ACTA ARITHMETICA
XXXIX (1981)

Dedekind-Rademacher sums and lattice points in triangles and tetrahedra

by

KENNETH H. ROSEN* (Boulder, Colo.)

1. Introduction. In this paper it will be shown how one may express the number of lattice points in certain triangles and tetrahedra in terms of Dedekind–Rademacher sums. One of the principal benefits of doing so is that the reciprocity law for Dedekind–Rademacher sums may be used in conjunction with the Euclidean algorithm to rapidly compute the exact number of lattice points in these triangles and tetrahedra. Formulae of a somewhat different nature for the number of lattice points in triangles and tetrahedra have been discovered by Ehrhart ([4], [5], [6], [7], [8]).

The results that appear below generalize the following beautiful result of Mordell ([14]; [18], pp. 39–44), which expresses the number of lattice points in certain tetrahedra in terms of classical Dedekind sums.

THEOREM 1.1. *Let p, q and r be three pairwise coprime integers, and let $N(p, q, r)$ denote the number of lattice points in the tetrahedron*

$$0 \leqslant x < p,\ 0 \leqslant y < q,\ 0 \leqslant z < r,\ 0 < \frac{x}{p} + \frac{y}{q} + \frac{z}{r} < 1.$$

Then

$$N(p, q, r) = \frac{1}{6}pqr + \frac{1}{4}(pq + qr + pr) + \frac{1}{4}(p + q + r) +$$
$$+ \frac{1}{12}\left(\frac{qr}{p} + \frac{pr}{q} + \frac{pq}{r}\right) + \frac{1}{12pqr} - 2 - \big(s(qr, p) + s(pr, q) + s(pq, r)\big).$$

It should be remarked that Hirzebruch [12] has used the curious connection between Dedekind sums and the topology of certain manifolds to provide another proof of Theorem 1.1. Also note that Carlitz [3] has generalized Theorem 1.1 to obtain three-term relations for generalized Dedekind–Rademacher sums.

* Supported by National Science Foundation Grant MCS77-02171.

In the last section of this paper it will be shown that a conjecture of Rademacher concerning the parity of the number of lattice points in certain higher dimensional tetrahedra is false.

2. Dedekind–Rademacher sums. Let h and k be relatively prime integers with k positive, and let x and y be real numbers. The Dedekind–Rademacher sum is defined to be

$$s(h,k;x,y) = \sum_{\mu \,(\mathrm{mod}\, k)} \left(\left(h\,\frac{\mu+y}{k}+x\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right) \tag{2.1}$$

where as usual

$$((x)) = \begin{cases} x-[x]-\frac{1}{2} & \text{if} \quad x \notin \mathbf{Z}, \\ 0 & \text{if} \quad x \in \mathbf{Z}. \end{cases}$$

It is easy to see that $s(h,k;x,y)$ is periodic with period one in both x and y. When x and y are both integers, the Dedekind–Rademacher sum $s(h,k;x,y)$ reduces to the classical Dedekind sum

$$s(h,k) = \sum_{\mu \,(\mathrm{mod}\, k)} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right). \tag{2.2}$$

An explicit formula proved in [17] may be given for Dedekind–Rademacher sums when $h=1$ and $x=0$.

LEMMA 2.3. *Let k be a positive integer and y a real number. Then*

$$s(1,k;0,y) = \begin{cases} \dfrac{k}{12}+\dfrac{1}{6k}-\dfrac{1}{4} & \text{if} \quad y \in \mathbf{Z}, \\[2ex] \dfrac{k}{12}+\dfrac{1}{k}\bar{B}_2(y) & \text{if} \quad y \notin \mathbf{Z} \end{cases}$$

where $\bar{B}_2(y) = (y-[y])^2-(y-[y])+\frac{1}{6}$, the second periodic Bernoulli function evaluated at y.

Apostol [1] has given the following easily derived explicit formula for $s(2,k)$.

LEMMA 2.4. *If k is odd,*

$$s(2,k) = \frac{(k-1)(k-5)}{24k}.$$

The fundamental fact about Dedekind–Rademacher sums is that they satisfy the following reciprocity law, proved by Rademacher [17].

THEOREM 2.5. *When x and y are not both integers*

$$s(h,k;x,y)+s(k,h;y,x)$$
$$= ((x))((y))+\frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(y)+\frac{k}{h}\bar{B}_2(x)+\frac{1}{hk}\bar{B}_2(hy+kx)\right\}.$$

However, when x and y are both integers the reciprocity law for classical Dedekind sums remains in effect:

$$s(h,k;x,y)+s(k,h;x,y) = s(h,k)+s(k,h) = -\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right).$$

The reciprocity law, together with the following lemma, also proved in [17], may be used to compute Dedekind–Rademacher sums, using the Euclidean algorithm.

LEMMA 2.6. *Let m be an integer. Then*

$$s(h,k;x,y) = s(h-mk,k;x+my,y).$$

Several properties of the function $((x))$ will be needed later. It is easy to see that $((x+1)) = ((x))$ and that $((-x)) = -((x))$. The following properties of $((x))$ are often useful.

LEMMA 2.7. *Let k be a positive integer and let w be a real number. Then*

$$\sum_{\mu \,(\mathrm{mod}\, k)} \left(\left(\frac{\mu+w}{k}\right)\right) = ((w)).$$

LEMMA 2.8. *Let n be a positive integer and x be a real number. Then*

$$\sum_{j \,(\mathrm{mod}\, n)} \left(\left(x+\frac{j}{n}\right)\right) = ((nx)).$$

A proo of Lemma 2.7 may be found in [19]. Lemma 2.8 follows from Lemma 2.7 by taking $k=n$ and $w=nx$.

In certain special cases, Dedekind–Rademacher sums may be expressed in terms of class cal Dedekind sums. The following result of this kind will be useful later.

LEMMA 2.9. *Let h and k be positive integers. Then*

$$s(h,k;h/2,0) = \begin{cases} s(h,k) & \text{if} \quad h \text{ is even}, \\ s(2h,k)-s(h,k) & \text{if} \quad h \text{ is odd}. \end{cases}$$

PROOF. If h is even, $s(h,k;h/2,0) = s(h,k)$ since $h/2$ is an integer. When h is odd, $s(h,k;h/2,0) = s(h,k;1/2,0)$. Now

$$s(h,k;\tfrac{1}{2},0)+s(h,k) = \sum_{\mu \,(\mathrm{mod}\, k)} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\left(\frac{h\mu}{k}\right)\right)+\left(\left(\frac{h\mu}{k}+\frac{1}{2}\right)\right)\right)$$
$$= \sum_{\mu \,(\mathrm{mod}\, k)} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{2h\mu}{k}\right)\right) = s(2h,k),$$

where the second equality is a consequence of Lemma 2.8.

3. Formulae for the number of lattice points in triangles. In this section a general formula involving Dedekind–Rademacher sums will be found for the number of lattice points inside certain triangles. Let C be a real number with $0 \leqslant C < 1$ and let p and q be relatively prime positive integers. Let $\mathscr{P} = (p, 0)$, $\mathscr{Q} = (0, q)$, $\mathscr{P}_C = (p(1-C), 0)$, $\mathscr{Q}_C = (0, q(1-C))$, and $\mathscr{O} = (0, 0)$. The lattice points (x, y) in the interior of the triangle $\mathscr{O}\mathscr{P}_C\mathscr{Q}_C$ and on its boundary, excluding points on the segment $\mathscr{P}_C\mathscr{Q}_C$, are precisely those pairs of integers (x, y) satisfying $0 \leqslant x < p$, $0 \leqslant y < q$, and $0 \leqslant x/p + y/q < 1-C$. Let $N(p, q; C)$ be the number of these lattice points; a formula will now be found for $N(p, q; C)$.

Let $F = x/p + y/q + C$. The points (x, y) to be counted are those for which $C \leqslant F < 1$. Note that for $0 \leqslant x < p$, $0 \leqslant y < q$, one has $C \leqslant F < 3$, since $C < 1$. To find a formula for $N(p, q; C)$ one starts with an identity similar to one used by Mordell to count lattice points in tetrahedra.

$$2N(p, q; C) = \sum_{x,y} ([F]-1)([F]-2) \tag{3.1}$$

where the summation is over the range of $0 \leqslant x < p$, $0 \leqslant y < q$.

Equality holds in (3.1) because the points being enumerated are those for which $[F] = 0$, while other points in the range give $[F] = 1$ or $[F] = 2$ and hence contribute zero to the sum. Since

$$[F] = \begin{cases} F - ((F)) - \frac{1}{2} & \text{if} \quad F \notin \mathbf{Z}, \\ F - ((F)) & \text{if} \quad F \in \mathbf{Z} \end{cases}$$

one has

$$\sum_{x,y} ([F]-1)([F]-2) \tag{3.2}$$
$$= \sum_{x,y} \left(F - ((F)) - \tfrac{3}{2}\right)\left(F - ((F)) - \tfrac{5}{2}\right) - \tfrac{7}{4}\delta_0 - \tfrac{3}{4}\delta_1 + \tfrac{1}{4}\delta_2$$

where δ_j is the number of solutions of $F = j \in \mathbf{Z}$. Note that $\delta_0 = \delta_1 = \delta_2 = 0$ if Cpq is not an integer.

From (3.1) and (3.2) one has

$$2N(p, q; C) + \tfrac{7}{4}\delta_0 + \tfrac{3}{4}\delta_1 - \tfrac{1}{4}\delta_2$$
$$= \sum_{x,y} (F - \tfrac{3}{2})(F - \tfrac{5}{2}) - 2\sum_{x,y} (F-2)((F)) + \sum_{x,y}' ((F))^2$$
$$= S_1 - 2S_2 + S_3.$$

Each of the sums S_1, S_2, and S_3 will now be evaluated. First consider S_1.

$$S_1 = \sum_{x,y} \left(\frac{x}{p} + \frac{y}{q} + C - \tfrac{3}{2}\right)\left(\frac{x}{p} + \frac{y}{q} + C - \tfrac{5}{2}\right)$$
$$= \sum_{x,y} \left(\frac{x^2}{p^2} + \frac{y^2}{q^2}\right) + 2\sum_{x,y} \frac{xy}{pq} + 2(C-2)\sum_{x,y}\left(\frac{x}{p} + \frac{y}{q}\right) + \sum_{x,y}' (C - \tfrac{3}{2})(C - \tfrac{5}{2})$$

$$= \frac{q(p-1)(2p-1)}{6p} + \frac{p(q-1)(2q-1)}{6q} + \frac{(p-1)(q-1)}{2} +$$
$$+ (C-2)\left((p-1)q + (q-1)p\right) + pq(C - \tfrac{3}{2})(C - \tfrac{5}{2})$$
$$= pq\left(C^2 - 2C + \frac{11}{12}\right) + (p+q)(1-C) + \frac{q}{6p} + \frac{p}{6q} + \frac{1}{2}.$$

Dedekind–Rademacher sums arise in the evaluation of S_2.

$$S_2 = \sum_{x,y} (F-2)((F))$$
$$= \sum_{x,y} \left(\frac{x}{p} - \frac{1}{2}\right)((F)) + \sum_{x,y} \left(\frac{y}{q} - \frac{1}{2}\right)((F)) + (C-1)\sum_{x,y} ((F)).$$

From Lemma 2.7 one has

$$\sum_{x,y} ((F)) = \sum_{x,y} \left(\left(\frac{x}{p} + \frac{y}{q} + C\right)\right) = \sum_{t=0}^{pq-1} \left(\left(\frac{t + Cpq}{pq}\right)\right) = ((Cpq)).$$

Using Lemma 2.8 and recalling definition (2.1),

$$\sum_{x,y} \left(\frac{x}{p} - \frac{1}{2}\right)((F)) = \sum_{x,y} \left(\left(\frac{x}{p}\right)\right)\left(\left(\frac{x}{p} + \frac{y}{q} + C\right)\right) - \tfrac{1}{2}\sum_{y} \left(\left(\frac{y}{q} + C\right)\right)$$
$$= \sum_{x} \left(\left(\frac{x}{p}\right)\right)\left(\left(\frac{qx}{p} + Cq\right)\right) - \tfrac{1}{2}((Cq)) = s(q, p; Cq, 0) - \tfrac{1}{2}((Cq)).$$

Likewise

$$\sum_{x,y} \left(\frac{y}{q} - \tfrac{1}{2}\right)((F)) = s(p, q; Cp, 0) - \tfrac{1}{2}((Cp)).$$

Hence

$$S_2 = s(q, p; Cq, 0) + s(p, q; Cp, 0) - \tfrac{1}{2}((Cp)) - \tfrac{1}{2}((Cq)) + (C-1)((Cpq)).$$

Finally to find S_3, write

$$S_3 = \sum_{x,y} \left(\left(\frac{x}{p} + \frac{y}{q} + C\right)\right)^2 = \sum_{x,y} \left(\left(\frac{qx + py}{pq} + C\right)\right)^2$$
$$= \sum_{t=0}^{pq-1} \left(\left(\frac{t + Cpq}{pq}\right)\right)^2 = s(1, pq; 0, Cpq)$$
$$= \begin{cases} \dfrac{pq}{12} + \dfrac{1}{6pq} - \dfrac{1}{4} & \text{if} \quad Cpq \in \mathbf{Z}, \\ \dfrac{pq}{12} + \dfrac{1}{pq}\bar{B}_2(Cpq) & \text{if} \quad Cpq \notin \mathbf{Z}, \end{cases}$$

using Lemma 2.3.

Consequently, by inserting the values of S_1, S_2, and S_3 into (3.3) one obtains

THEOREM 3.4. *Let p and q be relatively prime positive integers and let $0 \leqslant C < 1$. Then*

$$2N(p,q;C)+\tfrac{7}{4}\delta_0+\tfrac{3}{4}\delta_1-\tfrac{1}{4}\delta_2$$

$$=(1-C)^2pq+(1-C)(p+q)+\frac{q}{6p}+\frac{p}{6q}+\frac{1}{2}-$$

$$-2\big(s(p,q;Cp,0)+s(q,p;Cq,0)\big)+((Cp))+((Cq))+2(1-C)((Cpq))$$

$$+\begin{cases}\dfrac{1}{6pq}-\dfrac{1}{4} & \text{if} \quad Cpq\in \boldsymbol{Z},\\[2mm] \dfrac{1}{pq}\,\overline{B}_2(Cpq) & \text{if} \quad Cpq\notin \boldsymbol{Z}.\end{cases}$$

In the special case $C=0$ one has $\delta_0=1$ and $\delta_1=\delta_2=0$. So

$$2N(p,q;0)+\tfrac{7}{4}$$

$$=pq+p+q+\frac{q}{6p}+\frac{p}{6q}+\frac{1}{6pq}+\frac{1}{2}-2\big(s(p,q)+s(q,p)\big)-\frac{1}{4}$$

$$=pq+p+q+\tfrac{3}{4}.$$

The last equality is a consequence of the reciprocity law for Dedekind sums. Hence

$$N(p,q;0)=\frac{(p+1)(q+1)}{2}-1.$$

This is the expected value since $N(p,q;0)$ equals the number of lattice points inside or on the rectangle with vertices $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, and (p,q), and below the diagonal $\mathcal{P}\mathcal{Q}$. By the correspondence $x'=p-x$, $y'=q-y$, it is easy to see that the number of lattice points above the diagonal equals the number below the diagonal. Since $(p,q)=1$, there are exactly 2 lattice points, namely $\mathcal{P}$ and $\mathcal{Q}$, on the diagonal. Since there are $(p+1)(q+1)$ total lattice points inside or on the rectangle, one concludes

$$N(p,q;0)=\frac{(p+1)(q+1)}{2}-1.$$

The formula for $N(p,q;C)$ when $C=\frac{1}{2}$ simplifies considerably and will be needed later. In this case, there is another geometric interpretation of $N(p,q;C)$; it equals the number of lattice points with even coordinates inside the triangle $\mathcal{O}\mathcal{P}\mathcal{Q}$ or on its boundary excluding the segment $\mathcal{P}\mathcal{Q}$, i.e., $N(p,q;\frac{1}{2})$ is the number of solutions of $0\leqslant x/p+y/q<1$ with $0\leqslant x<p$ and $0\leqslant y<q$, with both x and y even. The values of $N(p,q;\frac{1}{2})$ are related to quadratic reciprocity and arise in topological investigations. See the work of Hirzebruch and Mayer [13] for a discussion of these topics.

A consequence of Theorem 3.4 is

COROLLARY 3.5. *If p is odd and q is even and if $(p,q)=1$, then*

$$N(p,q;\tfrac{1}{2})=\frac{pq}{8}+\frac{p+q}{4}-s(2p,q)+2s(p,q).$$

If p and q are both odd and if $(p,q)=1$, then

$$N(p,q;\tfrac{1}{2})=\frac{pq}{8}+\frac{p+q}{4}+\frac{q}{6p}+\frac{p}{6q}+\frac{1}{24pq}-s(2p,q)-s(2q,p).$$

Proof. When p is odd and q is even, $\delta_0=\delta_2=0$, but $\delta_1=1$, and $\frac{1}{2}pq\in\boldsymbol{Z}$. Hence from Theorem 3.4,

$$2N(p,q;\tfrac{1}{2})=\frac{pq}{4}+\frac{p+q}{2}+\frac{q}{6p}+\frac{p}{6q}+\frac{1}{2}- \tag{3.6}$$

$$-2\left(s\left(p,q;\frac{p}{2},0\right)+s\left(q,p;\frac{q}{2},0\right)\right).$$

But from Lemma 2.9 and from Dedekind sum reciprocity

$$s(p,q;p/2,0)+s(q,p;q/2,0)=s(2p,q)-s(p,q)+s(q,p) \tag{3.7}$$

$$=s(2p,q)-2s(p,q)-\frac{1}{4}+\frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{1}{pq}\right).$$

The desired formula is obtained by substituting (3.7) into (3.6) and then dividing by 2.

When both p and q are odd, $\delta_0=\delta_1=\delta_2=0$ and $\frac{1}{2}pq\notin\boldsymbol{Z}$. Note $\overline{B}_2(\frac{1}{2}pq)=\overline{B}_2(\frac{1}{2})=-\frac{1}{12}$. Hence Theorem 3.4 implies

$$2N(p,q;\tfrac{1}{2})=\frac{pq}{4}+\frac{p+q}{2}+\frac{q}{6p}+\frac{p}{6q}+\frac{1}{2}-\frac{1}{12pq}- \tag{3.8}$$

$$-2\left(s\left(p,q;\frac{p}{2},0\right)+s\left(q,p;\frac{q}{2},0\right)\right).$$

From Lemma 2.9 and using the reciprocity law for Dedekind sums one has

$$s\left(p,q;\frac{p}{2},0\right)+s\left(q,p;\frac{q}{2},0\right)=s(2p,q)-s(p,q)+s(2q,p)-s(q,p) \tag{3.9}$$

$$=s(2p,q)+s(2q,p)+\frac{1}{4}-\frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{1}{pq}\right).$$

The desired formula is obtained by substitution of (3.9) into (3.8), and upon division by 2. ∎

In special cases, the Dedekind sums occurring in the formula for $N(p,q;\frac12)$ may be explicitly evaluated. For example one has

COROLLARY 3.10. *If $p\equiv 1 \pmod q$ and p is odd, then*

$$N(p,q;\tfrac12)=\frac{(p+3)(q+2)}{8}+\begin{cases}-1 & \text{if } q \text{ is even},\\ \dfrac{p-1}{8q}-\dfrac12 & \text{if } q \text{ is odd}.\end{cases}$$

Proof. If q is even, Corollary 3.5 implies

$$N(p,q;\tfrac12)=\frac{pq}{8}+\frac{p+q}{4}-s(2p,q)+2s(p,q). \tag{3.11}$$

Using Lemma 2.3 one has

$$s(2p,q)=s(2,q)=s\left(1,\frac q2\right)=\frac{(q-2)(q-4)}{24q} \tag{3.12}$$

and

$$s(p,q)=s(1,q)=\frac{(q-1)(q-2)}{12q}. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) yields the desired formula for even q.

For q odd, Corollary 3.5 gives

$$N(p,q;\tfrac12)=\frac{pq}{8}+\frac{p+q}{4}+\frac{q}{6p}+\frac{p}{6q}+\frac{1}{24pq}-s(2p,q)-s(2q,p). \tag{3.14}$$

Lemma 2.4 shows that

$$s(2p,q)=s(2,q)=\frac{(q-1)(q-5)}{24q}. \tag{3.15}$$

From Dedekind sum reciprocity and from Lemma 2.5, since $p\equiv 1 \pmod{2q}$,

$$\begin{aligned} s(2q,p) &= -\frac14+\frac1{12}\left(\frac{p}{2q}+\frac{2q}{p}+\frac{1}{2pq}\right)-s(p,2q)\\ &= -\frac14+\frac1{12}\left(\frac{p}{2q}+\frac{2q}{p}+\frac{1}{2pq}\right)-s(1,2q)\\ &= -\frac14+\frac1{12}\left(\frac{p}{2q}+\frac{2q}{p}+\frac{1}{2pq}\right)-\frac{(2q-1)(q-1)}{12q}. \end{aligned} \tag{3.16}$$

Substituting the values found for $s(2p,q)$ and $s(2q,p)$ into (3.14) gives the result for odd q.

4. Formulae for the number of lattice points in tetrahedra. In this section formulae for the number of lattice points in certain tetrahedra will be found. Let p, q, and r be pairwise coprime positive integers and let C be a non-negative real number with $C<\frac1p+\frac1q+\frac1r$. Let $N(p,q,r;C)$ denote the number of lattice points inside or on the tetrahedron bounded by the three coordinate planes and by the plane $\frac xp+\frac yq+\frac zr=1-C$, excluding points on the face where $\frac xp+\frac yq+\frac zr=1-C$. Hence $N(p,q,r;C)$ equals the number of solutions in integers x, y, and z of the system of inequalities $0\leqslant x<p$, $0\leqslant y<q$, $0\leqslant z<r$ and $0\leqslant \frac xp+\frac yq+\frac zr+C<1$. Let $E=\frac xp+\frac yq+\frac zr+C$. Then

$$2N(p,q,r;C)=\sum_{x,y,z}([E]-1)([E]-2) \tag{4.1}$$

where the summation is over the set of (x,y,z) with $0\leqslant x<p$, $0\leqslant y<q$, and $0\leqslant z<r$. Since $0\leqslant C<\frac1p+\frac1q+\frac1r$, one has $0\leqslant E<3$, and so the only points which give a nonzero contribution to the sum are those points for which $0\leqslant E<1$, and each of these contribute 2 to the sum.

To evaluate this sum, note that

$$\begin{aligned} &\sum_{x,y,z}([E]-1)([E]-2)\\ &\quad=\sum_{x,y,z}\left(E-((E))-\tfrac32\right)\left(E-((E))-\tfrac52\right)-\tfrac74\delta_0-\tfrac34\delta_1+\tfrac14\delta_2 \end{aligned} \tag{4.2}$$

where δ_j equals the number of solutions of $E=j$. Note that $\delta_0=\delta_1=\delta_2=0$ if $Cpqr$ is not an integer.

To compute $N(p,q,r;C)$ it is necessary to evaluate

$$\begin{aligned} S &= \sum_{x,y,z}\left(E-((E))-\tfrac32\right)\left(E-((E))-\tfrac52\right)\\ &= \sum_{x,y,z}(E-\tfrac32)(E-\tfrac52)-2\sum_{x,y,z}(E-2)((E))+\sum_{x,y,z}((E))^2\\ &= S_1-2S_2+S_3. \end{aligned} \tag{4.3}$$

The three sums S_1, S_2, and S_3 will now be considered.

$$S_1 = \sum_{x,y,z}(E-\tfrac{3}{2})(E-\tfrac{5}{2}) = \sum_{x,y,z}\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+C-\frac{3}{2}\right)\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+C-\frac{5}{2}\right)$$

$$= \sum_{x,y,z}\left(\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}\right)+2\sum_{x,y,z}\left(\frac{xy}{pq}+\frac{xz}{pr}+\frac{yz}{qr}\right)+$$

$$+(2C-4)\sum_{x,y,z}\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)+\sum_{x,y,z}(C-\tfrac{3}{2})(C-\tfrac{5}{2})$$

$$= \frac{qr(p-1)(2p-1)}{6p}+\frac{pr(q-1)(2q-1)}{6q}+\frac{pq(r-1)(2r-1)}{6r}+$$

$$+\frac{(p-1)(q-1)r}{2}+\frac{(p-1)(r-1)q}{2}+\frac{(q-1)(r-1)p}{2}+$$

$$+(C-2)\big((p-1)qr+(q-1)pr+(r-1)pq\big)+pqr(C-\tfrac{3}{2})(C-\tfrac{5}{2})$$

$$= \left(C^2-C+\frac{1}{4}\right)pqr+\left(\frac{1}{2}-C\right)(pq+qr+pr)+$$

$$+\frac{1}{2}(p+q+r)+\frac{1}{6}\left(\frac{pr}{q}+\frac{pq}{r}+\frac{qr}{p}\right).$$

The sum S_2 is the term where Dedekind–Rademacher sums arise.

$$S_2 = \sum_{x,y,z}(E-2)((E)) = \sum_{x,y,z}\left(\frac{x}{p}-\frac{1}{2}\right)((E))+\sum_{x,y,z}\left(\frac{y}{q}-\frac{1}{2}\right)((E))+$$

$$+\sum_{x,y,z}\left(\frac{z}{r}-\frac{1}{2}\right)((E))-\frac{1}{2}\sum_{x,y,z}((E)).$$

First from Lemma 2.6,

$$\sum_{x,y,z}((E)) = \sum_{x,y,z}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+C\right)\right)$$

$$= \sum_{x,y,z}\left(\left(\frac{qrx+pry+pqz}{pqr}+C\right)\right)$$

$$= \sum_{t=0}^{pqr-1}\left(\left(\frac{t+Cpqr}{pqr}\right)\right) = ((Cpqr)).$$

Next note that applying Lemma 2.8 shows that

$$\sum_{x,y,z}\left(\frac{x}{p}-\frac{1}{2}\right)((E)) = \sum_{x,y,z}\left(\left(\frac{x}{p}\right)\right)\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+C\right)\right)-\tfrac{1}{2}\sum_{y,z}\left(\left(\frac{y}{q}+\frac{z}{r}+C\right)\right)$$

$$= \sum_{x,y}\left(\left(\frac{x}{p}\right)\right)\left(\left(\frac{rx}{p}+\frac{ry}{q}+Cr\right)\right)-\tfrac{1}{2}\sum_{y}\left(\left(\frac{ry}{q}+Cr\right)\right)$$

$$= \sum_{x}\left(\left(\frac{x}{p}\right)\right)\left(\left(\frac{qrx}{p}+Cqr\right)\right)-\tfrac{1}{2}((Cqr))$$

$$= s(qr,p;Cqr,0)-\tfrac{1}{2}((Cqr)).$$

The other two terms are computed in a similar manner. One finds that

$$S_2 = s(qr,p;Cqr,0)+s(pr,q;Cpr,0)+s(pq,r;Cpq,0)-$$
$$-\tfrac{1}{2}((Cpq))-\tfrac{1}{2}((Cpr))-\tfrac{1}{2}((Cqr))-\tfrac{1}{2}((Cpqr)).$$

Finally, to evaluate S_3 one uses Lemma 2.3.

$$S_3 = \sum_{x,y,z}((E))^2 = \sum_{x,y,z}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+C\right)\right)^2$$

$$= \sum_{x,y,z}\left(\left(\frac{qrx+pry+pqz}{pqr}+C\right)\right)^2 = \sum_{t=0}^{pqr-1}\left(\left(\frac{t+Cpqr}{pqr}\right)\right)^2$$

$$= s(1,pqr;0,Cpqr) = \begin{cases}\dfrac{pqr}{12}+\dfrac{1}{6pqr}-\dfrac{1}{4} & \text{if}\quad Cpqr\in\mathbf{Z},\\[2ex] \dfrac{pqr}{12}+\dfrac{1}{pqr}\bar{B}_2(Cpqr) & \text{if}\quad Cpqr\notin\mathbf{Z}.\end{cases}$$

Substituting the values found for S_1, S_2, and S_3 into (4.3) and using (4.1) and (4.2) yields

THEOREM 4.4. *If p, q, and r are pairwise coprime positive integers and if $0\leqslant C<\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$, then*

$$2N(p,q,r;C)+\tfrac{7}{4}\delta_0+\tfrac{3}{4}\delta_1-\tfrac{1}{4}\delta_2$$

$$= (C^2-C+\tfrac{1}{3})pqr+(\tfrac{1}{2}-C)(pq+qr+pr)+\tfrac{1}{2}(p+q+r)+\tfrac{1}{6}\left(\frac{pr}{q}+\frac{pq}{r}+\right.$$

$$\left.+\frac{qr}{p}\right)-2\big(s(qr,p;Cqr,0)+s(pr,q;Cpr,0)+s(pq,r;Cpq,0)\big)+$$

$$+((Cpq))+((Cpr))+((Cqr))+((Cpqr))+\begin{cases}\dfrac{1}{6pqr}-\dfrac{1}{4} & \textit{if}\quad Cpqr\in\mathbf{Z},\\[2ex] \dfrac{1}{pqr}\bar{B}_2(Cpqr) & \textit{if}\quad Cpqr\notin\mathbf{Z}.\end{cases}$$

When $C = 0$, Theorem 4.4 becomes

$$2N(p,q,r;0)+\tfrac{7}{4} = \tfrac{1}{3}pqr+\tfrac{1}{2}(pq+qr+pr)+\tfrac{1}{2}(p+q+r)+$$
$$+\frac{1}{6}\left(\frac{pr}{q}+\frac{pq}{r}+\frac{qr}{p}\right)-2\big(s(qr,p)+s(pr,q)+s(pq,r)\big)+\frac{1}{6pqr}-\frac{1}{4}.$$

This agrees with Mordell's result, Theorem 1.1, since the origin is included in $N(p,q,r;0)$ but not in Mordell's enumeration $N(p,q,r)$.

The special case of $C = \frac{1}{2}$ will now be considered. Note that $N(p,q,r;\frac{1}{2})$ equals the number of lattice points (x,y,z) with all three of x, y, and z even inside or on the tetrahedron described by the inequalities $0 \leqslant x < p$, $0 \leqslant y < q$, $0 \leqslant z < r$ and $0 \leqslant \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 1$, excluding points on the face $\frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 1$. When $C = \frac{1}{2}$, the restriction $C < \frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ may be lifted. Consider the sum

$$S = \sum_{x,y,z}([E]-1)([E]-2) \tag{4.5}$$

where $E = \frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{1}{2}$.

The $N = N(p,q,r;\frac{1}{2})$ points with $\frac{1}{2} \leqslant E < 1$ and the N' points with $3 < E < \frac{7}{2}$ each contribute 2 to S, whereas points with $1 \leqslant E < 3$ contribute zero to S; there are never any points with $E = 2$ or $E = 3$. Hence

$$S = 2N+2N'. \tag{4.6}$$

Let N'' be the number of lattice points with $0 \leqslant x < p$, $0 \leqslant y < q$, $0 \leqslant z < r$ satisfying $0 \leqslant \frac{x}{p}+\frac{y}{q}+\frac{z}{p} < \frac{1}{2}$ with $xyz = 0$. Then excluding these N'' points there is a one-to-one correspondence between the remaining $N-N''$ points with $\frac{1}{2} \leqslant E < 1$ and the N' points with $3 < E < \frac{7}{2}$, given by $x' = p-x$, $y' = q-y$, $z' = r-z$. Hence

$$N' = N-N''. \tag{4.7}$$

Clearly $N(p,q;\frac{1}{2})+N(p,r;\frac{1}{2})+N(q,r;\frac{1}{2})$ equals the number of lattice points with $\frac{1}{2} \leqslant E < 1$ and with exactly one coordinate zero, plus twice the number of lattice points with $\frac{1}{2} \leqslant E < 1$ and with exactly two coordinates zero, plus 3 since the origin is counted thrice. Therefore

$$N'' = N(p,q;\tfrac{1}{2})+N(p,r;\tfrac{1}{2})+N(q,r;\tfrac{1}{2})-\left[\frac{p-1}{2}\right]-\left[\frac{q-1}{2}\right]-\left[\frac{r-1}{2}\right]-2. \tag{4.8}$$

Consequently from (4.5), (4.6), (4.7), and (4.8), it follows that

$$\begin{aligned} S &= 4N(p,q,r;\tfrac{1}{2})-2N'' \\ &= 4N(p,q,r;\tfrac{1}{2})-2N(p,q;\tfrac{1}{2})-2N(p,r;\tfrac{1}{2})-2N(q,r;\tfrac{1}{2})+ \\ &\qquad +2\left[\frac{p-1}{2}\right]+2\left[\frac{q-1}{2}\right]+2\left[\frac{r-1}{2}\right]+4. \end{aligned} \tag{4.9}$$

As before one finds from (4.3) and the previous evaluations of S_1, S_2, and S_3 that

$$\begin{aligned} S+\tfrac{3}{4}\delta_1 &= \tfrac{1}{12}pqr+\tfrac{1}{2}(p+q+r)+\tfrac{1}{6}(pr/q+pq/r+qr/p)- \\ &\quad -2\big(s(qr,p;qr/2,0)+s(pr,q;pr/2,0)+s(pq,r;pq/2,0)\big)+ \\ &\quad +\begin{cases} \dfrac{1}{6pqr}-\dfrac{1}{4} & \text{if } pqr \text{ is even} \\ -\dfrac{1}{12pqr} & \text{if } pqr \text{ is odd.} \end{cases} \end{aligned} \tag{4.10}$$

The fact that $\bar{B}_2(pqr/2) = \bar{B}_2(\frac{1}{2}) = -\frac{1}{12}$ if pqr is odd has been used.

The case where pqr is even will be considered first. Since p, q, and r are coprime, at most one of p, q, and r is even. Assume p is even and q and r are odd. Then $\delta_1 = 1$. Lemma 2.9 implies

$$\begin{aligned} &s(qr,p;qr/2,0)+s(pr,q;pr/2,0)+s(pq,r;pq/2,0) \\ &\qquad = s(2qr,p)-s(qr,p)+s(pr,q)+s(pq,r). \end{aligned} \tag{4.11}$$

One also finds from Corollary 3.5 that

$$\begin{aligned} N(p,q;\tfrac{1}{2}) &= \frac{pq}{8}+\frac{p+q}{4}-s(2q,p)+2s(q,p), \\ N(p,r;\tfrac{1}{2}) &= \frac{pr}{8}+\frac{p+r}{4}-s(2r,p)+2s(r,p), \\ N(q,r;\tfrac{1}{2}) &= \frac{qr}{8}+\frac{q+r}{4}+\frac{q}{6r}+\frac{r}{6q}+\frac{1}{24qr}- \\ &\qquad -s(2q,r)-s(2r,q). \end{aligned} \tag{4.12}$$

When p, q, and r are all odd, one has $\delta_1 = 0$. From Lemma 2.9 it follows that

$$\begin{aligned} &s(qr,p;qr/2,0)+s(pr,q;pr/2,0)+s(pq,r;pq/2,0) \\ &\qquad = s(2qr,p)-s(qr,p)+s(2pr,q)-s(pr,q)+s(2pq,r)-s(pq,r). \end{aligned} \tag{4.13}$$

From Corollary 3.5 one has

(4.14)
$$N(p,q;\tfrac{1}{2}) = \frac{pq}{8}+\frac{p+q}{4}+\frac{p}{6q}+\frac{q}{6p}+\frac{1}{24pq}-s(2p,q)-s(2q,p),$$
$$N(p,r;\tfrac{1}{2}) = \frac{pr}{8}+\frac{p+r}{4}+\frac{p}{6r}+\frac{r}{6p}+\frac{1}{24pr}-s(2p,r)-s(2r,p),$$
$$N(q,r;\tfrac{1}{2}) = \frac{qr}{8}+\frac{q+r}{4}+\frac{q}{6r}+\frac{r}{6q}+\frac{1}{24qr}-s(2q,r)-s(2r,q).$$

Hence from (4.9)–(4.14) one concludes

THEOREM 4.15. *Let p, q, r be pairwise coprime positive integers. Then if p is even and q and r are odd*

$$\begin{aligned}(4.16)\quad N(p,q,r;\tfrac{1}{2}) = {} & \frac{1}{48}pqr+\frac{1}{16}(pq+qr+pr)+\frac{1}{8}(p+q+r)+\\ & +\frac{1}{24}\left(\frac{pq}{r}+\frac{pr}{q}+\frac{qr}{p}\right)+\frac{1}{12}\left(\frac{q}{r}+\frac{r}{q}\right)+\frac{1}{48qr}+\\ & +\frac{1}{24pqr}-\frac{1}{4}-\frac{1}{2}\big(s(2qr,p)-s(qr,p)+\\ & +s(pr,q)+s(pq,r)+s(2q,p)-2s(q,p)+\\ & +s(2r,p)-2s(r,p)+s(2q,r)+s(2r,q)\big).\end{aligned}$$

If p, q, and r are all odd, then

$$\begin{aligned}(4.17)\quad N(p,q,r;\tfrac{1}{2}) = {} & \frac{1}{48}pqr+\frac{1}{16}(pq+qr+pr)+\frac{1}{8}(p+q+r)+\\ & +\frac{1}{24}\left(\frac{pq}{r}+\frac{pr}{q}+\frac{qr}{p}\right)+\frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{q}{r}+\right.\\ & \left.+\frac{r}{q}+\frac{p}{r}+\frac{r}{p}\right)+\frac{1}{48}\left(\frac{1}{pq}+\frac{1}{pr}+\frac{1}{qr}\right)-\\ & -\frac{1}{48pqr}+\frac{1}{4}-\frac{1}{2}\big(s(2qr,p)+s(2pr,q)+\\ & +s(2pq,r)-s(qr,p)-s(pr,q)-s(pq,r)+\\ & +s(2p,q)+s(2q,p)+s(2p,r)+s(2r,p)+\\ & +s(2q,r)+s(2r,q)\big).\end{aligned}$$

In special cases, one can find a formula for $N(p,q,r;\frac{1}{2})$ not involving Dedekind sums. For instance, when p is even, $q = p-1$, and $r = p+1$, one can find a formula for $N(p,q,r;\frac{1}{2})$, depending on the residue of p modulo 4, which is a rational function of p. Indeed, this formula arises by explicitly evaluating the Dedekind sums that occur in (4.16), by means of Lemmata 2.3 and 2.4, and by using the reciprocity law for Dedekind sums to find a formula for $s(4,k)$, which depends on the residue of k modulo 4.

5. A conjecture of Rademacher. At the 1963 Number Theory Conference in Boulder, Rademacher [15] made the following conjecture.

Let $\{a_j\}$, $j = 1, 2, \ldots, n$, be pairwise coprime positive integers and let $N_m(a_j)$ be the number of solutions of

$$0 < \sum_{j=1}^{m}\frac{x_j}{a_j} < 1,$$

in non-negative integers x_j. Then

$$(5.1)\qquad N_m(a_j) \equiv \frac{1}{2^{m-1}}\prod_{j=1}^{m}(a_j+1) \pmod 2.$$

This conjecture is true for $m = 1, 2$, and 3. The case of $m = 3$ has been proved by Rademacher [16], who started with Mordell's Theorem 1.1 and used congruences for Dedekind sums and by Артюхов [2], whose proof is elementary and geometric. The author [21] also has used congruence (5.1) for $m = 3$ to give an elementary solution of a problem posed by Hasse [9], [10] concerning the parity of the number of lattice points in certain tetrahedra. This problem was solved by non-elementary methods by Hirzebruch [11]. Hasse needed this result in his investigations of class numbers of abelian number fields.

For $m \geq 4$, however, Rademacher's conjecture is false. Indeed, for $m = 4$, let $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, and $a_4 = 7$. Then

$$N_4(a_j) = 73 \not\equiv \tfrac{1}{8}(4\cdot 5\cdot 6\cdot 8) \equiv 0 \pmod 2.$$

A counterexample for $m > 4$ is provided by $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, $a_4 = 7$, and $a_j = 1$ for $j > 4$. Then

$$N_m(a_j) = 73 \not\equiv \frac{1}{2^{m-1}}(4\cdot 5\cdot 6\cdot 8\cdot 2^{m-4}) \equiv 0 \pmod 2.$$

Exactly how the parity of $N_m(a_j)$ depends on the a_j is still undetermined. The author will examine the parity of the number of lattice points in four dimensional tetrahedra further in another paper [11].

Acknowledgement. The author wishes to thank the referree for his careful reading of the manuscript and his helpful suggestions.

References

[1] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Springer-Verlag, New York 1976.

[2] M. M. Артюхов, *К вопросу о числе целых точек в некоторых тетраэдрах* (*On the number of lattice points in a certain tetrahedra*), Мат. сб. 57 (1962), pp. 3–12.

[3] L. Carlitz, *A reciprocity theorem and a three-term relation for generalized Dedekind–Rademacher sums*, Acta Arith. 37 (1980), pp. 117–132.

[4] E. Ehrhart, *Nombre de points entiers d'un tétraèdre*, C. R. Acad. Sci. Paris, 258 (1964), pp. 3945–3948.

[5] — *Polynômes arithmétiques et Méthode de Polyèdres en Combinatoire*, Birkhäuser Verlag, Basel 1977.

[6] — *Sur le nombre de solutions non négatives d'une équation diophantienne lineaire*, C. R. Acad. Sci. Paris, 256 (1963), pp. 4566–4569.

[7] — *Sur le nombre de solutions non négatives d'une équation diophantienne lineaire*, ibid. 256 (1963), pp. 4801–4803.

[8] — *Sur les points entiers d'un tétraèdre*, ibid. 258 (1964), pp. 4182–4183.

[9] H. Hasse, *Aufgabe 327*, Jber. Deutsch. Math.-Verein. 53 (1943), p. 241.

[10] — *Über die Klassenzahl abelschen Zahlkörper*, Akademie-Verlag, Berlin 1952.

[11] F. Hirzebruch, *Lösung der Aufgabe 327*, Jber. Deutsch. Math.-Verein. 72 (1971), pp. 29–32.

[12] — *The signature theorem: reminiscences and recreation*, Prospects in Mathematics, Annals of Mathematics Studies, 70, pp. 3–31, Princeton University Press, Princeton, New Jersey 1971.

[13] F. Hirzebruch and K. H. Mayer, *O(n)-Mannigfaltigkeiten exotische Sphären und Singularitaten*, Lecture Notes in Mathematics 57, Springer-Verlag, Berlin 1968.

[14] L. J. Mordell, *Lattice points in a tetrahedron and generalized Dedekind sums*, Journ. Indian Math. Soc. 15 (1951), pp. 41–46.

[15] *Proceedings of the 1963 Number Theory Conference*, University of Colorado, Boulder, Colorado, 1963, *Problem 98*; *Collected Papers of Hans Rademacher*, MIT Press, Cambridge, MA. Vol. II, p. 626.

[16] H. Rademacher, *On Dedekind sums and lattice points in a tetrahedron*, Studies in Mathematics and Mechanics, Presented to Richard von Mises, Academic Press, New York 1954, pp. 49–53; *Collected Papers of Hans Rademacher*, Vol. II, pp. 391–395.

[17] — *Some remarks on certain generalized Dedekind sums*, Acta Arith. 9 (1964), pp. 97–105; *Collected Papers of Hans Rademacher*, Vol. II, pp. 534–542.

[18] H. Rademacher and E. Grosswald, *Dedekind Sums*, The Carus Mathematical Monographs, Number 16, Mathematical Association of America, Washington, D. C., 1972.

[19] H. Rademacher and A. L. Whiteman, *Theorems on Dedekind Sums*, Amer. J. Math. 63 (1941), pp. 377–407; *Collected Papers of Hans Rademacher*, Vol. II, pp. 220–250.

[20] K. H. Rosen, *Lattice points in four-dimensional tetrahedra and a conjecture of Rademacher* J. Reine Angew. Math. 307/308 (1979), pp. 264–275.

[21] — *On the parity of the number of lattice points in certain tetrahedra*, ibid. 307/308 (1979), pp. 276–278.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO
Boulder, Colorado 80309, U.S.A.

Received on 11. 5. 1978
and in revised form on 23. 11. 1978 (1069)