Principe de la démonstration: Soit Θ la suite construite dans la preuve du théorème 1. Comme il est remarqué dans [1], on peut supposer que φ est non décroissante. On applique alors à Θ l'algorithme décrit dans [1], pour construire U par „blocs".

Soit $m_0 = \operatorname{Inf}\{n \in N \mid \varphi(n) \geqslant |\theta_0|\}$. Pour $n < m_0$, on pose $u_n = 0$.

Soit $m_1 = \operatorname{Inf}\{n \in N \mid \varphi(n) \geqslant |\theta_1|$ et $n \geqslant m_0+1\}$. Pour $n \in \{m_0, \ldots, m_1-1\}$, on pose $u_n = \theta_0$.

On définit, par récurrence sur k, la suite (m_k) suivante:

$$\forall k \geqslant 1, \quad m_{k+1} = \operatorname{Inf}\{n \in N \mid \varphi(n) \geqslant |\theta_{k+1}| \text{ et } n \geqslant (1+k)m_k \text{ et } n - m_k \equiv 0 \bmod k+1\}$$

de sorte que (m_k) est strictement croissante.

On pose, pour $n \in \{m_k, \ldots, m_{k+1}-1\}$,

$$u_n = \theta_j$$

où j est le reste de la division euclidienne de $n-m_k$ par $k+1$.

Il est clair que $U = (u_n)$ satisfait à la condition 1) du théorème 2.

Le fait que $x\Theta$ et xU soient complètement équiréparties ou non équiréparties pour les mêmes valeurs de x est démontré dans [1].

IV.2. La méthode utilisée pour démontrer le théorème 1 se généralise à la répartition selon une mesure μ dans un compact S métrisable ([3]).

Addendum: le théorème 1 a été amélioré par l'auteur dans un article paru aux Annales de la Faculté des Sciences de Toulouse (vol. 2, 1980, p. 137–155).

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Integers without large prime factors, II

by

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1. Let a and q be relatively prime positive integers. For m a positive integer, let $P(m)$ (respectively $Q(m)$) denote its largest (resp. smallest) prime factor. This paper is concerned with the sum

$$S = S(X, Y, a, q) = \sum_{\substack{m \leqslant X \\ m \equiv a \pmod q \\ P(m) \leqslant Y}} 1,$$

where $X \geqslant Y \geqslant 2$.

In the case where q is less than a fixed power of $\log Y$, fairly precise results are known (see [8]) and, in particular, if $q = 1$ and $u = \frac{\log X}{\log Y} < (\log Y)^{1/2}$ (this restriction can be weakened somewhat), then $S \sim X\varrho(u)$, where $\varrho(u)$ is the Dickman function (see [1] for details).

The situation is less satisfactory in the case where q is larger, say a fixed power of Y. In [4] a result was given which yielded a non-trivial lower bound provided that $\beta = \log Y/\log q$ was fixed, exceeding $\sqrt{e}/(\sqrt{e}-1)$.

In this paper we give the following upper bound.

Theorem. *If $X \geqslant q^2 Y^5$ and $u \leqslant (\log Y)^{1/4}$, then*

$$\sum_{\substack{m \leqslant X \\ m \equiv a \pmod q \\ P(m) \leqslant Y}} 1 \ll \frac{X}{q} \varrho(u - \beta^{-1} - 4).$$

The constants $2, 5, \frac{1}{4}$ and 4 are all susceptible to improvement. Unlike the result in [4], the above estimate remains nontrivial when $Y < q$.

An estimate in [10], stated without proof, is in fact superior to the above. Unfortunately, the method of proof in that paper is flawed and, barring a completely new idea, seems incapable of being modified to give the stated results. Our method of proof does, however, owe its origin to one of the ideas in that paper as well as to the earlier work [7] of Hmyrova.

One interest of the sum S is due to its connection with the problem of the least kth power non-residue. For $k \geqslant 2$ an integer, and $q \equiv 1 \pmod{k}$ a prime, let $g_k(q)$ be the least kth power nonresidue (mod q).

Defining $a_k = \overline{\lim_{q\to\infty}} \dfrac{\log g_k(q)}{\log q}$, the conjecture of Vinogradov states that $a_k = 0$ for each k. The result [9] of Vinogradov that $a_k \ll \dfrac{\log\log k}{\log k}$ remains, to this day, essentially the best known estimate, subsequent improvements by Davenport and Erdös [3] and by Burgess [2], important as they are, affecting only the value of the implied constant.

A connection between this problem and the sum S can be described as follows. For a given prime $q \equiv 1 \pmod{k}$, let X be some fixed power of q and let $Y < g_k(q)$. Let $\bar{S}$ be an upper bound for S which holds for each a prime to q. The sum S vanishes for each a other than the $(q-1)/k$ classes which are kth power residues. Thus

$$\psi(X, Y) = \sum_{\substack{m\leqslant X\\ P(m)\leqslant Y}} 1 < \frac{q}{k}\bar{S}.$$

This inequality, when combined with well-known lower bounds for $\psi(X, Y)$, leads to an upper bound for Y. Using this idea in conjunction with the bound for S provided by the theorem gives only another proof that $a_k \ll \dfrac{\log\log k}{\log k}$. A refinement of the theorem sufficient to improve this latter estimate appears to lie rather deep. The result stated in [10], if true, would yield $a_k \ll \dfrac{\log\log\log k}{\log k}$.

2. Lemma. *Let* $\psi_q(X, Y) = \sum_{\substack{m\leqslant X\\ P(m)\leqslant Y\\ (m,q)=1}} 1.$

(A) *For* $\log^2 X \leqslant Y \leqslant X$,

$$\psi_q(X, Y) \ll X\log^2 X \exp\big(-u\log u + O(u\log\log u)\big).$$

(B) *For* $Y < \log^2 X$,

$$\psi_q(X, Y) \leqslant \psi_q(X, \log^2 X).$$

(C) *For* $q \leqslant X$, $u \leqslant (\log Y)^{2/7}$,

$$\psi_q(X, Y) \ll \frac{\varphi(q)}{q} X\varrho(u).$$

Proof. (A) is an immediate consequence of a result of de Bruijn, (1.6 of [1]), and (B) is a triviality.

(C) is a weak consequence of a result due to Hazlewood (2.26 of [6]) by which we have

$$\psi_q(X, Y) = \frac{\varphi(q)}{q} X\varrho(u) + O\big(X(\nu(q)+1)(\log Y)^M \exp(-\sqrt{\log Y})\big) + \\ + O(X^{1/2}2^{\nu(q)}) + O\left(\frac{X}{\log Y}\frac{\sigma(q)}{q}\varrho(u-2)\right),$$

where M is absolute and φ, ν, σ each have their usual meaning.

Since, for $u \geqslant 1$,

$$u\varrho(u) = \int_{u-1}^{u} \varrho(t)\,dt > \int_{u-1}^{u-2/3} \varrho(t)\,dt \geqslant \tfrac{1}{3}\varrho(u-\tfrac{2}{3}),$$

it follows that $\varrho(u) \gg u^{-3}\varrho(u-2)$.

Using the well-known estimates,

$$\nu(q) \ll \frac{\log 3q}{\log\log 3q}, \quad \frac{\varphi(q)}{q} \gg (\log\log 3q)^{-1} \quad \text{and} \quad \frac{\sigma(q)}{q} \ll \log\log 3q,$$

(C) follows.

3. Proof of the theorem. Let $V = Xq^{-1}Y^{-4}$. For bounded V the result is trivial and we shall make use of the fact that V can be taken to be large.

For $m > V$ contributing to S, we may write $m = m_1 m_2$ where

$$Q(m_2) \geqslant P(m_1) \quad \text{and} \quad \frac{m_1}{P(m_1)} \leqslant V < m_1.$$

Thus,

$$S \ll Vq^{-1} + \sum_{\substack{p\leqslant Y\\ p\nmid q}} \sum_{\substack{V<m_1\leqslant pV\\ P(m_1)=p\\ (m_1,q)=1}} \sum_{\substack{m_2\leqslant X/m_1\\ Q(m_2)\geqslant p\\ m_1m_2\equiv a \,(\mathrm{mod}\, q)}} 1.$$

Applying the upper bound sieve (cf. Theorem 3.4, P104, of [5]), the inner sum satisfies

$$\sum_{m_2} \ll \frac{X}{m_1\varphi(q)\log p} + p^2,$$

so that,

$$S \ll Vq^{-1} + Y^3\psi(V, Y) + \frac{X}{\varphi(q)} \sum_{\substack{p\leqslant Y\\ p\nmid q}} \frac{1}{p\log p} \sum_{\substack{V/p<m\leqslant V\\ P(m)\leqslant p\\ (m,q)=1}} \frac{1}{m}.$$

Defining $W = \exp((\log V)^{7/9})$, and applying partial summation to (C) of the lemma,

$$\sum_{W<p\leqslant V}\frac{1}{p\log p}\sum_{\substack{V/p<m\leqslant V\\ P(m)\leqslant p\\ (m,q)=1}}\frac{1}{m}\ll\frac{\varphi(q)}{q}\sum_{W<p\leqslant V}\frac{1}{p}\frac{\log V}{\log p}\varrho\left(\frac{\log V}{\log p}\right).$$

A routine computation, using the prime number theorem and partial summation, shows this to be

$$\ll\frac{\varphi(q)}{q}\varrho\left(\frac{\log V}{\log Y}\right).$$

On the other hand, using (A) and (B) of the lemma and partial summation, we have, for sufficiently large V,

$$\sum_{p\leqslant W}\frac{1}{p\log p}\sum_{\substack{V/p<m\leqslant V\\ P(m)\leqslant p\\ (m,q)=1}}\frac{1}{m}\ll\log^3 X\exp\left(-\frac{\log V}{2\log W}\right).$$

Collecting our estimates,

$$S\ll Vq^{-1}+Y^3\psi(V,Y)+\frac{X}{q}\varrho\left(\frac{\log V}{\log Y}\right)+\frac{X}{\varphi(q)}\log^3 X\exp\left(-\frac{\log V}{2\log W}\right)$$
$$\ll\frac{X}{q}\varrho\left(\frac{\log V}{\log Y}\right)$$

for sufficiently large V.

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