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Class number formulas for quaternary quadratic forms

by

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Introduction. This paper may be regarded as a sequel to [4]. Unless otherwise indicated, the notation and terminology are taken from [4], especially § 1, § 3 and § 5.

We recapitulate some of the results on class numbers derived in [3], [4]. Let V be a definite quadratic space of dimension four over the field of rational numbers $\boldsymbol{Q}$. Let $\mathfrak{I}$ be an idealcomplex of maximal lattices on V (cf. [4], § 3). Let Δ denote the reduced discriminant of $\mathfrak{I}$ and H the number of proper similitude classes in $\mathfrak{I}$. In the case where V has square discriminant $\mathfrak{I}$ is uniquely determined, and an explicit formula for H was given in [3] (Theorem, p. 297).

If the discriminant $D(V)$ of V is not a square, we put $K = \boldsymbol{Q}(\sqrt{D(V)})$, and denote the discriminant of K by Δ_K. It was shown in [4] (Prop. 7) that

$$\Delta = \Delta_K (p_1 \ldots p_e)^2 (q_1 \ldots q_f)^2,$$

where $q_1, \ldots, q_f$ are the anisotropic finite primes of V; $q_1, \ldots, q_f$ split in K, and $p_1, \ldots, p_e$ are distinct rational primes which remain prime in K. In § 6 of [4] explicit formulas were obtained for H (Theorems 1, 2) under the following conditions:

(i) $f = 0$,

(ii) The fundamental unit of K has norm -1.

In this paper we obtain such formulas for H without making either of these restrictions. As a result, we completely solve the problem of determining the proper class number of an arbitrary idealcomplex of maximal quaternary lattices (cf. [4], Prop. 11, for the indefinite case). As a special case of these formulas we obtain, in the classical language, a formula for the number of proper classes of positive definite integral quaternary forms of discriminant Δ_K.

By scaling, we may assume that $\mathfrak{I}$ contains the maximal integral lattices of V. When $D(V)$ is a nonsquare, there is a unique quaternion

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algebra $\mathfrak{A}$ over $\boldsymbol{Q}$ such that V is isometric to $\{a \in \mathfrak{A}_K \mid \bar{a}^* = a\}$ ([4], Prop. 2). Then $p_1, \ldots, p_e$; $q_1, \ldots, q_f$ may be characterized as the set of all rational primes which ramify in $\mathfrak{A}$ but not in K ([4], Prop. 6(b)). We set $\delta_1 = p_1 \ldots \ldots p_e$, $\delta_2 = q_1 \ldots q_f$, $\delta = \delta_1\delta_2$. Then H may be identified with a generalized type number $t_s(\Omega)$, where Ω is an Eichler order of $\mathfrak{A}_K$ of level δ ([4], p. 22). If $\delta_2 = 1$, then $t_s(\Omega) = t(\Omega)$, the usual type number of Ω. Thus the formulas we obtain are also formulas for the type numbers of certain Eichler orders associated to $\mathfrak{A}_K$. Under the further restriction that $\delta = 1$, Vignéras [5] stated a formula for $t(\Omega)$ which is incorrect, in general. The correct formula is the special case $e = f = 0$ of our formulas in Section 5.

1. Let $a \mapsto a^*$ be the canonical involution of $\mathfrak{A}_K$ and $N\colon \mathfrak{A}_K \to K$ the reduced norm. The conjugation map of K extends to a $\boldsymbol{Q}$-automorphism $a \mapsto \bar{a}$ of $\mathfrak{A}_K$ having $\mathfrak{A}$ as its fixed ring. These mappings extend in the obvious manner to the completions of $\mathfrak{A}_K$ and to its idele group $J_{\mathfrak{A}_K}$. Let $J^1_{\boldsymbol{Q}}, J^1_K, J^1_{\mathfrak{A}_K}$ denote the norm 1 idele groups of $\boldsymbol{Q}, K, \mathfrak{A}_K$, respectively. Let Ω be an Eichler order of $\mathfrak{A}_K$ of level δ which is symmetric, in the sense that $\bar{\Omega} = \Omega$. The symmetric normalizer $\mathfrak{N}^1_s(\tilde{\Omega})$ is defined to be the group of all $\nu \in J^1_{\mathfrak{A}_K}$ such that $\nu\tilde{\Omega}\bar{\nu}^* = n\tilde{\Omega}$ for some $n \in J^1_{\boldsymbol{Q}}$ (cf. [4], p. 21). The usual normalizer $\mathfrak{N}^1(\tilde{\Omega})$ is defined to be the group of all $\nu \in J^1_{\mathfrak{A}_K}$ such that $\nu\tilde{\Omega}\nu^{-1} = \tilde{\Omega}$. One easily sees that $\mathfrak{N}^1_s(\tilde{\Omega})$ is a normal subgroup of $\mathfrak{N}^1(\tilde{\Omega})$ and $\mathfrak{N}^1(\tilde{\Omega})/\mathfrak{N}^1_s(\tilde{\Omega})$ is an elementary abelian 2-group of order 2^f ([4], p. 22). Set

$$G = \mathfrak{A}_K^\times/K^\times, \quad G^1_A = J^1_{\mathfrak{A}_K}/J^1_K, \quad G^1_s(\tilde{\Omega}) = \mathfrak{N}^1_s(\tilde{\Omega})/J^1_K.$$

The generalized type number $t_s(\Omega)$ is defined by

$$t_s(\Omega) = \operatorname{card}\big(G\backslash G^1_A/G^1_s(\tilde{\Omega})\big).$$

It was shown in § 4 of [4] that $H = t_s(\Omega)$ (Prop. 9).

For every rational prime p let $K_p = K \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$, $\mathfrak{A}_{K_p} = \mathfrak{A} \otimes_{\boldsymbol{Q}} K_p$. If K_p is a direct sum of two fields, then $\boldsymbol{Q}_p$ is identified with the diagonal of K_p. We denote the unit group of K_p by U_{K_p}. From the local description of the symmetric normalizer ([4], p. 22) it is readily seen that $\mathfrak{N}^1_s(\tilde{\Omega})$ is the set of all $\nu = (\nu_p) \in \mathfrak{N}^1(\tilde{\Omega})$ such that

(1) $$N(\nu_p) \in \boldsymbol{Q}_p^\times (K_p^\times)^2 U_{K_p} \quad \text{for all } p \mid \delta_2.$$

The latter norm description can be used to define $\mathfrak{N}^1_s(\tilde{\Omega})$, $t_s(\Omega)$ for any Eichler order of level δ, symmetric or not. Since all Eichler orders of level δ are locally conjugate, $t_s(\Omega)$ is independent of the choice of Ω. We denote this common value by t_s, and the usual type number of Eichler orders of level δ by t.

We fix Ω and let $L_2\big(G\backslash G^1_A/G^1_s(\tilde{\Omega})\big)$ denote the space of all complex-valued functions on G^1_A which are invariant under left multiplication by G and right multiplication by $G^1_s(\tilde{\Omega})$. Let $F_{\tilde{\Omega}}$ be the characteristic function of $G^1_s(\tilde{\Omega})$. If we normalize the Haar measure on G^1_A so that $G^1_s(\tilde{\Omega})$ has measure 1, then convolution with respect to $F_{\tilde{\Omega}}$ gives the identity operator on $L_2\big(G\backslash G^1_A/G^1_s(\tilde{\Omega})\big)$. Applying the Selberg trace formula as in § 8 of [4], we obtain

(2) $$H = \operatorname{Tr}(F_{\tilde{\Omega}}) = \sum_r \int_{G(r)\backslash G^1_A} \psi_r(g')\,d\lambda(g'),$$

where r runs over a complete set of representatives for the conjugacy classes of G, $G(r)$ is the centralizer of r in G, and

$$\psi_r(g') = F_{\tilde{\Omega}}(g^{-1}rg), \quad g \in G^1_A.$$

A representative r makes a non-zero contribution to the trace sum (2) if and only if $g^{-1}rg \in G^1_s(\tilde{\Omega})$ for some $g \in G^1_A$. Let $a \in \mathfrak{A}_K$ represent r, and $\gamma \in J^1_{\mathfrak{A}_K}$ represent g. Then

$g^{-1}rg \in G^1_s(\tilde{\Omega}) \Leftrightarrow \gamma^{-1}a\gamma \in \mathfrak{N}^1_s(\tilde{\Omega}) \Leftrightarrow \gamma^{-1}a\gamma \in \mathfrak{N}^1(\tilde{\Omega})$ and $N(\gamma_p^{-1}a_p\gamma_p) \in \boldsymbol{Q}_p^\times (K_p^\times)^2 U_{K_p}$ for all $p \mid \delta_2 \Leftrightarrow a \in \mathfrak{N}^1(\gamma\tilde{\Omega}\gamma^{-1})$ and $N(a_p) \in \boldsymbol{Q}_p^\times (K_p^\times)^2 U_{K_p}$ for all $p \mid \delta_2 \Leftrightarrow a \in \mathfrak{N}(\gamma\Omega\gamma^{-1})$ and the principal $\big(N(a)\big) = n\mathfrak{i}^2$, where n is a *rational* integer dividing δ, and $\mathfrak{i}$ is a fractional ideal of K.

2. We proceed now to determine all a which give non-zero contributions to the trace sum (2). Our approach is a modified form of the argument in § 7 of [4]. If ξ_1, ξ_2 are algebraic numbers, then $\xi_1 \simeq \xi_2$ will mean that, for some $x \in K^\times$, $x\xi_1$ and ξ_2 have the same minimal polynomial over K. If $a, \beta \in \mathfrak{A}_K$, the condition $a \simeq \beta$ is equivalent to the classes of a, β being conjugate in G.

Suppose $a \not\simeq 1$, which is to say $a \notin K$. We may assume a is integral over $\mathfrak{O}$, the ring of integers of K. Then $\big(N(a)\big) = n\mathfrak{i}^2$, where $\mathfrak{i}$ is an integral ideal of $\mathfrak{O}$ and n is a rational integer dividing δ. From $\mathfrak{i}^2 = \big(n^{-1}N(a)\big)$ it follows that $\mathfrak{i}$ belongs to an ambiguous ideal class of K. Let D denote the square-free kernel of Δ_K. Then one of the following must hold ([1], p. 190, Exs. 8–10)

(a) $\mathfrak{i}$ is equivalent to an ambiguous ideal,

(b) $D = a^2 + b^2$, where a, b are integers; every unit of $\mathfrak{O}$ has norm 1, and $\mathfrak{i} = x\mathfrak{a}$, where $x \in K$ and $\mathfrak{a}^2 = (b + \sqrt{D})$.

If (b) holds, then $\big(N(a)\big) = \big(nx^2(b + \sqrt{D})\big)$. Taking the norm of both sides, we obtain

$$n_{K/\boldsymbol{Q}}\big(N(a)\big) = n^2\big(n_{K/\boldsymbol{Q}}(x)\big)^2(b^2 - D),$$

since all units have norm 1. This implies

$$n_{K/Q}(N(\alpha)) = n^2(n_{K/Q}(x))^2(-\alpha^2),$$

contradicting the fact that $N(\alpha)$ is totally positive. Since (a) is the only possibility, the reasoning of § 7 in [4] is valid here, and we deduce that $(N(\alpha)) = n\mathfrak{j}^2$, where $\mathfrak{j}$ is ambiguous. Thus $N(\alpha) = mu$, where m is a square-free rational integer dividing $\delta\Delta_K$, and u is a totally positive unit of $\mathfrak{O}$.

The minimal polynomial of α over K must then be of the form $X^2 + bX + mu$, where $b \in \mathfrak{O}$. Arguing exactly as in § 7 of [4], we have $\alpha^* = \omega^{-1}\alpha$, where ω is a primitive nth root of 1 for one of the following values of n: 2, 3, 4, 5, 6, 8, 10, 12. If $n = 2$, then $b = 0$ and we have $\alpha \simeq \sqrt{-mu}$.

If $n > 2$, then $K(\alpha) = K(\omega)$ and $\alpha^2 = mu\omega$. It suffices to consider the equation $\alpha^2 = \pm mu\omega$ in $K(\omega)$ for $n = 3, 4, 5, 8, 12$. We note that D must be 5 if $n = 5$, $D = 2$ if $n = 8$, and $D = 3$ if $n = 12$. Put $\alpha = x + y\omega$, $x, y \in K$. Each possible ω satisfies a minimal polynomial over K having constant term 1. This implies $\alpha^2 = x^2 - y^2 + z\omega$, $z \in K$. Hence $x^2 - y^2 + z\omega = \pm mu\omega$, from which it follows that $x^2 = y^2$, $x = \pm y$. If $x = 0$, then $\alpha \simeq \omega$. If $x \neq 0$, then $\alpha \simeq 1 \pm \omega$. We consider the various possibilities:

If $n = 3$, then $\alpha \simeq \zeta$, a primitive cube root of 1, or $\alpha \simeq 1 - \zeta \simeq \zeta\sqrt{-3}$

If $n = 4$, then $\alpha \simeq \sqrt{-1}$ or $\alpha \simeq 1 + \sqrt{-1}$.

If $n = 5$, let η denote a primitive 5-th root of 1. We observe that $1 + \eta \simeq \eta$. Furthermore, $N(1-\eta) = \sqrt{5}v$ for some unit v of $\mathfrak{O}$, which means $1-\eta$ cannot normalize an Eichler order of level δ. Let η' denote another primitive 5-th root which is not conjugate to η over $Q(\sqrt{5})$. Then $\alpha \simeq \eta$ or $\alpha \simeq \eta'$.

If $n = 8$, then $\omega \simeq 1 + \sqrt{-1}$. Since $N(1+\omega) = \sqrt{2}v$ for some unit v of $\mathfrak{O}$, $1 + \omega$ cannot normalize an Eichler order of level δ. Hence $\alpha \simeq 1 + \sqrt{-1}$ is the only possibility.

If $n = 12$, then $\alpha \simeq \zeta\sqrt{-1}$ or $\alpha \simeq 1 \pm \zeta\sqrt{-1}$.

3. In the preceding section we determined the following list of possible α:

I. $\alpha \simeq \sqrt{-mu}$, where m is a square-free integer dividing $\delta\Delta_K$, and u is a totally positive unit of $\mathfrak{O}$.

II. $\alpha \simeq 1 + \sqrt{-1}$ provided $2 | \delta\Delta_K$.

III. $\alpha \simeq \zeta$; $\alpha \simeq \zeta\sqrt{-3}$ provided $3 | \Delta_K$.

IV. $\alpha \simeq \zeta\sqrt{-1}$ or $\alpha \simeq 1 \pm \zeta\sqrt{-1}$ provided $D = 3$.

V. $\alpha \simeq \eta, \eta'$ provided $D = 5$.

In this section we determine under what conditions such α actually occur in the normalizer of an Eichler order of level δ. Our first step is to reduce case I.

Lemma 1. *If $u \neq 1$, then $\sqrt{-mu} \simeq \sqrt{-m(\operatorname{tr} u - 2)}$.*

Proof. If $\alpha^2 = -mu$, put $\beta = \alpha(\bar{u} - 1)$. Then $\beta^2 = -m(\operatorname{tr} u - 2)$.

Lemma 2. *Assume $u \neq 1$. Then the square-free kernel of $\operatorname{tr} u - 2$ divides Δ_K.*

Proof. Put $u = a + b\sqrt{D}$, where $a, b \in \frac{1}{2}Z$, $b \neq 0$. Since $a^2 - b^2 D = 1$, we have

$$a^2 - 1 = (a-1)(a+1) = b^2 D. \tag{3}$$

If $D \equiv 2, 3 \pmod 4$, then $a, b \in Z$ and $(a-1, a+1) = 1, 2$, according as a is even or odd, resp. If a is odd, then (3) implies b is even and

$$((a-1)/2)((a+1)/2) = (b/2)^2 D.$$

Hence $a - 1 = c^2 d$ or $(a-1)/2 = c^2 d$ for some $c \in Z$, $d | D$. Then $\operatorname{tr} u - 2 = 2(a-1) = c^2(2d)$, or $\operatorname{tr} u - 2 = (4c^2)d$, for some $d | D$.

If $D \equiv 1 \pmod 4$ and $a, b \in Z$, then taking equation (3) mod 4, we see that a cannot be even. Hence $(a-1, a+1) = 2$ and $\operatorname{tr} u - 2 = (4c^2)d$, for some $c \in Z$, $d | D$. If $a, b \notin Z$, then $2a$, $2b$ are odd, $(\operatorname{tr} u - 2)(\operatorname{tr} u + 2) = (2b)^2 D$, and $(\operatorname{tr} u - 2, \operatorname{tr} u + 2) = 1$. It follows that $\operatorname{tr} u - 2 = c^2 d$, for some $c \in Z$, $d | D$.

As a consequence of the preceding two lemmas, we have $\sqrt{-mu} \simeq \sqrt{-m'}$ for some square-free divisor m' of $\delta\Delta_K$. Thus case I reduces to:

I. $\alpha \simeq \sqrt{-m}$, where m is a square-free integer dividing $\delta\Delta_K$.

We note that, in each of the cases I–IV, $K(\alpha)$ is a biquadratic extension of Q. The matter of whether one of these α occurs in $\mathfrak{A}_K$ is then easily settled by means of the Kronecker symbol. Suppose $K(\alpha) = K(\sqrt{-m})$. Let $\Delta(-m)$ denote the discriminant of $Q(\sqrt{-m})$. Then

$$X^2 + m = 0 \text{ solvable in } \mathfrak{A}_K \Leftrightarrow \left(\frac{\Delta(-m)}{p}\right) \neq 1 \text{ for all } p \mid \delta_2. \tag{4}$$

In particular, when $f = 0$, $X^2 + m = 0$ is always solvable in $\mathfrak{A}_K$.

Now suppose α occurs in $\mathfrak{A}_K$ and belongs to one of the cases I–IV. Then, by the criterion of Eichler ([2], p. 133), the only way α might not occur in the normalizer of some Eichler order of level δ is if one of the p_i, $i = 1, \ldots, e$, remains inert from K to $K(\alpha)$. Since $K(\alpha)$ is biquadratic over Q, this is impossible. We conclude that α will occur in some normalizer provided only that it occurs already in $\mathfrak{A}_K$.

As for case V, one readily sees that η occurs in $\mathfrak{A}_K \Leftrightarrow$ no q_j splits completely in $Q(\eta)$, $j = 1, \ldots, f \Leftrightarrow q_j \not\equiv 1 \pmod 5$, $j = 1, \ldots, f$. Arguing the same as on p. 36 of [4], we see that η occurs in some normalizer $\Leftrightarrow \eta$ occurs in $\mathfrak{A}_K$ and $e = 0$.

4. In this section we determine the contribution to the trace sum (2) of each α which occurs in the normalizer of some Eichler order of level δ. If $\alpha \simeq 1$, then its contribution is $2M(\mathfrak{J})$, where $M(\mathfrak{J})$ is the weight of $\mathfrak{J}$, an explicit formula for which can be found on p. 26 of [4].

Now suppose $\alpha \not\simeq 1$, so that $K_\alpha = K(\alpha)$ is an imaginary quadratic extension of K. We say that an order $\mathcal{O}$ of K_α is *admissible for* α if $\alpha \in \mathcal{O}$ and, for some Eichler order Ω of level δ, $\mathcal{O} = \Omega \cap K_\alpha$ and $\alpha \in \mathfrak{N}(\Omega)$. We let l_α denote the number of primes *of* K which divide δ and ramify in K_α. If l_1 is the number of p_i, $i = 1, \ldots, e$, and l_2 is the number of q_j, $j = 1, \ldots, f$, which ramify in K_α, then it is clear that

$$l_\alpha = l_1 + 2l_2.$$

Proceeding exactly as in § 8 of [4], we find that the contribution of α to the trace sum is

$$(5)\qquad \frac{2^{f-1-l_\alpha} h(K_\alpha)}{h(K)[E_\alpha : E_K]} \sum_{\mathcal{O}} [U^1_\alpha : U^1(\tilde{\mathcal{O}})] \quad \text{if } \alpha \text{ is pure},$$

$$(6)\qquad \frac{2^{f-l_\alpha} h(K_\alpha)}{h(K)[E_\alpha : E_K]} \sum_{\mathcal{O}} [U^1_\alpha : U^1(\tilde{\mathcal{O}})] \quad \text{if } \alpha \text{ is not pure},$$

where $\mathcal{O}$ ranges over all orders of K_α which are admissible for α, and the rest of the notation is as in § 8 of [4] (Note: $l_\alpha = l_{\mathcal{O}}$ of § 8 as the conductor condition in the definition of $l_{\mathcal{O}}$ is superfluous).

If n is a positive integer, let $h(-n)$ denote the class number of $\boldsymbol{Q}(\sqrt{-n})$. Let W_α denote the group of roots of unity contained in K_α. If α belongs to one of the case I–IV, then $K_\alpha = K(\sqrt{-m})$ for some $m \mid \delta\Delta_K$, and Bachmann's formula for the class number of an imaginary biquadratic number field shows that the contribution of α is

$$(7)\qquad \varepsilon(\alpha) 2^{f-1-l_\alpha} \frac{h(-m)h(-mD)}{\operatorname{card}(W_\alpha)} \sum_{\mathcal{O}} [U^1_\alpha : U^1(\tilde{\mathcal{O}})] \quad \text{if } \alpha \text{ is pure},$$

$$(8)\qquad \varepsilon(\alpha) 2^{f-l_\alpha} \frac{h(-m)h(-mD)}{\operatorname{card}(W_\alpha)} \sum_{\mathcal{O}} [U^1_\alpha : U^1(\tilde{\mathcal{O}})] \quad \text{if } \alpha \text{ is not pure},$$

where $\varepsilon(\alpha) = 2$ if $K_\alpha = \boldsymbol{Q}(\sqrt{-1}, \sqrt{-2})$ and $\varepsilon(\alpha) = 1$ otherwise, and $\mathcal{O}$ ranges over all orders admissible for α.

In order to evaluate the contributions (7), (8) explicitly, one proceeds as in § 8 of [4]. The first step is to determine all the admissible orders for a given α. This is done by first computing $n_{K/\boldsymbol{Q}}(\Delta_{L/K})$ ([4], Prop. 13, p. 37), where $L = K_\alpha$, using this to determine $\Delta_{L/K}$, and then comparing $\Delta_{L/K}$ with $\Delta(-m)$. In the present situation some caution must be exercised, as $n_{K/\boldsymbol{Q}}(\Delta_{L/K})$ can be divisible by q_j^2 for some j, and $\Delta_{L/K}$ could *a priori* be divisible by any product of the two primes of K lying above q_j. However, in all our cases $\Delta_{L/K}$ is invariant under the conjugation of K, which implies $\Delta_{L/K}$ is divisible by q_j if its norm is divisible by q_j^2. After all the admissible orders for a given α are determined, the unit indices $[U^1_\alpha : U^1(\tilde{\mathcal{O}})]$ are computed by means of Proposition 16 of [4], p. 43. The final results are summarized in the formulas of the next section. We omit the details of the computations, as they do not basically differ from the ones carried out in § 8 of [4].

If case V occurs, then $D = 5$ and $e = 0$. The primitive fifth roots η, η' each contribute $2^f/5$, so the term $2^{f+1}/5$ must be added to the trace sum in this case.

5. Let m be a positive integer. Let $\lambda(m)$ denote the number of primes *of* K which divide m. Define ε_m to be 1 if $X^2 + m = 0$ is solvable in $\mathfrak{A}_K$ and 0 otherwise.

Theorem. *Let $\mathfrak{J}$ be an idealcomplex of definite maximal $\boldsymbol{Z}$-lattices of rank four. Let H be the number of proper similitude classes in $\mathfrak{J}$. Assume that the reduced discriminant Δ of $\mathfrak{J}$ is not a perfect square and put $K = \boldsymbol{Q}(\sqrt{\Delta})$, so that $\Delta = \Delta_K \delta^2$ with δ square-free. Write $\delta = \delta_1\delta_2$, where δ_1 is only divisible by primes which are inert in K, and δ_2 is only divisible by primes which split in K. Let f be the number of rational primes which divide δ_2. Denote the square-free kernel of Δ by D.*

(a) *If* $D \equiv 1 \pmod 8$, *then*

$$(9)\qquad H = 2M(\mathfrak{J}) + 2^f\Big(c_3 h(-3D) + \sum_{n|\delta,\, d|D} \varepsilon_{nd} 2^{-\lambda(n)-\sigma(nd)} h(-nd) h(-nD/d)\Big),$$

where $nd \neq 1, 3$; $d < \sqrt{D}$,

$$c_3 = \begin{cases} \frac{1}{6}\varepsilon_3 & \text{if } 3 \nmid \Delta, \\ \frac{5}{6}\varepsilon_3 & \text{if } 3 \mid \delta_1, \\ \frac{5}{12}\varepsilon_3 & \text{if } 3 \mid \delta_2, \\ \frac{7}{3}\varepsilon_3 & \text{if } D \equiv 3 \pmod 9, \\ 2\varepsilon_3 & \text{if } D \equiv 6 \pmod 9, \end{cases}$$

and

$$\sigma(m) = \begin{cases} -2 & \text{if } m \equiv 3 \pmod 8, \\ 0 & \text{if } m \equiv 7 \pmod 8, \\ 2 & \text{if } m \equiv 2 \pmod 4, \\ 2 & \text{if } m \equiv 1 \pmod 4,\ 2 \nmid \delta_2, \\ 4 & \text{if } m \equiv 1 \pmod 4,\ 2 \mid \delta_2. \end{cases}$$

(b) *If* $D \equiv 5 \pmod 8$, $D \neq 5$, *then*

$$H = 2M(\mathfrak{J}) + 2^f\Big(e_1 h(-D) + e_3 h(-3D) + \sum_{n|\delta,\, d|D} \varepsilon_{nd} 2^{-\lambda(n)-\sigma(nd)} h(-nd)\, h(-nD/d)\Big), \tag{10}$$

where $nd \neq 1, 3$; $d < \sqrt{D}$,

$$e_1 = \begin{cases} \frac{1}{8}\varepsilon_1 & \text{if } 2 \nmid \delta_1, \\ \frac{3}{16}\varepsilon_1 & \text{if } 2 \mid \delta_1, \end{cases}$$

$$e_3 = \begin{cases} \frac{1}{6}\varepsilon_3 & \text{if } 3 \nmid \Delta, \\ \frac{1}{3}\varepsilon_3 & \text{if } 3 \mid \delta_1, \\ \frac{1}{6}\varepsilon_3 & \text{if } 3 \mid \delta_2, \\ \frac{4}{3}\varepsilon_3 & \text{if } D \equiv 3 \pmod 9, \\ \varepsilon_3 & \text{if } D \equiv 6 \pmod 9, \end{cases}$$

and

$$\sigma(m) = \begin{cases} 0 & \text{if } m \equiv 3 \pmod 4, \\ 2 & \text{if } m \equiv 2 \pmod 4, \\ 2 & \text{if } m \equiv 1 \pmod 4,\ 2 \nmid \delta_1, \\ 3 & \text{if } m \equiv 1 \pmod 4,\ 2 \mid \delta_1. \end{cases}$$

If $D = 5$, *then* $2^{f+1}/5$ *must be added to* (10) *when* $\delta_1 = 1$ *and no rational prime dividing* δ_2 *is* $\equiv 1 \pmod 5$.

(c) *If* $D \equiv 3 \pmod 4$, *then for every* $n \mid \delta$, $d \mid D$, *either* nd *or* nD/d *is* $\equiv 3 \pmod 4$; *if* $D \neq 3$, *then*

$$H = 2M(\mathfrak{J}) + 2^f\Big(e_1 h(-D) + e_3 h(-3D) + \sum_{\substack{n|\delta,\, d|D \\ nd>3}} e_{nd} 2^{-\lambda(n)-\sigma(nd)} \times h(-nd)\, h(-nD/d) + \sum_{\substack{n|\delta,\, d|D \\ d<\sqrt{D}}} \varepsilon_{2nd} 2^{-\lambda(n)-\sigma(2nd)} h(-2nd)\, h(-2nD/d)\Big), \tag{11}$$

where

$$e_1 = \begin{cases} \frac{9}{4}\varepsilon_1 & \text{if } D \equiv 3 \pmod 8, \\ \varepsilon_1 & \text{if } D \equiv 7 \pmod 8, \end{cases}$$

$$e_3 = \begin{cases} \frac{1}{6}\varepsilon_3 & \text{if } 3 \nmid \Delta, \\ \frac{7}{12}\varepsilon_3 & \text{if } 3|\delta_1, \\ \frac{7}{24}\varepsilon_3 & \text{if } 3|\delta_2, \\ \frac{11}{6}\varepsilon_3 & \text{if } D \equiv 3 \pmod 9, \\ \frac{3}{2}\varepsilon_3 & \text{if } D \equiv 6 \pmod 9, \end{cases}$$

for $m > 3$,

$$e_m = \begin{cases} 0 & \text{if } m \not\equiv 3 \pmod 4, \\ 5\varepsilon_m & \text{if } m \equiv 3 \pmod 8, \\ \varepsilon_m & \text{if } m \equiv 7 \pmod 8, \end{cases}$$

and

$$\sigma(m) = \begin{cases} 0 & \text{if } m \equiv 7 \pmod 8, \\ 1 & \text{if } m \equiv 3 \pmod 8, \\ 2 & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

If $D = 3$, then $\varepsilon_1 = \varepsilon_3$, and the term $2^f\big(e_1 h(-D) + e_3 h(-3D)\big)$ in (11) must be replaced by $2^f(\frac{17}{12}\varepsilon_1)$.

(d) *If* $D \equiv 2 \pmod 4$, *then*

$$H = 2M(\mathfrak{J}) + 2^f\Big(\tfrac{5}{8}\varepsilon_1 h(-D) + e_3 h(-3D) + \sum_{\substack{n|\delta,\, d|D \\ d \text{ odd}}} e_{nd} 2^{-\lambda(n)-\sigma(nd)} h(-nd)\, h(-nD/d)\Big), \tag{12}$$

where $nd > 3$, e_3 *is as in part* (c), *and for* $m > 3$,

$$e_m = \begin{cases} 5\varepsilon_m & \text{if } m \equiv 3 \pmod 8, \\ \varepsilon_m & \text{if } m \equiv 7 \pmod 8, \\ 3\varepsilon_m & \text{if } m \equiv 1 \pmod 4, \end{cases}$$

$$\sigma(m) = \begin{cases} 1 & \text{if } m \equiv 3 \pmod 8, \\ 0 & \text{if } m \equiv 7 \pmod 8, \\ 2 & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

Concluding remarks. The above formulas can be interpreted in the classical language of quadratic forms (cf. [4], p. 32–33). In particular, if $\delta = 1$, then H = the number of proper classes of positive definite integral quaternary forms with discriminant Δ_K.

In [5] Vignéras obtained formulas for the arithmetic genus of certain Hilbert modular varieties, and claimed (Théorème, p. 212) that, as a special case, one obtained a formula for the type number t in the case $\delta = 1$. Our results for the case $\delta = 1$ show that this formula is *not*, in general, a formula for t, as it does not contain the contributions of all the elements $\alpha \simeq \sqrt{-d}$, where $d|\Delta_K$. The source of the difficulty is that the ideal class number of $\mathfrak{A}_K$ divided by the proper class number of K is *not* the type number t, in general, even if $\mathfrak{A}_K$ is split at all finite primes of K.

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