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Fields with non-trivial Kaplansky's radical and finite square class number

by

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Let F be a field of characteristic not 2 and let $g(F) = F^*/F^{*2}$ be the group of square classes of F and $q = |g(F)|$ — the square class number. C. M. Cordes [1] classified the Witt groups of anisotropic quadratic forms over non-real fields with $q = 8$ and obtained 10 non-isomorphic groups for this class of fields. A little later K. Szymiczek [10] classified the Grothendieck groups of quadratic forms over all fields with $q \leqslant 8$ and his classification in the case $q = 8$ gives 7 non-isomorphic Grothendieck groups for non-real fields and 6 non-isomorphic groups for real fields. Having classified Grothendieck groups he was also able to classify Witt groups for all fields with $q = 8$ and for non-real fields he found the 10 groups confirming Cordes' classification and for real fields with $q = 8$ he got 6 non-isomorphic Witt groups. Thus there are at most 16 possible Witt groups for fields with $q = 8$. Both authors also supplied some examples of fields fitting the classifications. But there remained 4 cases (out of the 16 possible) left without any example of field and it was not clear whether the number of different Witt groups can be further lessened or not. In this paper we construct the four missing fields (cf. a remark added in proof, p. 418). The four fields are characterized by the following values of field invariants(cf. [10], Th. 3.2).

(A) $q = 8$, $s = 2$, $q_2 = 8$, $u_2 = 2$ (the case (4.4) of Theorem 3.2 of [10]);

(B) $q = 8$, $s = \infty$, $q_2 = 4$, $u_2 = 1$ (the case (5.2));

(C) $q = 8$, $s = 1$, $q_2 = 8$, $u_2 = 2$ (the case (4.3));

(D) $q = 8$, $s = 2$, $q_2 = 4$, $u_2 = 2$ (the case (4.7)),

where s is the stufe (level) of the field (the minimal number of terms in a representation of -1 as the sum of squares), q_2 is the number of square classes whose elements are represented as the sum of two squares and u_2 is the number of equivalence classes of binary universal forms over the field.

We shall use the result of Szymiczek ([11], Chapter VI) asserting that a quadratic extension of a field satisfying (B) satisfies (A) and first we construct a field satisfying (B). These two examples (A) and (B) make complete the classification of Grothendieck groups for fields with $q \leqslant 8$ as described in [10], Theorems 2.1–2.5, and also solve the problems proposed in [12], p. 454 and in [11], Problem 9 at p. 61. Then we construct a pythagorean field with $q = 16$ and exactly 5 orderings and show that a suitable quadratic extension of it satisfies (C). Finally, a field fulfilling (D) is constructed.

Before going on into the constructions let us recall another context where the fields satisfying (A), (C), (D) are of importance. I. Kaplansky [7] introduced the notion of radical of the field F to be the set of square classes $a \in g(F)$ with the property that quadratic form with diagonalization $\langle 1, -a\rangle$ be universal over F. Thus u_2 is the number of radical elements of $g(F)$. In all known examples of non-real fields with finite q either $u_2 = 1$ or $u_2 = q$, that is, the radical is trivial: $\{1\}$ or $g(F)$. Cordes [2] constructed a field with non-trivial radical but the field has apparently infinite square class number q. It was shown by Cordes [2], [3] that, when dealing with quadratic forms over fields, in many results the group of squares F^{*2} can be replaced by the radical of the field.

Our constructions produce the first examples of non-real fields with finite q and non-trivial Kaplansky's radical. The method we use makes it possible to construct such fields for any $q = 2^n \geqslant 8$. This will be done in another paper.

We use the following notation and terminology. For a quadratic form φ we write $D(\varphi)$ to denote the set of square classes represented by φ. Thus $q_2 = |D(\langle 1, 1\rangle)|$ and $s = \min\{n \in N\colon\ -1 \in D(n \times \langle 1\rangle)\}$, provided the minimum exists; if not, the field is formally real and we write $s = \infty$. An ordering P of a real field F is a subset of F such that $P+P \subset P$, $P\cdot P \subset P$ and $P \cup -P = F^*$. Since $F^{*2} \subset P$, the group P/F^{*2} is a subgroup of $g(F)$ and we also regard P/F^{*2} to be the ordering of F. Let $v\colon F \to \Gamma \cup \{\infty\}$ be a valuation of the field F into an ordered group Γ. $\mathfrak{O}_v$, $\mathfrak{m}_v$ and $\bar{F} = \mathfrak{O}_v/\mathfrak{m}_v$ denote the valuation ring, the corresponding maximal ideal and the residue class field of the valuation v. For $x \in \mathfrak{O}_v$ we write $\bar{x}$ to denote $x+\mathfrak{m}_v \in \bar{F}$. If E is an extension field of F and w is a prolongation of v onto E, we denote by $e(w|v) = [w(E^*)\colon v(F^*)]$ and $f(w|v) = [\bar{E}:\bar{F}]$ the ramification index and the residue class degree, respectively. The valuation w is said to be an immediate extension of v if $e(w|v) = f(w|v) = 1$. For a discrete valuation v we write F_v for the completion of F with respect to v. In this case v has the unique extension w onto F_v and we have $e(w|v) = f(w|v) = 1$.

We begin with the following

LEMMA. *Let F be a field with a discrete valuation v such that $v(2) = 0$ and $v(F^*) = \mathbf{Z}$. Suppose E is an extension field of F and w is an arbitrary extension of v onto E.*

(a) *If $E = F(\sqrt{D})$, where $D \in F$, then $e(w|v) = 1$ iff $v(D) \in 2\mathbf{Z}$.*

(b) *If $E = F(\sqrt[2^n]{D}\colon n \in N)$, then $e(w|v) = 1$ iff $v(D) = 0$.*

(c) *If $E = F(\sqrt{D})$, then $f(w|v) = 1$ iff $v(D) \notin 2\mathbf{Z}$ or there is an element $a \in F$ with $a^{-2}D \equiv 1 \pmod{\mathfrak{m}_v}$.*

(d) *If $D \equiv 1 \pmod{\mathfrak{m}_v}$, then there exists an extension w of v onto the field $E = F(\sqrt[2^n]{D}\colon n \in N)$ such that $f(w|v) = 1$.*

Proof. (a) The implication $e(w|v) = 1 \Rightarrow v(D) \in 2\mathbf{Z}$ is trivial. Conversely, assume that $e(w|v) \neq 1$. Then there exist elements $a, b \in F$ such that $w(a+b\sqrt{D}) = \frac{1}{2}$. The element $x = a+b\sqrt{D}$ satisfies

$$x^2+(a^2-b^2D) = 2ax.$$

Since $w(2ax) \notin \mathbf{Z}$, so $w(x^2) = w(a^2-b^2D)$ and

$$2w(x) = w(a+b\sqrt{D})+w(a-b\sqrt{D}),$$

i.e., $w(x) = w(a-b\sqrt{D})$. Since $v(2) = 0$, we have

$$\begin{aligned} v(a) &= v(2a) = w[(a+b\sqrt{D})+(a-b\sqrt{D})] \\ &\geqslant \min[w(a+b\sqrt{D}), w(a-b\sqrt{D})] = \tfrac{1}{2}. \end{aligned}$$

But $v(a) \in \mathbf{Z}$, so $v(a) \geqslant 1$ and

$$\tfrac{1}{2} = w(a+b\sqrt{D}) \geqslant \min\big(v(a), w(b\sqrt{D})\big) = w(b\sqrt{D}).$$

From $v(a) > \frac{1}{2} \geqslant w(b\sqrt{D})$ we get $\frac{1}{2} = w(a+b\sqrt{D}) = w(b\sqrt{D})$ hence

$$v(D) = 1-2v(b) \notin 2\mathbf{Z}.$$

(b) Obviously, $e(w|v) = 1$ implies that $v(D) = 0$. Let $E_n = F(\sqrt[2^n]{D})$ and $w_n = w|_{E_n}$. From (a) it follows that $e(w_1|v) = 1$. Assume that $e(w_{n-1}|v) = 1$. Since $E_n = E_{n-1}\big((D^{1/2^{n-1}})^{1/2}\big)$ and $w_{n-1}(D^{1/2^{n-1}}) = 0$, so by (a) we obtain $e(w_n|w_{n-1}) = 1$. Using the induction hypothesis and the equality

$$e(w_n|v) = e(w_n|w_{n-1})\,e(w_{n-1}|v)$$

we get $e(w_n|v) = 1$.

The field $E = \bigcup\{E_n\colon n \in N\}$, so $w(E^*) = \bigcup\{w_n(E_n^*)\colon n \in N\} = \mathbf{Z}$. This proves that $e(w|v) = 1$.

(c) If $f(w|v) = 1$ and $v(D) \in 2\mathbf{Z}$, we choose $p \in F$ such that $v(p) = 1$. Then $\overline{p^{-v(D)}D} \in \bar{F}^{*2}$. Hence there is an element $b \in F^*$ with $\bar{b}^2 = \overline{p^{-v(D)}D}$. Thus for $a = bp^{v(D)/2}$, $a^{-2}D \equiv 1 \pmod{\mathfrak{m}_v}$. Conversely, since $v(2) = 0$, the field E is a Galois extension of F. Therefore ramification indices and residue class degrees of all prolongations of the valuation v onto E are equal and $[E:F] = e(w|v)f(w|v)s$, where s is the number of different extensions of v onto E (cf. [9], Chapter XII, § 6, Corollary 2). If $v(D) \notin 2\mathbf{Z}$ then from (a) we have $e(w|v) > 1$. Hence $f(w|v) = 1$. Now suppose $a \in F$ and $a^{-2}D \equiv 1 \pmod{\mathfrak{m}_v}$. Since $F(\sqrt{D}) = F(\sqrt{D^{-1}})$ we can assume that $v(D) \geqslant 0$. By Hensel's Lemma the polynomial $x^2 - D$ has a root in the completion F_v of F. Therefore one can embed the field E into F_v. Hence E has an immediate extension of the valuation v, and since E is a Galois extension of F, every extension w of v onto E has $f(w|v) = 1$.

(d) Let u be the unique extension of valuation v onto the completion F_v of the field F. By Hensel's Lemma for every $n \geqslant 1$ the polynomial $x^{2^n} - D$ has a unique root a_n in F_v such that $a_n \equiv 1 \pmod{\mathfrak{m}_u}$ and $a_n^2 = a_{n-1}$. Therefore the field $E' = F(a_n\colon n \in \mathbf{N})$ is a subfield of the field F_v isomorphic with the field E. Let $\iota\colon E \to E'$ be the isomorphism, and $w' = u|_{E'}$ be the valuation of the field E', then $w = \iota^{-1}w'\iota$ is a valuation of E and it fulfils $f(w|v) = 1$.

THEOREM. *If F is a formally real field with ordering P_0 and discrete valuation v such that $v(2) = 0$, then there exists an algebraic, formally real extension E of the field F with ordering P such that $P \cap F = P_0$ and with a valuation w, which is an immediate extension of the valuation v. Moreover, if $a \in P$ and $\bar{a} \in \bar{E}^{*2}$, then $a \in E^{*2}$.*

Proof. Let R be a real closure of the field F, with ordering $\tilde{P}$ such that $\tilde{P} \cap F = P_0$. Consider the family $\mathscr{F}$ of all pairs (K, w) such that $F \subset K \subset R$ and the valuation w is an immediate extension of v onto the field K. $\mathscr{F}$ is non-empty because $(F, v) \in \mathscr{F}$. In the family $\mathscr{F}$ we define a partial ordering $\leqslant$ by putting

$$(K_1, w_1) \leqslant (K_2, w_2) \quad \text{if and only if} \quad K_1 \subset K_2 \text{ and } w_2|_{K_1} = w_1.$$

We verify that every chain $\mathscr{L}$ in $\mathscr{F}$ is bounded. Namely, for the field $L = \bigcup\{K\colon (K, w) \in \mathscr{L}\}$ we define a valuation u of the field L in the following way. For $a \in L$ there is a pair $(K, w) \in \mathscr{L}$ with $a \in K$ and we put $u(a) = w(a)$. Obviously $(L, u) \in \mathscr{F}$ and produces an upper bound for $\mathscr{L}$. By Kuratowski-Zorn's Lemma we get a pair (E, w), which is a maximal element of the family $\mathscr{F}$. Since E is contained in real closed field R, with ordering $\tilde{P}$ it is also formally real field and $P = \tilde{P} \cap E$ is an ordering of E. To show that the triplet (E, P, w) fulfils the requirements of the theorem we need verify that if $a \in P$ and $\bar{a} \in \bar{E}^{*2}$ then a is a square of the field E. Let $a \in P$ and $\bar{a} \in \bar{E}^{*2}$. Obviously, the field $E(\sqrt{a}) \subset R$.

From lemma (a), (c) it follows that if valuation w' is an extension of w onto $E(\sqrt{a})$ then $e(w'|w) = f(w'|w) = 1$, hence $(E(\sqrt{a}), w') \in \mathscr{F}$ and $(E, w) \leqslant (E(\sqrt{a}), w')$. Since (E, w) is a maximal element of $\mathscr{F}$, so $E = E(\sqrt{a})$ and $a \in E^{*2}$. This finishes the proof of the theorem.

Now we use the theorem in constructing fields having the properties described in the introduction.

EXAMPLE 1. Let F be the field of rational numbers $\mathbf{Q}$, P_0 be the natural ordering of $\mathbf{Q}$, and v let be the 5-adic valuation. According to the theorem there exists an algebraic formally real extension E of $\mathbf{Q}$, with and ordering P such that $P \cap \mathbf{Q} = P_0$ and with an immediate extension w of the valuation v. Since the valuation w is discrete, every element $a \in E^*$ can be written in the form

$$a = (-1)^i 5^j \varepsilon$$

where i, j are integers and $\varepsilon \in P$, $w(\varepsilon) = 0$. Since w is an immediate extension of v, the residue class field $\bar{E}$ is $\mathbf{F}_5$. Thus from the theorem we conclude that if $\varepsilon \equiv 1$ or $4 \pmod{\mathfrak{m}_w}$, then $\varepsilon \in E^{*2}$ and if $\varepsilon \equiv 2$ or $3 \pmod{\mathfrak{m}_w}$, then $\frac{1}{2}\varepsilon \in E^{*2}$. It follows that every element $a \in E^*$ can be uniquely represented as

$$a = (-1)^i 5^j 2^k \eta,$$

where $i, k \in \{0, 1\}$, j is an integer and η is a square of the field E. Consequently, $g(E) = \{1, -1\} \times \{1, 2\} \times \{1, 5\}$ and $q(E) = 8$.

The form $\langle 1, 1\rangle$ is positive definite in the ordering P, so it represents only positive elements of the field, hence

$$q_2 = |D(\langle 1, 1\rangle)| \leqslant 4.$$

On the other hand, 2 and 5 are sums of two squares, hence $q_2 = 4$. The forms $\langle 1, 2\rangle$, $\langle 1, 5\rangle$, $\langle 1, 10\rangle$ are positive definite, so they are not universal. It is easy to show that if one of the forms $\langle 1, -2\rangle$, $\langle 1, -5\rangle$, $\langle 1, -10\rangle$ is universal then all of of them are universal. But the form $\langle 1, -5\rangle$ is not universal, because it does not represent 2 (otherwise, 2 would be a square in the field $\mathbf{F}_5$), hence $u_2 = 1$. Thus E satisfies (B).

EXAMPLE 2. If E is the field from Example 1, then $E(\sqrt{-2})$ satisfies (A), according to Theorem 1 of [11] (Chapter VI).

EXAMPLE 3. Let $\mathrm{Ac}\mathbf{Q}$ be an algebraic closure of the rational field $\mathbf{Q}$, and $R_1, R_2 \subset \mathrm{Ac}\mathbf{Q}$ be two real closures of the field $\mathbf{Q}(\sqrt{2})$ that induce different orderings on $\mathbf{Q}(\sqrt{2})$ and assume that $R_1 \subset \mathbf{R}$, where $\mathbf{R}$ denotes the real numbers. Then the field $k = R_1 \cap R_2$ is a pythagorean field contained in $\mathbf{R}$ and it has two orderings (cf. [4], p. 1187), hence $g(k) = \{1, -1\} \times \{1, \sqrt{2}\}$. Let $F = k(x)$ and suppose v is the valuation determined by the polynomial $x \in k(x)$ and

$$P_0 = \left\{ \frac{f(x)}{g(x)} \in k(x) \colon \frac{f(\pi)}{g(\pi)} > 0 \right\}$$

is the ordering determined by the transcendental number π. By the theorem, for (F, P_0, v) there exists a triplet (E, P, w) with the properties described in the theorem. Therefore every element $a \in E^*$ can be written in the form $a = (-1)^i x^j \varepsilon$ where $\varepsilon \in P$, $w(\varepsilon) = 0$ and i, j are integers. The elements of the set $C = \{1, x-1, y, (x-1)y\}$ where $y = x\left(\frac{1-\sqrt{2}}{2}\right)^2 + \sqrt{2}$, are positive in the ordering P. Moreover for every element $r \in C$, $v(r) = 0$ and the elements $1, \overline{x-1}, \overline{y}, \overline{(x-1)y} \in k$ represent all cosets of the group $g(k)$. So for a suitable $r \in C$ the element $\varepsilon \cdot r^{-1}$ is a square of the field E. Hence for $a \in E^*$ we can write $a = (-1)^i x^j r\eta$, where $i = 0, 1$; j is integer, $r \in C$ and $\eta \in E^{*2}$. This shows that

$$g(E) = \{1, -1\} \times \{1, x\} \times \{1, y\} \times \{1, x-1\} \quad \text{and} \quad q = 16.$$

The form $\langle 1, x-1 \rangle$ is positive definite in the ordering P and the identities

$$x = 1^2 + (x-1)1^2, \qquad y = \left(\frac{1+\sqrt{2}}{2}\right)^2 + (x-1)\left(\frac{1-\sqrt{2}}{2}\right)^2$$

hold, so we conclude that $|D(\langle 1, x-1 \rangle)| = 8$.

If $P_2, P_1 \subset k = \bar{E}$ are different orderings of the field k, then the sets

$$P_{i1} = \{x^j \varepsilon \in E^* \colon \bar{\varepsilon} \in P_i\}, \qquad i = 1, 2,$$
$$P_{i2} = \{x^j \varepsilon \in E^* \colon (-1)^j \bar{\varepsilon} \in P_i\}, \qquad i = 1, 2,$$

are orderings of the field E.

This shows, that the field E has at least five orderings and by Proposition 1.12 of [10] it is a pythagorean field. Considering the signs of elements of the group $g(E)$ it is easy to check that E has no further orderings, since otherwise from Corollary 2.2 of [8] it follows that $|D(\langle 1, x-1 \rangle)| \leqslant 4$, contrary to the above arguments.

Thus E is a pythagorean field with $q = 16$ and has exactly five orderings. We will use the field in the next example.

Example 4. We shall show that the field $K = E(\sqrt{-1})$, where E is the field constructed in the Example 3, satisfies (C).

Obviously $s = 1$ and the form $\langle 1, 1 \rangle$ is universal. According to Corollary 7 of [6] or Lemma 4.3 of [10],

$$g(K) = \{1, x\} \times \{1, x-1\} \times \left\{1, \left(\frac{1-\sqrt{2}}{2}\right)^2 x + \sqrt{2}\right\}, \quad \text{so} \quad q = 8.$$

According to the arguments used above in the Example 3, the form $\langle 1, x-1 \rangle$ is a second universal form, hence $u_2 \geqslant 2$. Since the field E has five orderings, so according to Corollary 1.3 and Theorem 4.1 of [8] we get $u_2 < 8$. Using Lemma 1.8 of [10] we obtain $u_2 = 2$. Thus K satisfies (C).

Modifying a little the above construction we can get a field, which fulfils (D).

Example 5. Take the field $F = Q(\sqrt[2^n]{7} \colon n \in N)$, and the 3- and 5-adic valuations v_3 and v_5 of the field Q. Of course, $v_3(7) = v_5(7) = 0$ and $\bar{7} = 1$ in the field F_3, hence using lemma (b), (d) we obtain some prolongations w_3 and w_5 of the valuations v_3 and v_5, respectively, onto F, with $e(w_3|v_3) = f(w_3|v_3) = 1$ and $e(w_5|v_5) = 1$. One can show that the residue class field of valuation w_5 is $F_5(\sqrt[2^n]{2} \colon n \in N)$. Using Corollary 7 of [6] or Lemma 4.3 of [10] we obtain that $g(F_5(\sqrt[2^n]{2} \colon n \in N)) = \{1\}$.

Fix an algebraic closure $\mathrm{Ac}F$ of the field F and consider the family of all triplets (K, u_3, u_5) such that $F \subset K \subset \mathrm{Ac}F$, the valuation u_i is an immediate extension of valuation v_i for $i = 3, 5$. We introduce the partial ordering $\leqslant$ by

$$(K, u_3, u_5) \leqslant (K', u_3', u_5') \quad \text{if and only if} \quad K \subset K',\ u_3'|_K = u_3,\ u_5'|_K = u_5.$$

Similarly as in the proof of the theorem one can show that in the family there exists a maximal element (E, u_3, u_5) with the property: if $a \in E^*$, $u_3(a) = u_5(a) = 0$, and $\bar{a}$ is a square in the residue class field F_3 then $a \in E^{*2}$. Since valuations u_3 and u_5 are discrete, so every element $a \in E^*$ can be written in the form $a = (-1)^i 3^j 5^l \varepsilon$ where i, j, l are integers and $\varepsilon \in E^{*2}$. Hence

$$g(E) = \{1, -1\} \times \{1, 3\} \times \{1, 5\}$$

and $g(E) = 8$.

The form $\langle 1, 1 \rangle$ is not universal (it does not represent 3, because otherwise the form $\langle 1, 1 \rangle$ would be isotropic over F_3) and the equalities

$$5 = 1^2 + 2^2, \qquad -1 = 1^2 + (\sqrt{-2})^2$$

show $q_2 = 4$ ($\sqrt{-2} \in E$ because $u_3(-2) = u_5(-2) = 0$ and -2 is a square in the residue class field of valuation u_3). Hence we get $q_2 = 4$, $s = 2$, and $u_2 < 8$.

From the identities $3^2 + 5(\sqrt{-2})^2 = -1$, $(\sqrt{-2})^2 + 5 \cdot 1^2 = 3$ it follows that there are two universal forms $\langle 1, 5 \rangle$, $\langle 1, -1 \rangle$, so that $u_2 \geqslant 2$. Using Lemma 1.8 of [10] we obtain $u_2 = 2$. Thus E satisfies (D).

Added in proof: After I had sent this paper to editor, L. Szczepanik notice a mistake in Theorem 2.5 of [10]. Actually, there exist exactly seven Grothendiec groups and seven Witt groups for formally real fields with 8 square classes. Complet classification of formally real field with $q = 8$ can be found in: M. Kula, L. Szczep nik, K. Szymiczek, *Quadratic forms over formally real fields with eight square classe* Manuscripta Math. 29 (1979), pp. 295–303.

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