4.1) by a recursive argument. We have

$$\Big|\sum_{n=1}^{N} f_M(x_n)\Big| \leqslant \Big|\sum_{n=1}^{N} f_{M-1}(x_n)\Big| + \Big|\sum_{n=1}^{N}\{f_M(x_n)-f_{M-1}(x_n)\}\Big| \tag{4.3}$$

where

$$\begin{aligned}\Big|\sum_{n=1}^{N}\{f_M(x_n)-f_{M-1}(x_n)\}\Big| &\leqslant \sum_{l=0}^{p^{M-1}-1}\Big|\sum_{n=1}^{N}\varphi(l,M-1,x_n)\{f_M(x_n)-f_{M-1}(l)\}\Big|\\ &= \sum_{l=0}^{p^{M-1}-1}\Big|\sum_{j=0}^{p-1}\sum_{n=1}^{N}\varphi(l+p^Mj,M,x_n)\{f_M(l+p^Mj)-f_{M-1}(l)\}\Big|.\end{aligned} \tag{4.4}$$

In view of the identity

$$\sum_{j=0}^{p-1}\{f_M(l+p^Mj)-f_{M-1}(l)\} = \sum_{j=0}^{p-1}\Big\{p^M\int_{S_M^{(l+p^Mj)}} f(y)\,d\mu(y) - p^{M-1}\int_{S_{M-1}^{(l)}} f(y)\,d\mu(y)\Big\} = 0,$$

the right-hand side of (4.4) is

$$\begin{aligned}&= \sum_{l=0}^{p^{M-1}-1}\Big|\sum_{j=0}^{p-1}\{f_M(l+p^Mj)-f_{M-1}(l)\}\Big\{\sum_{n=1}^{N}\varphi(l+p^Mj,M,x_n)-Np^{-M}\Big\}\Big|\\ &\leqslant \Delta_N(M)\sum_{l=0}^{p^{M-1}-1}\sum_{j=0}^{p-1}|f_M(l+p^Mj)-f_{M-1}(l)|\\ &\leqslant \Delta_N(M)\sum_{l=0}^{p^{M-1}-1}\sum_{j=0}^{p-1}p^{2M-1}\int_{S_M^{(l+p^Mj)}}\int_{S_{M-1}^{(l)}}|f(x)-f(y)|\,d\mu(y)\,d\mu(x)\\ &\leqslant \Delta_N(M)\,p\sum_{l=0}^{p^{M-1}-1} v(l,M-1,f) \leqslant \Delta_N(M)\,p\,V^*(f).\end{aligned} \tag{4.5}$$

It follows from (4.3), (4.4) and (4.5) that

$$\begin{aligned}\Big|\sum_{n=1}^{N} f_M(x_n)\Big| &\leqslant \Big|\sum_{n=1}^{N} f_{M-1}(x_n)\Big| + pV^*(f)\,\Delta_N(M)\\ &\leqslant \Big|\sum_{n=1}^{N} f_{M-2}(x_n)\Big| + pV^*(f)\{\Delta_N(M)+\Delta_N(M-1)\}\end{aligned}$$

and so

$$\Big|\sum_{n=1}^{N} f_M(x_n)\Big| \leqslant pV^*(f)\sum_{J=1}^{M}\Delta_N(J). \tag{4.6}$$

Thus (4.1), (4.2) and (4.6) combine to establish the inequality (1.4). ■

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Class number formulas for quaternary quadratic forms

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Introduction. This paper may be regarded as a sequel to [4]. Unless otherwise indicated, the notation and terminology are taken from [4], especially § 1, § 3 and § 5.

We recapitulate some of the results on class numbers derived in [3], [4]. Let V be a definite quadratic space of dimension four over the field of rational numbers $\boldsymbol{Q}$. Let $\mathfrak{I}$ be an idealcomplex of maximal lattices on V (cf. [4], § 3). Let Δ denote the reduced discriminant of $\mathfrak{I}$ and H the number of proper similitude classes in $\mathfrak{I}$. In the case where V has square discriminant $\mathfrak{I}$ is uniquely determined, and an explicit formula for H was given in [3] (Theorem, p. 297).

If the discriminant $D(V)$ of V is not a square, we put $K = \boldsymbol{Q}(\sqrt{D(V)})$, and denote the discriminant of K by Δ_K. It was shown in [4] (Prop. 7) that

$$\Delta = \Delta_K (p_1 \ldots p_e)^2 (q_1 \ldots q_f)^2,$$

where $q_1, \ldots, q_f$ are the anisotropic finite primes of V; $q_1, \ldots, q_f$ split in K, and $p_1, \ldots, p_e$ are distinct rational primes which remain prime in K. In § 6 of [4] explicit formulas were obtained for H (Theorems 1, 2) under the following conditions:

(i) $f = 0$,

(ii) The fundamental unit of K has norm -1.

In this paper we obtain such formulas for H without making either of these restrictions. As a result, we completely solve the problem of determining the proper class number of an arbitrary idealcomplex of maximal quaternary lattices (cf. [4], Prop. 11, for the indefinite case). As a special case of these formulas we obtain, in the classical language, a formula for the number of proper classes of positive definite integral quaternary forms of discriminant Δ_K.

By scaling, we may assume that $\mathfrak{I}$ contains the maximal integral lattices of V. When $D(V)$ is a nonsquare, there is a unique quaternion

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