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On the distribution of the values of Riemann's Zeta-function

by

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1. Introduction. The theory of the distribution of the values of Riemann's Zeta-function ζ has its origin in the Riemann Hypothesis. Many results on zero-free regions and on the density of the zeros in certain domains have been proved since its appearance. The study of the *non-zero* values of ζ was initiated by H. Bohr early this century. His results, together with zero-density estimates, reveal that ζ has many more w-roots (solutions of $\zeta(s) = w$), w non-zero complex, than zeros in closed half-planes on the right of the critical line. Many classical results on the distribution of the non-zero values of ζ have recently been sharpened or extended. This will also be the theme of this paper, where we shall exclusively be concerned with the behaviour of ζ in zero-free regions. Until now, there are two types of results dealing with this behaviour. To one of them, recent contributions have been made by S. Voronin [11], [12], to the other by H. L. Montgomery [6].

Voronin proved in particular the following:

(V1) *Let $\frac{1}{2} < \sigma < 1$ and $0 < r < \min(1-\sigma, \sigma-\frac{1}{2})$. Let K be a positive integer and $w_1, \ldots, w_K$ non-zero complex numbers. Then the number of positive integers $n \leqslant T$ such that there are complex numbers z_k, $1 \leqslant k \leqslant K$, satisfying*

$$\zeta(\sigma+in+z_k) = w_k, \quad |z_k| \leqslant r,$$

for $k = 1, \ldots, K$ exceeds cT for $T \geqslant T_0$ where c and T_0 are positive numbers depending on $w_1, \ldots, w_K$, σ and r.

(V2) *Let σ and K be as in* (V1). *Denote by $\zeta^{(k)}$, $k = 1, 2, \ldots$, the k-th derivative of ζ. Then the sequence of points*

$$\left(\zeta(\sigma+in), \zeta^{(1)}(\sigma+in), \ldots, \zeta^{(K-1)}(\sigma+in)\right), \quad n = 1, 2, \ldots,$$

is everywhere dense in C^K, the K-dimensional complex vector space.

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(V3) *Let $r < 1/4$ and $\varepsilon > 0$. Let f be a non-vanishing function which is continuous on the closed disc of radius r around the origin and analytic in its interior. Then there are real numbers T depending on ε and f such that*

$$\max_{|z|\leqslant r} |\zeta(\tfrac{3}{4}+iT+z)-f(z)| < \varepsilon.$$

In case $K = 1$, (V1) is due to Bohr [1] and (V2) to Bohr and Courant [2]. There is no classical version of (V3) which can be considered as an infinite dimensional analogue of (V2).

Montgomery showed that *for $1/2 < \sigma < 1$ and any real θ*

$$\text{(M1)} \qquad \operatorname{Re}\big(e^{i\theta}\log\zeta(\sigma+it)\big) = \Omega\big((\log t)^{1-\sigma}(\log\log t)^{-\sigma}\big), \qquad t\to\infty,$$

where Re *denotes the real part.* This implies in particular that

$$\text{(M2)} \qquad \log|\zeta(\sigma+it)| = \Omega_{\pm}\big((\log t)^{1-\sigma}(\log\log t)^{-\sigma}\big), \qquad t\to\infty.$$

See [6] for a historical account of Ω-results for $\log|\zeta(\sigma+it)|$.

Comparing (V1)–(V3) with (M1)–(M2) one notices at once that they are different in nature. In Voronin's theorems, for instance, only such values of ζ are involved as lie in a previously fixed compact subset of $C-\{0\}$. No information is given about the questions, how soon $\zeta(\sigma+it)$ gets large or small in (V1), or how soon $\zeta(\sigma+it)$ approximates very well in (V2) and (V3), when $1/2 < \sigma < 1$ and $t\to\infty$. So (V1)–(V3) only imply that the left-hand sides of (M1), (M2) are $\Omega_{\pm}(1)$. On the other hand, (M1) and (M2) do not tell us much about the shape of the sets

$$\{\zeta(\sigma+it+z) \mid |z|\leqslant r\}, \qquad 0 < r < \min(1-\sigma, \sigma-\tfrac{1}{2}), \qquad t > 0.$$

This paper is aimed at the investigation of these and similar questions. In particular, quantitative versions of (V1)–(V3) will be proved. The precise statement of the main results will follow in the next section. But first let us add some remarks about the above mentioned theorems and their proofs.

Montgomery's paper is likely to mark an achievement of double importance for the theory of the distribution of the values of ζ. First (M2), though not too big an improvement on earlier results, appears to be best possible from a probabilistic point of view. Secondly a unified approach to problems of type (V1)–(V3) and (M1)–(M2) can be given now. Earlier proofs of Ω-results for $\log|\zeta(s)|$ and the proofs for (V1)–(V3) had not much in common.

To study the behaviour of ζ in zero-free regions, it is most convenient to work with $\log\zeta$ which can be uniquely defined in a suitably slit complex plane. The proof of (M1)–(M2) and of our results now show the following general pattern. In a first step it is proved that $\log\zeta$ approximates some auxiliary functions very often very well. These auxiliary functions are simpler to study than $\log\zeta$. But since they come close to $\log\zeta$, their properties carry over to $\log\zeta$ with only minor modifications. In a second step these auxiliary functions are investigated. Voronin's proofs also split into two parts. The main differences between his and our procedure are as follows:

First Voronin shows by means of a mean value theorem for ζ and Weyl's criterion for uniformly distributed sequences that certain finite products often come close to ζ. We, however, approximate $\log\zeta$ by using a zero-density estimate, high moments for $\sum p^{-s}$, where the summation runs over all primes in a finite interval and quantitative forms of Dirichlet's theorem on diophantine approximation as well as of the just mentioned criterion of Weyl. The approximation process Voronin used can also be carried out in a quantitative manner. Nevertheless the results obtained in that way from our present day knowledge of ζ are inferior to those we get by approximating to $\log\zeta$.

In the second step Voronin's main tool is a generalization of the following lemma to finite and infinite dimensional Hilbert spaces: The summands of a conditionally but not absolutely convergent series of real numbers can be rearranged in such a way that the rearranged series converges to any real number given in advance. It is rather obvious that quantitative results cannot be obtained only with this tool. What we use, instead, are convexity arguments such as Hadamard's three-circle-theorem and a consequence of the theorem of Hahn–Banach.

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2. Statement of the results. Before we can write down our theorems, we have to introduce several notations. These will be kept throughout the paper.

Let ε always denote a small positive number. Then $c_1, c_2, \ldots$ will denote positive numbers which depend on no other parameters except possibly on ε. The $c_1, c_2, \ldots$ need not have the same meaning at different occurrences. The constants implicitly given by O-symbols are either absolute or depend only on ε. We sometimes suppose that certain parameters remain bounded. Since such parameters then affect the results only by a bounded factor, we absorb this factor into the c_j or the O-symbol by assuming, without loss of generality, that those parameters are bounded by $1/\varepsilon$.

The letter p always stands for a prime number. As usual, $\pi(\varrho)$ denotes the number of primes not exceeding ϱ and

$$\Lambda_1(m) = \begin{cases} 1/j, & \text{if} \quad m = p^j,\ j = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $s = \sigma+it$, $w = u+iv$ and $z = x+iy$ complex variables with corresponding real parts σ, u, x and imaginary parts t, v, y. For positive r we put

$$D_r = \{z|\ |z| \leqslant r\}.$$

If S is a subset of the complex plane, and f an analytic function defined in a neighborhood of S, we set

$$s+S = \{s+z|\ z \text{ in } S\}$$

and

$$f(S) = \{f(z)|\ z \text{ in } S\}.$$

The kth derivative of f is denoted by $f^{(k)}$.

Let G be the open set of complex numbers s in $\sigma > 1/2$ such that $\zeta(s+\eta)$ is a non-zero complex number for all non-negative real η. Let $\log\zeta$ denote the (unique) analytic function on G which is representable as

$$\log\zeta(s) = \sum_{m=1}^{\infty} \Lambda_1(m) m^{-s}$$

in the halfplane $\sigma > 1$.

Now we are able to begin with the list of our results.

THEOREM 1. *Let R be positive and $1/2+\varepsilon+R \leqslant \sigma \leqslant 1-\varepsilon-R$. Then there are $c_1, \ldots, c_6$, $c_5 < c_6$, such that for positive v, r and T satisfying*

$$c_1 \leqslant T, \quad c_2 \leqslant v \leqslant \log T \quad \text{and} \quad c_3/\log v \leqslant r \leqslant R$$

we can find more than $T\exp(-c_4 v/\log v)$ positive numbers $t_n \leqslant T$, $t_{n+1} > t_n+2r$ for $n = 1, 2, \ldots$, such that G contains $\sigma+it_n+D_r$ and $\log\zeta(\sigma+it_n+D_r)$ contains

$$\left\{w\Big|\ c_5 \frac{v^{1-\sigma+r}}{\log v} \leqslant |w| \leqslant c_6 \frac{v^{1-\sigma+r}}{\log v}\right\}.$$

Theorem 1 has a corollary which sharpens (V1) in several directions.

COROLLARY 1. *Let the conditions of Theorem 1 be fulfilled. Then there exists a c_7 and more than $T\exp(-c_4 v/\log v)$ positive numbers $t_n \leqslant T$, $t_{n+1} > t_n+2r$ for $n = 1, 2, \ldots$, such that the annulus*

$$\left\{w\Big|\ \exp\left(-\frac{c_7 v^{1-\sigma+r}}{\log v}\right) \leqslant |w| \leqslant \exp\left(\frac{c_7 v^{1-\sigma+r}}{\log v}\right)\right\}$$

is contained in $\zeta(\sigma+it_n+D_r)$.

Remark. Montgomery conjectures that Theorem 1 holds for $c_2 \leqslant v \leqslant \log T \log\log T$. This would be the complete analogue of his results (M1).

It would follow from his conjecture [6]

$$\log|\zeta(\sigma+it)| = O\left((\log t)^{1-\sigma}(\log\log t)^{-\sigma}\right), \quad t \to \infty,\ 1/2+\varepsilon \leqslant \sigma \leqslant 1-\varepsilon,$$

by an obvious modification of the proof of Theorem 1.

THEOREM 2. *Let a be a positive number such that*

$$(1) \qquad 1 \leqslant a \leqslant 2 \quad \text{and} \quad \left|e^{2\pi l/a} - \frac{m}{n}\right| \geqslant \frac{\varepsilon}{(nl\log(1+|n|))^2}$$

for all non-zero integers l, m and n. Let R be positive, $1/2+\varepsilon+R \leqslant \sigma \leqslant 1-\varepsilon-R$ and $\sigma_1 = \sigma-R$. Then there are $c_1, \ldots, c_6$ such that the following is true for $T \geqslant c_1$ and all positive integers $K \leqslant c_2\log\log T$: Let

$$(2) \qquad \begin{gathered} c_3 \leqslant V \leqslant \varrho^{1/2}\min\left((\log\varrho)^{-1/2}, \varepsilon\exp\left(-K\frac{\sigma_1-\frac12}{1-\sigma}\right)\right), \\ \varrho = c_4(\log T\log^{1/2}V)^{1/2}(V\log^{1/2}V)^{-\frac{5}{2(2\sigma_1-1)}} \end{gathered}$$

and w_k, $0 \leqslant k \leqslant K$ complex numbers such that

$$|w_k| \leqslant \frac{1}{10}\varrho^{1-\sigma}(\log\varrho)^{k-K-1}k!(K-k)!K^{-3}\left(2e(1-\sigma)\right)^{-K}$$

for $0 \leqslant k \leqslant K$. Then there are more than $\frac15 TV^{-\pi(\varrho)}$ points $s_n = \sigma+ina$, $1 \leqslant n \leqslant T$, such that

$$|(\log\zeta)^{(k)}(s_n) - w_k| = O\left(k!R^{-k}\frac{\varrho^{1-\sigma_1}}{V\log\varrho}\right)$$

for $0 \leqslant k \leqslant K$. In particular for a fixed K, $K \leqslant 1/\varepsilon$ say, and

$$(3) \qquad \begin{gathered} \varrho = (\log T)^{\frac{2(2\sigma-1)}{8\sigma+1}}(\log\log T)^{1/4}, \\ |w_k| \leqslant c_5\varrho^{1-\sigma}(\log\varrho)^{k-K-1}, \quad 0 \leqslant k \leqslant K, \end{gathered}$$

there are more than $T\exp(-c\,\varrho)$ points $s_n = \sigma+ina$, $1 \leqslant n \leqslant T$, such that

$$|(\log\zeta)^{(k)}(s_n) - w_k| = O\left(\varrho^{1/2-\sigma}(\log\varrho)^{k-1/2}\right)$$

for $0 \leqslant k \leqslant K$.

For numbers a satisfying (1) see the remarks to Lemma 7. The corollary to Theorem 2 sharpens (V2).

COROLLARY 2. *Let a, R, σ; σ_1, T and K be as in Theorem 2. Let V and ϱ be given by (2) and assume in addition that*

$$R \geqslant 40e(1-\sigma)\left(\frac{\varrho^{1-\sigma}}{5K^2\log\varrho}\right)^{-1/K}.$$

Let w_k', $0 \leqslant k \leqslant K$, be complex numbers satisfying

$$|w_0'| \leqslant \frac{\varrho^{1-\sigma}K!}{10K^3\log\varrho}\big(2e(1-\sigma)\log\varrho\big)^{-K},$$

$$|w_k'| \leqslant \frac{1}{3K}\left(\frac{\varrho^{1-\sigma}}{5K^2\log\varrho}\right)^{k/K}\big(4e(1-\sigma)\big)^{-k}, \qquad 1 \leqslant k \leqslant K.$$

Then there are more than $\frac{1}{5}TV^{-\pi(\varrho)}$ points $s_n = \sigma+ina$, $1 \leqslant n \leqslant T$, such that

$$|\log\zeta(s_n)-w_0'| = O\left(\frac{\varrho^{1-\sigma_1}}{V\log\varrho}\right),$$

$$\left|\frac{\zeta^{(k)}(s_n)}{k!\,\zeta(s_n)}-w_k'\right| = O\left(\frac{\varrho^{1-\sigma_1}}{RV\log\varrho}\left(\frac{\varrho^{1-\sigma}}{5K^2\log\varrho}\right)^{(k-1)/K}\big(4e(1-\sigma)\big)^{1-k}\right), \qquad 1 \leqslant k \leqslant K.$$

In particular for a fixed K, $K \leqslant 1/\varepsilon$ say, ϱ as in (3) and

$$|w_0'| \leqslant c_5\varrho^{1-\sigma}(\log\varrho)^{-K-1},$$

$$|w_k'| \leqslant c_7\left(\frac{\varrho^{1-\sigma}}{\log\varrho}\right)^{k/K}, \qquad 1 \leqslant k \leqslant K,\ c_7 \text{ suitably chosen},$$

there are more than $T\exp(-c_6\varrho)$ points $s_n = \sigma+ina$, $1 \leqslant n \leqslant T$, such that

$$|\log\zeta(s_n)-w_0'| = O\left(\frac{\varrho^{1/2-\sigma}}{(\log\varrho)^{1/2}}\right),$$

$$\left|\frac{\zeta^{(k)}(s_n)}{\zeta(s_n)}-w_k'\right| = O\left(\varrho^{1/2-\sigma}\left(\frac{\varrho^{1-\sigma}}{\log\varrho}\right)^{(k-1)/K}(\log\varrho)^{1/2}\right), \qquad 1 \leqslant k \leqslant K.$$

Our next theorem deals with the infinite dimensional analogue of what Theorem 1 is in the one-dimensional case. A finite dimensional version of the theorem follows from it. A sharper finite dimensional form could be proved directly by combining some results obtained in the proofs of Theorem 1 and 2. However, since the conception of the proof is essentially the same in the finite and in the infinite dimensional case, we restrict ourselves here to treating the latter. Finally, Theorem 4 contains the infinite dimensional analogue of what Theorem 2 is in the finite dimensional case. Before these theorems can be stated, some more notations are needed.

Let U be an open set in the complex s-plane. Let $L_2(U)$ denote the Hilbert space of equivalence classes of complex-valued functions which are square integrable on U with respect to the Lebesgue measure in the plane. Let $P_2(U)$ denote the closed subspace in $L_2(U)$ generated by the polynomials in s, i.e. the closure of the subspace spanned by the functions

(4) $$s \mapsto s^n, \quad n = 0, 1, \ldots$$

Note that $P_2(U)$ may (e.g. $U = \{s|\ |s|<1\}$) or may not (e.g. $U = \{s|\ \varepsilon < |s| < 1\}$) coincide with $H_2(U)$, the closed subspace of analytic functions in $L_2(U)$. Let C be a compact subset of U. Denote by $P_\infty(C)$ the set of continuous functions on C which are uniformly approximable on C by finite linear combinations of the functions listed in (4).

Our aim is to study the "curve"

$$\Phi\colon\ z \mapsto \big(s \mapsto \log\zeta(z+s),\ s \text{ in } U\big)$$

in $L_2(U)$, at least when z belongs to the open set

$$G_U = \{z|\ z+s \text{ in } G \text{ for all } s \text{ in } \overline{U}\},$$

where $\overline{U}$ denotes the closure of U in $\mathbf{C}$. Theorem 3 will tell us what hyperplanes in $L_2(U)$ are hit by the curve Φ. Since $L_2(U)$ is a Hilbert space, a hyperplane in $L_2(U)$ is characterized by an element f in $L_2(U)$ and a complex number w. Moreover, Φ hits the hyperplane corresponding to f and w, if and only if there is a z in G_U such that

$$\Phi_f(z) = w,$$

where

(5) $$\Phi_f(z) = \int_U \bar{f}(s)\log\zeta(z+s)\,d\sigma\,dt$$

and $\bar{f}(s)$ is the complex conjugate of $f(s)$.

THEOREM 3. *Let $\frac{1}{2}+\varepsilon < \sigma_1 < \sigma_2 < 1-\varepsilon$. Let U be an open set in the complex s-plane such that $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ and $|t| \leqslant 1/\varepsilon$ whenever s belongs to U. Let f be in $L_2(U)$ and Φ_f be given in G_U by (5).*

If f is orthogonal to $P_2(U)$, then $\Phi_f(z) = 0$ for all z in G_U. If f is not orthogonal to $P_2(U)$ and $\delta > 0$, then there are c_1, σ_0 and a sequence $(T_j)_{j=1}^{\infty}$ such that $\sigma_1 \leqslant \sigma_0 \leqslant \sigma_2$, $T_j \to \infty$ as $j \to \infty$ and for $T = T_j$, $j = 1, 2, \ldots$, the following is true:

If $0 < R < \sigma_1-\frac{1}{2}-\varepsilon$, $\tau = (\log T)^2$ and

$$W = \{z|\ |y| \leqslant \tau,\ x \geqslant -R\} \cup \{z|\ x \geqslant \tfrac{1}{2}\},$$

then there are more than $T\exp\left(-c_1\dfrac{\log T}{\log\log T}\right)$ positive integers $n \leqslant T$ such that G_U contains $3n\tau i+W$ and $\Phi_f(3n\tau i+W)$ contains

$$\{w|\ |w| \leqslant (\log T)^{1-\sigma_0+R-\delta}\}.$$

Theorem 4 deals with the question, how close the curve Φ can come to a given element of $L_2(U)$. Thus Theorem 3 and 4 are dual to each other in some sense. But neither can directly be deduced from the other.

THEOREM 4. *Let a satisfy (1). Let σ_1, σ_2 and U be as in Theorem 3. Denote by $\Gamma_{\lambda\varrho}$, $\lambda < \varrho$, the set of functions g in $L_2(U)$ which are representable as,*

$$g(s) = \sum_{\lambda < p \leqslant \varrho} z_p p^{-s}, \quad s \text{ in } U,$$

where the z_p, $\lambda < p \leqslant \varrho$, are complex numbers and $|z_p| \leqslant 1$. Then there are c_1, c_2, c_3 such that the following is true:

(i) *There exists a function f_0 which is analytic in $\sigma > 1/2$ such that for $T \geqslant c_1$,*

$$c_2 \leqslant V \leqslant \left(\frac{\varrho}{\log \varrho}\right)^{1/2}, \qquad \varrho \leqslant c_3 (\log T \log^{1/2} V)^{1/2} (V \log^{1/2} V)^{-\frac{5}{2(2\sigma_1 - 1)}}$$

and for every g in $\Gamma_{\lambda\varrho}$ there are more than $\frac{1}{5} T V^{-\pi(\varrho)}$ positive integers $n \leqslant T$ satisfying

$$\log \zeta(s + ina) = f_0(s) + g(s) + O\left(\frac{\varrho^{1-\sigma}}{V \log \varrho} + \frac{\lambda^{1/2-\sigma}}{\log \lambda}\right)$$

for all s in U.

(ii) *Each $\Gamma_{\lambda\varrho}$ is a convex circled set in $L_2(U)$, i.e. if g_1, g_2 are in $\Gamma_{\lambda\varrho}$, then also $z_1 g_1 + z_2 g_2$ for all complex numbers z_1, z_2 with $|z_1| + |z_2| \leqslant 1$. The projection of $\Gamma_{\lambda\varrho}$ with respect to any f in $L_2(U)$, i.e. the set of complex numbers*

$$\int_U \bar{f}(s) g(s) \, d\sigma dt, \qquad g \text{ in } \Gamma_{\lambda\varrho},$$

is a closed disc, say $D_{r_f(\lambda,\varrho)}$. Its radius $r_f(\lambda, \varrho) = 0$ for all $\varrho > \lambda$, if and only if f is orthogonal to $P_2(U)$. For every $\delta > 0$, $\lambda > 0$ and f not orthogonal to $P_2(U)$, there is a sequence $(\varrho_j)_{j=1}^{\infty}$ such that $\varrho_j \to \infty$ as $j \to \infty$ and

$$r_f(\lambda, \varrho_j) \geqslant \varrho_j^{1-\sigma_2-\delta}, \qquad j = 1, 2, \ldots$$

If $\Gamma_\lambda(U)$ denotes the closure of the union

$$\bigcup_{\varrho > \lambda} \Gamma_{\lambda\varrho}$$

in $L_2(U)$, then $\Gamma_\lambda(U) = P_2(U)$ for all $\lambda > 0$.

(iii) *Let C be a compact subset of U. Then the functions*

$$s \mapsto \log \zeta(s + ina),$$

where n is an integer and ina belongs to G_U, form everywhere dense subsets in $P_2(U)$ and $P_\infty(C)$.

(iv) *Let β, r' and R' be such that*

$$0 < \beta, r', R' < 1, \qquad \varepsilon \leqslant \beta + R' < 1 \qquad \text{and} \qquad r' < \frac{\delta}{e} e^{-1/\delta},$$

where

$$\delta = \frac{1 - R' - \beta}{\log\left(\dfrac{2e}{R'}\right)}.$$

Let $r = r'(1-\sigma_0)$, $R = R'(1-\sigma_0)$ and $\frac{1}{2} + \varepsilon + r \leqslant \sigma_0 \leqslant 1 - \varepsilon - r$. Let T, ϱ, V be as in part (i) *and let f denote a function of the form*

$$f(z) = \sum_{k=0}^{\infty} w_k z^k$$

where w_k, $k = 0, 1, \ldots$, are complex numbers such that

$$|w_k| \leqslant \frac{\varrho^{\beta(1-\sigma_0)}}{\log^4 \varrho} R^{-k}, \qquad k = 0, 1, \ldots$$

Then $a = \delta \log\left(\dfrac{\delta}{er'}\right) - 1$ is positive and the number of positive integers $n \leqslant T$ such that

$$\log \zeta(s + ina) = f_0(s) + f(s - \sigma_0) + O\left(\frac{\varrho^{1-\sigma}}{V \log \varrho} + \frac{\varrho^{(1-\delta)(1/2-\sigma)}}{\log \varrho} + \frac{\varrho^{-a(1-\sigma_0)}}{\log \varrho}\right)$$

for $|s - \sigma_0| \leqslant r$ exceeds $\frac{1}{5} T V^{-\pi(\varrho)}$.

Remark. Since $P_2(D_r) = H_2(D_r)$ and D_r is simply connected, (V3) follows from Theorem 4 (iii) by exponentiation. Part (i) and (ii) show more precisely, how the approximation in (iii) takes place. From a quantitative point of view it would be desirable to have a more explicit description of $\Gamma_{\lambda\varrho}$ as $\varrho \to \infty$ than that given in (ii). Theorem 4 (iv) is an instance of what such a description should look like. It shows that considerably more than (V3) is true, if r is sufficiently small.

3. Auxiliary lemmas.

LEMMA 1. *Let G be as in § 2 and set $\tau = \log^2 T$ for $T > 0$. Then the following holds:*

(i) *Suppose that $0 < d \leqslant \frac{1}{2}$, $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $0 \leqslant t \leqslant T$. If*

$$\{w \mid \sigma \leqslant u \text{ and } |v - t| \leqslant 2\tau\}$$

is contained in G, then

$$\int_{-\tau}^{\tau} \log \zeta(s + iy) \left(\frac{\sin(dy)}{y}\right)^2 (1 + \cos(y \log \nu)) \, dy$$

$$= \frac{\pi}{4} \sum_{|\log(m/\nu)| \leqslant d} \Lambda_1(m) m^{-s} \left(d - \left|\log \frac{m}{\nu}\right|\right) + O(\nu \tau^{-1}), \qquad t \to \infty,$$

for $\nu \geqslant 1$.

(ii) *Let* $\mu = \tau^{1/\varepsilon}$ *and* $0 < t < T$. *Define* φ *by*

$$\varphi(\varrho) = \min\{1, \max(2-\varrho, 0)\}, \qquad \varrho \geqslant 0.$$

If

$$\{w|\ \tfrac{1}{2}+\varepsilon \leqslant u,\ |v-\tau| \leqslant 2\tau^{1/2+1/\varepsilon}\}$$

is contained in G, *it follows that*

$$\log\zeta(w) = \sum_{m=1}^{\infty} \Lambda_1(m) m^{-w} \varphi\left(\frac{m}{\mu}\right) + O(\tau^{-1/2}\log\tau), \qquad t \to \infty,$$

for $\frac{1}{2}+2\varepsilon \leqslant u \leqslant 1$ *and* $|v-\tau| \leqslant \tau^{1/2+1/\varepsilon}$.

Proof. Part (i) follows from equation (13) in [6] as (16) in [6]. Since we have

$$\varphi(\varrho) = 2-\varrho-\max(1-\varrho, 0)$$

for $0 \leqslant \varrho \leqslant 2$, part (ii) follows, if we show that under the conditions in (ii)

$$\log\zeta(w) = \sum_{m \leqslant \mu} \Lambda_1(m) m^{-w}\left(1-\frac{m}{\mu}\right) + O(\tau^{-1/2}\log\tau), \qquad t \to \infty \tag{6}$$

for $\frac{1}{2}+2\varepsilon \leqslant u \leqslant 1$ and $|v-\tau| \leqslant \tau^{1/2+1/\varepsilon}$. Now a version of Perron's formula asserts that

$$\sum_{m \leqslant \mu} \Lambda_1(m) m^{-w}\left(1-\frac{m}{\mu}\right) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \log\zeta(w+z)\frac{\mu^z}{z(z+1)}\,dz \tag{7}$$

for such w. Let $\mathscr{C}$ denote the path in the z-plane from $1-i\infty$ to $1+i\infty$ which passes through the vertices $1-i\infty$, $1-\frac{i}{2}\tau^{1/2+1/\varepsilon}$, $\frac{1}{2}+\varepsilon-u+\frac{1}{\tau}-\frac{i}{2}\tau^{1/2+1/\varepsilon}$, $\frac{1}{2}+\varepsilon-u+\frac{1}{\tau}+\frac{i}{2}\tau^{1/2+1/\varepsilon}$, $1+\frac{i}{2}\tau^{1/2+1/\varepsilon}$, $1+i\infty$ in this order and is linear in between. Since $u \leqslant 1$ and $|v-t| \leqslant \tau^{1/2+1/\varepsilon}$ in (7), Cauchy's theorem implies that

$$\log\zeta(w) = \sum_{m \leqslant \mu} \Lambda_1(m) m^{-w}\left(1-\frac{m}{\mu}\right) + \frac{1}{2\pi i}\int_{\mathscr{C}} \log\zeta(w+z)\frac{\mu^z}{z(z+1)}\,dz \tag{8}$$

for $\tau > 1/\varepsilon$. Denote by I_j, $j = 0, \pm1, \pm2$, the contribution to the integral in (8) coming from the integration over that part of $\mathscr{C}$ which connects the $(3+j)$th with the $(4+j)$th vertex. Since $\log\zeta(w+z)$ is bounded on $x = 1$, we conclude that

$$I_{\pm2} = O(\mu\tau^{-1/2+1/\varepsilon}) = O(\tau^{-1/2}), \qquad t \to \infty. \tag{9}$$

Theorem 9.2 and 9.6 (B) in [10] imply that

$$\log\zeta(w+z) = O\left(\tau^{1/2}\log\left(\frac{2}{u+x-\frac{1}{2}-\varepsilon}\right)\right), \qquad t \to \infty,$$

for $\frac{1}{2}+\varepsilon \leqslant u+x \leqslant 2$ and $|v-t| \leqslant 2\tau^{1/2+1/\varepsilon}-1$. Hence it follows that

$$I_{\pm1} = O\left(\mu\tau^{1/2-1-2/\varepsilon}\int_{1/\tau}^{2} \log\left(\frac{2}{\eta}\right)d\eta\right) = O(\tau^{-1/2}), \qquad t \to \infty, \tag{10}$$

and

$$I_0 = O\left(\mu^{1/2+\varepsilon-u+1/\tau}\tau^{1/2}\log\tau\int_{0}^{\infty}\frac{dy}{1+y^2}\right) = O(\mu^{-\varepsilon}\tau^{1/2}\log\tau) \tag{11}$$

$$= O(\tau^{-1/2}\log\tau), \qquad t \to \infty.$$

Thus (8)–(11) imply (6) and the lemma is proved.

LEMMA 2. *There are positive numbers* $c_1, \ldots, c_4$ *such that the following is true for* $c_1 \leqslant T$, $\tau = (\log T)^2$, $\mu = \tau^{1/\varepsilon}$, $c_2 \leqslant \nu \leqslant \mu$ *and* l *a positive integer*:

(i) *The number of positive integers* $n \leqslant T/\tau$ *satisfying*

$$\max_{|t| \leqslant \tau+1} |\log\zeta(s+3n\tau i)| \leqslant \nu^{1-\sigma}/\log\nu$$

for $\frac{1}{2}+2\varepsilon \leqslant \sigma \leqslant 1-2\varepsilon$ *exceeds*

$$\frac{T}{\tau} - O\left(T^{1-\varepsilon} + Tl!\left(c_3\frac{\log\nu}{\nu}\right)^l + l!\left(c_3\tau^{1/\varepsilon}\frac{\log\nu}{\nu}\right)^l\right).$$

(ii) *If* $1 \leqslant a \leqslant 2$, $\frac{1}{2}+2\varepsilon \leqslant \sigma_1$, $X > 0$, $0 < Y \leqslant \nu$ *and* φ *is as in Lemma* 1 (ii), *the number of positive integers* $n \leqslant T$ *such that*

$$\left|\sum_{p \geqslant \nu} p^{-s-ina}\varphi\left(\frac{p}{\mu}\right)\right| \leqslant XY^{-\sigma}$$

for $\sigma_1 \leqslant \sigma \leqslant 1-2\varepsilon$ *and* $|t| \leqslant 1/\varepsilon$ *exceeds*

$$T - O\left(\log^2\nu\, l!\left(\frac{c_4}{X^2}\right)^l\left\{T\left(\frac{\nu(Y/\nu)^{2\sigma_1}}{\log\nu}\right)^l + (Y\tau^{1/\varepsilon})^l\right\}\right).$$

Proof. Let the coefficients $a_m = a_m(l, \mu, \nu)$ be defined by

$$\left(\sum_{p \geqslant \nu} p^{-w}\varphi(p/\mu)\right)^l = \sum_{m=1}^{\infty} a_m m^{-w}. \tag{12}$$

Since $a_m \leqslant l!$, we infer from Corollary 3 in [7] that

$$(13)\quad \int_0^T \Big|\sum_{m=1}^{\infty} a_m m^{-w}\Big|^2 dv = T\sum_{m=1}^{\infty} a_m^2 m^{-2u} + O\Big(\sum_{m=1}^{\infty} a_m^2 m^{1-2u}\Big)$$
$$\leqslant Tl!\Big(\sum_{\nu\leqslant p\leqslant\mu} p^{-2u}\Big)^l + O\Big(l!\Big(\sum_{\nu\leqslant p\leqslant\mu} p^{1-2u}\Big)^l\Big)$$
$$\leqslant Tl!\left(c_5\frac{\nu^{1-2u}}{\log\nu}\right)^l + l!\left(c_5\frac{\mu^{2-2u}}{\log\mu}\right)^l,\quad \tfrac12+\varepsilon\leqslant u\leqslant 1-\varepsilon,$$

for a suitable c_5. Now let f be an analytic function in D_δ, $\delta > 0$. Then we have by Cauchy's theorem and Hölder's inequality

$$(14)\quad |f(0)|^{2l} = \left|\frac{1}{\pi\delta^2}\int_0^\delta\int_0^{2\pi} f(re^{i\varphi})r\,d\varphi\,dr\right|^{2l} \leqslant (\pi\delta^2)^{-2l}\int_{|z|\leqslant\delta}|f(z)|^{2l}dxdy\,(\pi\delta^2)^{2l-1}$$
$$= \frac{1}{\pi\delta^2}\int_{|z|\leqslant\delta}|f(z)|^{2l}dxdy.$$

Suppose that $1\leqslant\eta\leqslant T$, $\delta\leqslant\varepsilon$, $X>0$ and $0<Y\leqslant\nu$. Then we get from (12)–(14)

$$(15)\quad \sum_{1\leqslant n\leqslant T/\eta}\ \max_{\substack{\sigma_1\leqslant\sigma\leqslant 1-2\varepsilon\\ |t|\leqslant\eta/\varepsilon}}\left|\sum_{p\geqslant\nu} p^{-s-in\eta}\varphi\Big(\frac{p}{\mu}\Big)\frac{Y^s}{X}\right|^{2l}$$
$$\leqslant \frac{2}{\pi\delta^2\varepsilon}\int_{\sigma_1-\delta}^{1-\varepsilon}\int_{-T/\varepsilon}^{2T/\varepsilon}\left|\sum_{p\geqslant\nu} p^{-w}\varphi\Big(\frac{p}{\mu}\Big)\frac{Y^w}{X}\right|^{2l} dv\,du$$
$$\leqslant \frac{2l!}{\pi\delta^2\varepsilon}\left(\frac{c_5}{X^2}\right)^l\left\{\frac{3T}{\varepsilon}\left(\frac{\nu}{\log\nu}\right)^l\int_{\sigma_1-\delta}^{1-\varepsilon}\left(\frac{Y}{\nu}\right)^{2ul}du + \left(\frac{\mu^2}{\log\mu}\right)^l\int_{\sigma_1-\delta}^{1-\varepsilon}\left(\frac{Y}{\mu}\right)^{2ul}du\right\}$$
$$\leqslant \frac{l!}{\pi\delta^2\varepsilon}\left(\frac{c_5}{X^2}\right)^l\left\{\frac{3T}{\varepsilon}\left(\frac{\nu}{\log\nu}\right)^l\left(\frac{Y}{\nu}\right)^{2l(\sigma_1-\delta)} + \left(\frac{\mu Y}{\log\mu}\right)^l\right\}$$

for $\frac12+2\varepsilon\leqslant\sigma_1\leqslant 1-2\varepsilon$. Hence part (ii) follows from (15) with $\eta = a$ and $\delta = 1/\log\nu$. If we choose $\eta = 3\tau$, $\delta = \varepsilon$, $X = c_6\nu/\log\nu$ and $Y = \nu$, (15) also implies that for a sufficiently large c_6 the number of positive integers $n\leqslant T/\tau$ satisfying

$$(16)\quad \max_{|t|\leqslant\tau+1}\left|\sum_{p\geqslant\nu} p^{-s-3n\tau i}\varphi\Big(\frac{p}{\mu}\Big)\right| \leqslant c_6\frac{\nu^{1-\sigma}}{\log\nu}\qquad\text{for}\qquad \tfrac12-2\varepsilon\leqslant\sigma\leqslant 1-2\varepsilon$$

exceeds

$$\frac{T}{\tau} - O\left(Tl!\left(\frac{c_5\log\nu}{c_6^2\nu}\right)^l + l!\left(\frac{c_5\tau^{1/\varepsilon}}{c_6^2\nu}\log\nu\right)^l\right).$$

Now theorem 9.19 (B) in [10] tells us that

$$N(\tfrac12+\varepsilon, T) = O(T^{3(1/2-\varepsilon)/(3/2-\varepsilon)}\log^5 T) = O(T^{1-4\varepsilon/3}\tau^{5/2}),\qquad T\to\infty,$$

where $N(\sigma, T)$ denotes the number of zeros of $\zeta(w)$ in the rectangle $\sigma\leqslant u\leqslant 1$ and $|v|\leqslant T$. Hence the number of positive integers $n\leqslant T/\tau$ such that

$$(17)\qquad \{w\mid \tfrac12+\varepsilon\leqslant u \text{ and } |v-3n\tau|\leqslant 2\tau^{1/2+1/\varepsilon}\}\subset G$$

exceeds

$$(18)\qquad T/\tau - O(T^{1-4\varepsilon/3}\tau^{2+1/\varepsilon}) = T/\tau - O(T^{1-\varepsilon}).$$

Now

$$(19)\qquad \sum_{\substack{m\leqslant\varrho\\ m\neq p}} \Lambda_1(m) = O\left(\frac{\varrho^{1/2}}{\log\varrho}\right),\qquad \varrho\to\infty.$$

Thus, if (17) holds for n, we infer from Lemma 1 (ii) that

$$(20)\qquad \log\zeta(s+3n\tau i) = \sum_{m=1}^{\infty}\Lambda_1(m)m^{-s-3n\tau i}\varphi\left(\frac{m}{\mu}\right) + O(\tau^{-1/2}\log\tau)$$
$$= \sum_p p^{-s-3n\tau i}\varphi\left(\frac{p}{\mu}\right) + O(1)$$

for $\frac12+2\varepsilon\leqslant\sigma\leqslant 1$ and $|t|\leqslant\tau+1$. Hence part (i) of the lemma follows from (16)–(18) and (20) by a suitable choice of c_3.

The next lemma is essentially due to Montgomery [6].

Lemma 3. *There are $c_1, \ldots, c_3$ such that for $c_1\leqslant T$, $\tau = (\log T)^2$ and $c_2\leqslant\nu\leqslant\tau$ the number of positive integers $n\leqslant T/\tau$ satisfying*

$$\max_{|t|\leqslant\tau}|\log\zeta(s+3n\tau i)|\geqslant\frac{\nu^{1-\sigma}}{\log\nu}$$

for $\frac12+2\varepsilon\leqslant\sigma\leqslant 1-2\varepsilon$ exceeds

$$\frac{T}{\tau}\exp\left(-c_3\frac{\nu}{\log\nu}\right) - O(T^{1-\varepsilon}).$$

Proof. It follows from Lemma 1 (i), (17) and (18) that

$$(21)\qquad \int_{-\tau}^{\tau}\log\zeta(\sigma+i(t+3n\tau))\left(\frac{\sin t/2}{t}\right)^2(1+\cos(t\log\nu))\,dt$$
$$= \frac{\pi}{4}\sum_{|\log\frac{m}{\nu}|\leqslant\frac12}\Lambda_1(m)m^{-\sigma-3n\tau i}\left(\tfrac12-\left|\log\frac{m}{\nu}\right|\right) + O(\nu\tau^{-1}),\qquad \tfrac12+2\varepsilon\leqslant\sigma\leqslant 1,$$

for more than $T/\tau - O(T^{1-\varepsilon})$ positive integers $n \leqslant T/\tau$. Dirichlet's theorem on the simultaneous diophantine approximation (p. 152 in [10]) says that the number of positive integers $n \leqslant T/\tau$ such that

$$\min_{l \text{ integral}} \left| 3n\tau \frac{\log p}{2\pi} - l \right| \leqslant \frac{1}{6} \tag{22}$$

for all p with $\left|\log \frac{p}{\nu}\right| \leqslant \frac{1}{2}$ is at least

$$\frac{T}{\tau} \exp\{\log 6\, (\pi(e^{-1/2}\nu) - \pi(e^{1/2}\nu))\} \geqslant \frac{T}{\tau} \exp\left(-c_3 \frac{\nu}{\log \nu}\right). \tag{23}$$

The inequality in (23) follows from the prime number theorem. Suppose now that (21) and (22) hold. Then we obtain by appealing to (19) and once more to the prime number theorem

$$\begin{aligned} (24)\quad & \max_{|t|\leqslant\tau} |\log\zeta(s+3n\tau i)| \\ & \geqslant \max_{|t|\leqslant\tau} |\log\zeta(s+3n\tau i)| \frac{1}{\pi} \int_{-\tau}^{\tau} \left(\frac{\sin t/2}{t}\right)^2 (1+\cos(t\log\nu))\,dt \\ & \geqslant \frac{1}{\pi} \left| \int_{-\tau}^{\tau} \log\zeta(s+3n\tau i) \left(\frac{\sin t/2}{t}\right)^2 (1+\cos(t\log\nu))\,dt \right| \\ & \geqslant \frac{1}{4} \left| \sum_{|\log(p/\nu)|\leqslant\frac{1}{2}} p^{-\sigma-3n\tau i} \left(\frac{1}{2} - \left|\log\frac{p}{\nu}\right|\right) \right| - O(1) \\ & \geqslant \frac{1}{8} \sum_{|\log(p/\nu)|\leqslant\frac{1}{2}} p^{-\sigma} \left(\frac{1}{2} - \left|\log\frac{p}{\nu}\right|\right) - O(1) \geqslant c_4 \frac{\nu^{1-\sigma}}{\log\nu} \end{aligned}$$

for $\frac{1}{2} + 2\varepsilon \leqslant \sigma \leqslant 1 - 2\varepsilon$. The lemma now follows from (21)–(24).

The following lemma was proved by J. F. Koksma in [5]. It contains the quantitative form of Weyl's criterion we referred to in § 1.

LEMMA 4. *Let L be a positive integer and $a_1, \ldots, a_L$; $b_1, \ldots, b_L$ be real numbers such that $a_l < b_l \leqslant a_l + 1$ for $l = 1, \ldots, L$. Let Q denote the set of points $\boldsymbol{x} = (x_1, \ldots, x_L)$ in $\boldsymbol{R}^L$, the L-dimensional real vector-space, satisfying $a_l \leqslant x_l < b_l$ for $l = 1, \ldots, L$. Let $\boldsymbol{a} = (a_1, \ldots, a_L)$ be an L-tupel of real numbers and let $N(Q)$ denote the number of points $n\boldsymbol{a} = (na_1, \ldots, na_L)$, $1 \leqslant n \leqslant N$, which, modulo 1, lie in Q. Let H_l, $l = 1, \ldots, L$, be greater than 1 and*

$$\gamma_{h,l} = \begin{cases} b_l - a_l + 75/H_l, & \text{for } h = 0, \\ \min\left(\gamma_{0,l}, \dfrac{30}{|h|}\right), & \text{for } h \neq 0. \end{cases}$$

Then we have

$$\left| \frac{N(Q)}{N} - \prod_{l=1}^{L} (b_l - a_l) \right| \leqslant \prod_{l=1}^{L} (b_l - a_l) \left\{ \prod_{l=1}^{L} \left(1 + \frac{75}{H_l(b_l - a_l)}\right) - 1 \right\} + \frac{1}{N} \sum_{(\boldsymbol{h})}{}^{*} \left| \sum_{n=1}^{N} \exp\left(2\pi i n \sum_{l=1}^{L} a_l h_l\right) \right| \prod_{l=1}^{L} \gamma_{h_l, l},$$

where the sum $\sum_{(\boldsymbol{h})}^{*}$ extends over all $\boldsymbol{h} = (h_1, \ldots, h_L) \neq (0, \ldots, 0)$ such that h_l is integral and $|h_l| \leqslant H_l(1 + \min(\log H_l, \log L))$, $l = 1, \ldots, L$.

LEMMA 5. *Let a be a real number satisfying (1). Then there exist c_1, c_2 such that, if $\frac{1}{2} + 2\varepsilon \leqslant \sigma_1 \leqslant 1$, $1 \leqslant V \leqslant \varrho < \nu$, $c_1 \leqslant \varrho$, $V\nu^2(\nu/\varrho)^{1-\sigma_1} \leqslant c_2 \log T$ and θ_p, $p \leqslant \varrho$, are any real numbers,*

$$\left| \sum_{p\leqslant\nu} p^{-s-ina} - \sum_{p\leqslant\varrho} p^{-s} e^{-2\pi i\theta_p} \right| = O\left(\frac{\varrho^{1/2-\sigma}}{\log\varrho} \left(\log^{1/2}\varrho + \frac{\varrho^{1/2}}{V} \right) \right),$$

$$\sigma_1 \leqslant \sigma \leqslant 1 - \varepsilon \text{ and } |t| \leqslant 1/\varepsilon,$$

or more than $\frac{1}{4} T V^{-\pi(\varrho)}$ positive integers $n \leqslant T$.

Proof. Let J denote the integral part of $V(\nu/\varrho)^{1-\sigma_1}$. With $\boldsymbol{\theta} = (\theta_p)_{p\leqslant\nu}$, where the θ_p are real, we associate the set

$$Q(\boldsymbol{\theta}) = \left\{ \boldsymbol{x} = (x_p)_{p\leqslant\nu} \,\middle|\, \begin{array}{l} \theta_p - \dfrac{1}{2V} \leqslant x_p < \theta_p + \dfrac{1}{2V} \text{ for } p \leqslant \varrho \text{ and} \\ \theta_p - \dfrac{1}{2J} \leqslant x_p < \theta_p + \dfrac{1}{2J} \text{ for } \varrho < p \leqslant \nu \end{array} \right\} \tag{25}$$

in $\boldsymbol{R}^{\pi(\nu)}$. We now apply the preceding lemma with $L = \pi(\nu)$, $Q = Q(\boldsymbol{\theta})$ and $\boldsymbol{a} = \frac{a}{2\pi}(\log p)_{p\leqslant\nu}$. Thus we have

$$N(Q(\boldsymbol{\theta})) \geqslant \tfrac{1}{3} T V^{-\pi(\varrho)} J^{\pi(\varrho)-\pi(\nu)}, \tag{26}$$

provided that the inequalities

$$\left| \prod_{p\leqslant\varrho} \left(1 + \frac{75V}{H_p}\right) \prod_{\varrho<p\leqslant\nu} \left(1 + \frac{75J}{H_p}\right) - 1 \right| \leqslant \frac{1}{3} \tag{27}$$

and

$$\sum_{(\boldsymbol{h})}{}^{*} \left| \sum_{1\leqslant n\leqslant T} \exp\left(ina \sum_{p\leqslant\nu} h_p \log p\right) \right| \prod_{p\leqslant\nu} \gamma_{h_p, p} \leqslant \tfrac{1}{3} T V^{-\pi(\varrho)} J^{\pi(\varrho)-\pi(\nu)} \tag{28}$$

hold with some positive numbers $H_p > 1$, $p \leqslant \nu$. We set $H_p = H/\log\nu$ for $p \leqslant \nu$ and assume that $H \geqslant \nu$. Using simple inequalities we see that the left-hand side of (27) is smaller than

$$75J\frac{\log\nu}{H}\pi(\nu)\exp\left(75J\frac{\log\nu}{H}\pi(\nu)\right).$$

Hence (27) is true, provided that

$$(29)\qquad J\nu/H\leqslant c_3$$

for a sufficiently small c_3. On the other hand, let l denote an integer of minimal distance from $\frac{a}{2\pi}\sum_{p\leqslant\nu}h_p\log p$. Then it follows from (1) that

$$|l|\leqslant\frac{2}{\pi}\sum_{p\leqslant\nu}|h_p|\log p+\frac{1}{2}$$

and

$$(30)\qquad \Big|\sum_{1\leqslant n\leqslant T}\exp\Big(ina\sum_{p\leqslant\nu}h_p\log p\Big)\Big|\leqslant\Big|\sin\Big(\frac{a}{2}\sum_{p\leqslant\nu}h_p\log p\Big)\Big|^{-1}$$

$$\leqslant\frac{\pi}{a}\Big|\sum_{p\leqslant\nu}h_p\log p-\frac{2\pi l}{a}\Big|^{-1}\leqslant\frac{\pi}{a}e^{\pi/a}\Big|e^{2\pi l/a}\prod_{p\leqslant\nu}p^{-h_p}-1\Big|^{-1}$$

$$=O\Big(\prod_{p\leqslant\nu}p^{|h_p|}\Big(\sum_{p\leqslant\nu}|h_p|\log p\Big)^4\Big),$$

if $l\neq 0$ and not all h_p are zero. But, since

$$\Big|\sum_{p\leqslant\nu}h_p\log p\Big|\geqslant\prod_{p\leqslant\nu}p^{-|h_p|},$$

if not all h_p are zero, (30) also holds in case of $l=0$. By our choice of H_p we have

$$H_p\big(1+\min(\log H_p,\log\pi(\nu)\big)\leqslant 2H$$

for ν sufficiently large. Thus we conclude from Lemma 4 and (30) that the left-hand side of (28) is

$$(31)\qquad O\Big((H\nu)^4\sum_{(h)}^{*}\prod_{p\leqslant\nu}\gamma_{h_p,p}p^{|h_p|}\Big)=O\Big((H\nu)^4\prod_{p\leqslant\nu}\Big(1+60\sum_{1\leqslant h_p\leqslant 2H}\frac{p^{|h_p|}}{h_p}\Big)\Big)$$

$$=O\Big((H\nu)^4\prod_{p\leqslant\nu}\Big(c_4\frac{p^{2H}}{H}\Big)\Big)=O\big(\exp(c_5H\nu)\big),\qquad \nu\to\infty.$$

It follows from our assumptions on V and J that

$$J^{\pi(\nu)-\pi(\varrho)}V^{\pi(\varrho)}=O(e^{3\nu}),\qquad \nu\to\infty.$$

Hence (31) implies that (28) holds for $\nu\geqslant c_6$ provided that

$$(32)\qquad 2c_5H\nu\leqslant\log T.$$

Thus (27)–(29) and (32) show that (26) is true, if

$$(33)\qquad c_6\leqslant\nu,\qquad V\nu^2(\nu/\varrho)^{1-\sigma_1}\leqslant c_2\log T$$

and c_2, c_6 are suitably chosen. If $\sum_{(j)}$ denotes the summation over all $j=(j_p)_{\varrho<p\leqslant\nu}$, where the j_p are positive integers not exceeding J, we get from (14) with $l=1$ and $\delta=\varepsilon$

$$(34)\qquad \sum_{(j)}\max_{\substack{1/2+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon\\ |t|\leqslant 1/\varepsilon}}\Big|\varrho^s\sum_{\varrho<p\leqslant\nu}p^{-s}e^{-2\pi ij_p/J}\Big|^2$$

$$\leqslant\frac{1}{\pi\varepsilon^2}\sum_{(j)}\int_{-2/\varepsilon}^{2/\varepsilon}\int_{1/2+\varepsilon}^{1}\Big|\varrho^s\sum_{\varrho<p\leqslant\nu}p^{-s}e^{-2\pi ij_p/J}\Big|^2d\sigma dt$$

$$=\frac{1}{\pi\varepsilon^2}J^{\pi(\nu)-\pi(\varrho)}\int_{-2/\varepsilon}^{2/\varepsilon}\int_{1/2+\varepsilon}^{1}\varrho^{2\sigma}\sum_{\varrho<p\leqslant\nu}p^{-2\sigma}d\sigma dt=O\Big(\frac{\varrho}{\log\varrho}J^{\pi(\nu)-\pi(\varrho)}\Big),\qquad \varrho\to\infty.$$

From now on we assume that the θ_p, $\varrho<p\leqslant\nu$, are of the form j_p/J, where the positive integers j_p do not exceed J. Hence, if $\varrho>c_1$, (25) and (34) imply that we can find more than $\frac{3}{4}J^{\pi(\nu)-\pi(\varrho)}$ disjoint sets $Q(\theta)$ which agree in all the θ_p for $p\leqslant\varrho$ and satisfy

$$(35)\qquad \Big|\sum_{\varrho<p\leqslant\nu}p^{-s}e^{-2\pi i\theta_p}\Big|=O\Big(\frac{\varrho^{1/2-\sigma}}{(\log\varrho)^{1/2}}\Big)$$

for $\frac{1}{2}+2\varepsilon\leqslant\sigma\leqslant 1-\varepsilon$ and $|t|\leqslant 1/\varepsilon$. Suppose now that $\frac{na}{2\pi}(\log p)_{p\leqslant\nu}$ belongs to $Q(\theta)$ modulo 1. Then it follows from (25) that

$$(36)\qquad \Big|\sum_{p\leqslant\nu}p^{-s-ina}-\sum_{p\leqslant\nu}p^{-s}e^{-2\pi i\theta_p}\Big|=O\Big(\frac{1}{V}\sum_{p\leqslant\varrho}p^{-\sigma}+\frac{1}{J}\sum_{\varrho\leqslant p\leqslant\nu}p^{-\sigma}\Big)$$

$$=O\Big(\frac{\varrho^{1-\sigma}}{V\log\varrho}\Big)$$

for $\sigma_1\leqslant\sigma\leqslant 1-\varepsilon$. Combining (26), (33), (35) and (36) we get what we want.

LEMMA 6. *Let K and L be positive integers and $K\leqslant L$. Let a_{kl} and b_k, $1\leqslant k\leqslant K$, $1\leqslant l\leqslant L$, denote complex numbers. Suppose that the system of equations*

$$(37)\qquad \sum_{l=1}^{L}a_{kl}z_l=b_k,\qquad 1\leqslant k\leqslant K,$$

has a solution $(z_1, \ldots, z_L)$ *belonging to*

$$\Delta^L = \{(z_1, \ldots, z_L)|\ z_l \text{ complex and } |z_l| \leqslant 1 \text{ for } 1 \leqslant l \leqslant L\}.$$

Then (37) *also has a solution* $(z_1', \ldots, z_L')$ *in* Δ^L *such that* $|z_l'| = 1$ *for at least* $L-K$ *positive integers* $l \leqslant L$.

Proof. We proceed by induction on L. There is nothing to prove for $K = L$, and so the lemma holds for $L = 1$. If $1 \leqslant K < L$, the solutions of (37) form a linear manifold of (complex) dimension $\geqslant L-K > 0$. This linear manifold has a non-empty intersection with Δ^L by assumption. Hence its intersection with the boundary of Δ^L is non-empty as well, i.e. without loss of generality we may assume the existence of a real number θ such that the system

$$\sum_{l=1}^{L-1} a_{kl} z_l = b_k - a_{kL} e^{i\theta}, \quad k = 1, \ldots, K, \tag{38}$$

has a solution $(z_1, \ldots, z_{L-1})$ in Δ^{L-1}. Now (38) is a systen of K equations in $L-1$ unknowns. It has a solution $(z_1', \ldots, z_{L-1}')$ in Δ^{L-1} with $|z_l'| = 1$ for at least $L-1-K$ positive integers $l \leqslant L-1$ by the induction hypothesis. Hence the lemma is proved.

LEMMA 7. *Let a be a real number satisfying* (1). *Then there exist* c_1, c_2, c_3 *and a function* f_0 *which is analytic in* $\sigma > \frac{1}{2}$ *such that the following is true:*
If $c_1 \leqslant T$, $1 \leqslant \lambda$, $\sigma_1 \geqslant \frac{1}{2} + 2\varepsilon$, $c_2 \leqslant V \leqslant (\varrho/\log\varrho)^{1/2}$,

$$\varrho \leqslant c_3 (\log T \log^{1/2} V)^{1/2} (V \log^{1/2} V)^{-\frac{5}{2(2\sigma_1 - 1)}}$$

and z_p, $\lambda < p \leqslant \varrho$, *are complex numbers of modulus* $\leqslant 1$, *then there are more than* $\frac{1}{5} T V^{-\pi(\varrho)}$ *positive integers* $n \leqslant T$ *such that*

$$\log\zeta(s+ina) = f_0(s) + \sum_{\lambda < p \leqslant \varrho} z_p p^{-s} + O\left(\frac{\varrho^{1-\sigma}}{V \log\varrho} + \frac{\lambda^{1/2-\sigma}}{\log\lambda}\right)$$

for $\sigma_1 \leqslant \sigma \leqslant 1 - 2\varepsilon$, $|t| \leqslant 1/\varepsilon$.

Proof. Let $\tau = (\log T)^2$ and $\mu = \tau^{1/\varepsilon}$. Then it follows in the same way as (17) and (20) that the number of positive integers $n \leqslant T$ with

$$\log\zeta(s+ina) = \sum_{m=1}^{\infty} \Lambda_1(m) m^{-s-ina} \varphi\left(\frac{m}{\mu}\right) + O(\tau^{-1/2}\log\tau), \tag{39}$$

$$\tfrac{1}{2} + 2\varepsilon \leqslant \sigma \leqslant 1, \quad |t| \leqslant 1/\varepsilon,$$

exceeds $T - O(T^{1-\varepsilon})$. We also have by (19)

$$\sum_{\substack{m \geqslant \lambda \\ m \neq p}} \Lambda_1(m) m^{-s-ina} \varphi\left(\frac{m}{\mu}\right) = O\left(\frac{\lambda^{1/2-\sigma}}{\log\lambda}\right), \quad \lambda \to \infty, \tag{40}$$

for $\sigma \geqslant \frac{1}{2} + 2\varepsilon$. Thus we infer from Lemma 2 (ii), (39) and (40) that

$$\log\zeta(s+ina) = \sum_{\substack{m \leqslant \lambda \\ m \neq p}} \Lambda_1(m) m^{-s-ina} + \sum_{p \leqslant \nu} p^{-s-ina} + \tag{41}$$

$$+ O\left(\tau^{-1/2}\log\tau + \frac{\lambda^{1/2-\sigma}}{\log\lambda} + XY^{-\sigma}\right),$$

$$\sigma_1 \leqslant \sigma \leqslant 1 - 2\varepsilon, \ |t| \leqslant 1/\varepsilon,$$

for more than

$$T - O\left(T^{1-\varepsilon} + \log^2\nu\, l! \left(\frac{c_4}{X^2}\right)^l \left\{T\left(\frac{\nu (Y/\nu)^{2\sigma_1}}{\log\nu}\right)^l + (Y\tau^{1/\varepsilon})^l\right\}\right)$$

positive integers $n \leqslant T$, where $T \geqslant c_1$, $X > 0$, $c_5 \leqslant Y \leqslant \nu \leqslant \mu$, $1 \leqslant \lambda \leqslant \mu$ and l is a positive integer.

Next we choose real numbers θ_p^0, p prime, satisfying

$$\left|\sum_{p \leqslant \lambda} e^{-2\pi i \theta_p^0}\right| = O\left(\frac{\lambda^{1/2}}{\log\lambda}\right), \quad \lambda \to \infty. \tag{42}$$

It follows again from (19) that

$$\sum_{j=2}^{\infty} \frac{1}{j} \sum_{p} (e^{-2\pi i \theta_p^0} p^{-s})^j$$

is absolutely and uniformly convergent in $\sigma \geqslant \frac{1}{2} + 2\varepsilon$, whereas (42) implies the uniform convergence of

$$\sum_{p} e^{-2\pi i \theta_p^0} p^{-s}$$

on compact subsets of $\sigma > \frac{1}{2}$. Hence the function f_0 defined by

$$f_0(s) = \sum_{j=1}^{\infty} \frac{1}{j} \sum_{p} (e^{-2\pi i \theta_p^0} p^{-s})^j \tag{43}$$

is analytic in $\sigma > \frac{1}{2}$. Furthermore, if

$$\left|\theta_p^0 - \frac{na}{2\pi}\log p\right| \leqslant \frac{1}{2V} \quad \text{for} \quad p \leqslant \lambda, \tag{44}$$

it follows that

$$(45)\qquad \sum_{\substack{m\leqslant\lambda\\ m\neq p}} \Lambda_1(m)m^{-s-ina}+\sum_{p\leqslant\lambda} e^{-2\pi i\theta_p^0}p^{-s}$$

$$= f_0(s)+O\left(\Big|\sum_{p>\lambda} e^{-2\pi i\theta_p^0}p^{-s}\Big|\right)+O\left(\sum_{\substack{m>\lambda\\ m\neq p}}\Lambda_1(m)m^{-\sigma}\right)+O\left(\frac{1}{V}\sum_{\substack{m\leqslant\lambda\\ m\neq p}}\Lambda_1(m)m^{-\sigma}\right)$$

$$= f_0(s)+O\left(\frac{\lambda^{1/2-\sigma}}{\log\lambda}+\frac{1}{V}\right)$$

for $\frac{1}{2}+2\varepsilon\leqslant\sigma\leqslant 1$ and $|t|\leqslant 1/\varepsilon$.

Now Lemma 5 and its proof show that for any given real numbers θ_p, $\lambda<p\leqslant\varrho$, we can find more than $\frac{1}{4}TV^{-\pi(\varrho)}$ positive integers $n\leqslant T$ such that (44) holds and

(46)

$$\Big|\sum_{p\leqslant\nu}p^{-s-ina}-\sum_{p\leqslant\lambda}e^{-2\pi i\theta_p^0}p^{-s}-\sum_{\lambda<p\leqslant\varrho}e^{-2\pi i\theta_p}p^{-s}\Big| = O\left(\frac{\varrho^{1/2-\sigma}}{\log\varrho}\left(\log^{1/2}\varrho+\frac{\varrho^{1/2}}{V}\right)\right)$$

for $\sigma_1\leqslant\sigma\leqslant 1-2\varepsilon$, $|t|\leqslant 1/\varepsilon$, provided that the conditions of Lemma 5 are fulfilled. Combining (41), (45) and (46) we see that

$$(47)\qquad \log\zeta(s+ina) = f_0(s)+\sum_{\lambda<p\leqslant\varrho}e^{-2\pi i\theta_p}p^{-s}+$$

$$+O\left(\tau^{-1/2}\log\tau+\frac{\lambda^{1/2-\sigma}}{\log\lambda}+XY^{-\sigma}+\frac{1}{V}+\frac{\varrho^{1-\sigma}}{V\log\varrho}\right),\quad \sigma_1\leqslant\sigma\leqslant 1-2\varepsilon,\ |t|\leqslant 1/\varepsilon,$$

for more than

$$(48)\qquad \frac{1}{4}TV^{-\pi(\varrho)}-O\left(T^{1-\varepsilon}+\log^2\nu\, l!\left(\frac{c_4}{X^2}\right)^l\left\{T\left(\frac{\nu(Y/\nu)^{2\sigma_1}}{\log\nu}\right)^l+(Y\tau^{1/\varepsilon})^l\right\}\right)$$

positive integers $n\leqslant T$, if l is a positive integer and

$$(49)\qquad \begin{gathered} c_1\leqslant T,\quad X>0,\quad c_5\leqslant Y\leqslant\nu\leqslant\mu,\quad 1\leqslant\lambda\leqslant\varrho\leqslant\nu,\\ c_2\leqslant V\leqslant(\varrho/\log\varrho)^{1/2},\quad V\nu^2(\nu/\varrho)^{1-\sigma_1}\leqslant c_6\log T.\end{gathered}$$

Now let $X=\dfrac{\varrho}{V\log\varrho}$, $Y=\varrho$ and l be the integral part of $\dfrac{\varrho}{V^2\log\varrho}\times$ $\times(\varrho/\nu)^{1-2\sigma_1}$. Hence it follows from (49) that

$$l\leqslant\varrho\leqslant\nu\leqslant(c_6\log T)^{1/2}$$

and $\tau^{-1/2}\log\tau$ is the smallest term inside the O-symbol in (47). Thus we obtain from (47)–(49)

$$(50)\qquad \log\zeta(s+ina) = f_0(s)+\sum_{\lambda<p\leqslant\varrho}e^{-2\pi i\theta_p}p^{-s}+O\left(\frac{\lambda^{1/2-\sigma}}{\log\lambda}+\frac{\varrho^{1-\sigma}}{V\log\varrho}\right),$$

$$\sigma_1\leqslant\sigma\leqslant 1-2\varepsilon,\ |t|\leqslant 1/\varepsilon,$$

for more than $\frac{1}{5}TV^{-\pi(\varrho)}$ positive integers $n\leqslant T$, if

$$(51)\qquad \begin{gathered} c_1\leqslant T,\quad c_2\leqslant V\leqslant\left(\frac{\varrho}{\log\varrho}\right)^{1/2},\quad 1\leqslant\lambda\leqslant\varrho\leqslant\nu\leqslant\varrho^{c_7},\\ V\nu^2(\nu/\varrho)^{1-\sigma_1}\leqslant c_6\log T\quad\text{and}\quad V^2\log V\leqslant c_8(\nu/\varrho)^{2\sigma_1-1}.\end{gathered}$$

A simple calculation shows the existence of a positive ν such that all inequalities in (51) hold, if

$$(52)\qquad c_1\leqslant T,\quad 1\leqslant\lambda\leqslant\varrho,\quad c_2\leqslant V\leqslant\left(\frac{\varrho}{\log\varrho}\right)^{1/2}\quad\text{and}$$

$$\varrho\leqslant c_3(\log T\log^{1/2}V)^{1/2}(V\log^{1/2}V)^{-\frac{5}{2(2\sigma_1-1)}}$$

for a suitably chosen c_3.

To complete the proof of the lemma it remains to show that the $e^{-2\pi i\theta_p}$ in (50) can be replaced by arbitrary complex numbers of modulus $\leqslant 1$. To this end let $\eta>1$ and K be the integral part of $\dfrac{16}{\varepsilon}\log\eta$. If z_p, $\eta<p\leqslant 2\eta$, are complex numbers of modulus $\leqslant 1$ and $|s|\leqslant 2/\varepsilon$, we have

$$(53)\qquad \sum_{\eta<p\leqslant 2\eta}z_p p^{-s} = \sum_{0\leqslant k\leqslant K}s^k\sum_{\eta<p\leqslant 2\eta}z_p\frac{(-\log p)^k}{k!}+O\left(\sum_{k>K}\frac{\left(\frac{2}{\varepsilon}\log 2\eta\right)^k}{k!}\right)$$

$$= \sum_{0\leqslant k\leqslant K}s^k\sum_{\eta<p\leqslant 2\eta}z_p\frac{(-\log p)^k}{k!}+O(\eta^{-16/\varepsilon}),\quad \eta\to\infty.$$

Next we apply Lemma 6 to the system

$$b_k = \sum_{\eta<p\leqslant 2\eta}\frac{(-\log p)^k}{k!}z_p,\quad k=0,\ldots,K.$$

Thus for any complex numbers z_p with $|z_p|\leqslant 1$, $\eta<p\leqslant 2\eta$, there are complex numbers z_p' with $|z_p'|\leqslant 1$, $\eta<p\leqslant 2\eta$, and $|z_p'|<1$ for at most $K+1$ primes p such that

$$\sum_{\eta<p\leqslant 2\eta}\frac{(-\log p)^k}{k!}z_p = \sum_{\eta<p\leqslant 2\eta}\frac{(-\log p)^k}{k!}z_p',\quad k=0,\ldots,K.$$

Hence we have by (53)

$$\sum_{\eta<p\leqslant 2\eta} z_p p^{-s} = \sum_{\eta<p\leqslant 2\eta} z'_p p^{-s} + O(\eta^{-16/\varepsilon})$$

for $|s|\leqslant 2/\varepsilon$. Therefore real numbers θ_p can be found such that

$$(54)\qquad \sum_{\eta<p\leqslant 2\eta} z_p p^{-s} = \sum_{\eta<p\leqslant 2\eta} e^{-2\pi i\theta_p} p^{-s} + O(K\eta^{-\sigma}+\eta^{-16/\varepsilon})$$

for $|s|\leqslant 2/\varepsilon$. Using (54) with $\eta = 2^j\lambda$ for $1\leqslant j\leqslant 1+\log(\varrho/\lambda)/\log 2$, we see that for any z_p with $|z_p|\leqslant 1$, $\lambda<p\leqslant\varrho$, there are real numbers θ_p, $\lambda<p\leqslant\varrho$, such that

$$(55)\qquad \sum_{\lambda<p\leqslant\varrho} z_p p^{-s} = \sum_{\lambda<p\leqslant\varrho} e^{-2\pi i\theta_p} p^{-s} + O(\lambda^{-\sigma}\log\lambda)$$

for $|s|\leqslant 2/\varepsilon$. Thus the lemma follows from (43), (50)–(52) and (55).

Remark 1. Lemma 7 is not void, since for almost all positive real a (in the sense of Lebesgue) there exists an $\varepsilon>0$ such that (1) holds.

This follows as a special case from

Remark 2. Let ψ be a positive function of two integral variables such that

$$\psi(l, n)\leqslant \tfrac{1}{2}\quad\text{and}\quad \sum_{l,n=1}^{\infty}\psi(l, n)<\infty.$$

Then the set of those positive real numbers b for which

$$(56)\qquad |nb^l-m|\leqslant\psi(|l|, |n|)$$

has infinitely many solutions in non-zero integers l, m and n has Lebesgue measure zero. For the set of positive numbers b satisfying

$$|nb^l-m|\leqslant\psi(l, n)$$

for some positive integers l, m, n has measure $\leqslant \dfrac{2}{lm}\left(\dfrac{m}{n}\right)^{1/l}\psi(l, n)$. Hence for a fixed $\eta>1$ the set of numbers b with

$$1/\eta\leqslant b\leqslant\eta\quad\text{and}\quad |nb^l-m|\leqslant\psi(l, n)$$

for some $l\geqslant L$, $n\geqslant N$ and $m\geqslant 1$ has measure

$$\leqslant 2\sum_{l\geqslant L}\sum_{n\geqslant N}\sum_{1\leqslant m\leqslant 1+\eta^l n}\left(\frac{m}{n}\right)^{1/l}\frac{\psi(l, n)}{lm} = O\Big(\eta\sum_{l\geqslant L}\sum_{n\geqslant N}\psi(l, n)\Big).$$

This last expression tends to zero, if $L\to\infty$ and $N\to\infty$. Therefore the set of numbers b in the interval $[1/\eta, \eta]$ for which

$$|nb^l-m|\leqslant\psi(l, n)\quad\text{or}\quad |nb^{-l}-m|\leqslant\psi(l, n)$$

has infinitely many solutions in positive integers l, m and n has measure zero. Since η was arbitrary, (56) now follows.

Lemma 8. *Let h be a function which satisfies the following conditions:*

(i) *h is analytic in $\boldsymbol{R} = \{z|\ |x|\leqslant R_1,\ |y|\leqslant R_2+R_1\}$, where R_1 and R_2 are positive real numbers.*

(ii) *There is a $\nu>1$ such that*

$$|h(z)|\leqslant\nu^{-x}$$

for all z in $\boldsymbol{R}$.

(iii) *There is a positive number $d\leqslant 1$ such that*

$$\max_{|y|\leqslant R_2}|h(z)|\geqslant d\nu^{-x}$$

for all x with $|x|\leqslant R_1$.

If $c\geqslant 150$ and r is such that $R_1\geqslant r\geqslant 3c/\log\nu$, then (i)–(iii) *imply the existence of a real number t, $|t|\leqslant R_2$, such that $h(it+D_r)$ contains*

$$\{w|\ \nu^r e^{-2c}\leqslant|w|\leqslant d\nu^r e^{-c}\}.$$

Proof. We may assume that c and r satisfy the inequalities $e^{-c}\leqslant d$ and $R_1\geqslant r\geqslant 3c/\log\nu$. For otherwise there is nothing to prove. Now (iii) implies the existence of a real number t such that

$$(57)\qquad |t|\leqslant R_2\quad\text{and}\quad \left|h\left(-r+\frac{c}{2\log\nu}+it\right)\right|\geqslant de^{-c/2}\nu^r.$$

We assume now that the lemma is false. Then there is a complex number w such that

$$(58)\qquad \nu^r e^{-2c}\leqslant|w|\leqslant d\nu^r e^{-c}$$

and

$$(59)\qquad h(z)\neq w$$

for all z in $it+D_r$. Using (ii) we obtain for such an w

$$(60)\qquad \left|\sum_{k=1}^{\infty}\frac{h^k(z)w^{-k}}{k}\right|\leqslant\sum_{k=1}^{\infty}\frac{\nu^{-k(x+r)}e^{2kc}}{k}\leqslant\sum_{k=1}^{\infty}\frac{e^{-kc/2}}{k} = -\log(1-e^{-c/2}),$$

whenever z belongs to $\boldsymbol{R}^c = \boldsymbol{R}\cap\left\{z|\ x>-r+\dfrac{5c}{2\log\nu}\right\}$. Hence the function h_1 given by

$$(61)\qquad h_1(z) = \exp\left(-\frac{1}{c}\sum_{k=1}^{\infty}\frac{h^k(z)w^{-k}}{k}\right)-1$$

is analytic in $\boldsymbol{R}^c$. The assumption (59) implies that h_1 can be analytically continued to $it+D_r$. For this disc is contained in $\boldsymbol{R}$, has a non-empty intersection with $\boldsymbol{R}^c$ and

$$\exp\left(-\sum_{k=1}^{\infty}\frac{h^k(z)w^{-k}}{k}\right)=1-\frac{h(z)}{w}.$$

Again it follows from (ii) that for $c\geqslant 2$ and z in $it+D_r$

(62) $|h_1(z)|$

$$\leqslant 1+\exp\left(\frac{1}{c}\log\left|\frac{h(z)}{w}-1\right|\right)\leqslant 1+\exp\left(\frac{1}{c}\log\left(\frac{|h(z)|}{|w|}+1\right)\right)$$
$$\leqslant 1+\exp\left(\frac{1}{c}\log\left(\frac{\nu^r}{\nu^r e^{-2c}}+1\right)\right)=1+e^2\exp\left(\frac{1}{c}\log(1+e^{-2c})\right)\leqslant 10.$$

Using (60) and $|e^s-1|\leqslant|s|e^{|s|}$ in (61) we obtain

(63) $|h_1(z)|$

$$\leqslant-\frac{1}{c}\log(1-e^{-c/2})(1-e^{-c/2})^{-1/c}\leqslant e^{-c/2}(1-e^{-c/2})^{-1-1/c}\leqslant 3e^{-c/2},$$

if $x\geqslant -r+\dfrac{5c}{2\log\nu}$ and $c\geqslant 2$.

Now let M_j, $j=1,2,3$, denote the maximum modulus of h_1 in the discs of center $-r+\dfrac{3c}{\log\nu}+it$ and radii r_j, $j=1,2,3$, where

$$r_1=\frac{c}{2\log\nu},\qquad r_2=\frac{5c}{2\log\nu}\qquad\text{and}\qquad r_3=\frac{3c}{\log\nu}.$$

Hadamard's 3-circle-theorem tells us that

$$M_2<M_1^{1-\eta}M_3^{\eta}$$

for $\eta=\log(r_2/r_1)/\log(r_3/r_1)=\log 5/\log 6$. Hence we infer from (62) and (63) that

$$M_2\leqslant(3e^{-c/2})^{1-\log 5/\log 6}10^{\log 5/\log 6}\leqslant 30\exp\left(-\frac{c}{24}\right). \tag{64}$$

But it follows from (57) that for $c\geqslant 2$ and $z=-r+\dfrac{c}{\log\nu}+it$

(65) $M_2\geqslant|h_1(z)|$

$$\geqslant\left|\exp\left(\frac{1}{c}\log\left|\frac{h(z)}{w}-1\right|\right)-1\right|\geqslant\exp\left(\frac{1}{c}\log\left(\frac{|h(z)|}{|w|}-1\right)\right)-1$$
$$\geqslant\exp\left(\frac{1}{c}\log\left(\frac{d\nu^r e^{-c/2}}{d\nu^r e^{-c}}-1\right)\right)-1=\exp\left(\frac{1}{c}\log(e^{c/2}-1)\right)-1\geqslant\frac{3}{10}.$$

Therefore (64) and (65) imply that

$$3/10\leqslant 30\exp(-c/24)$$

or

$$c\leqslant 24\log(100)<150.$$

Hence (58) and (59) cannot hold, if $c\geqslant 150$, $R_1\geqslant r\geqslant 3c/\log\nu$ and $e^{-c}\leqslant d$. Thus the lemma is proved.

LEMMA 9. *Let $\Delta_{\lambda\varrho}$, $\lambda<\varrho$, denote the set of vectors $z=(z_p)_{\lambda<p\leqslant\varrho}$ with complex components z_p of modulus $\leqslant 1$. Let g_k, $k=0,1,\ldots$, be functions on $\Delta_{\lambda\varrho}$ defined by*

$$g_k(z)=\sum_{\lambda<p\leqslant\varrho}z_p p^{-\sigma}(-\log p)^k,\qquad \sigma\leqslant 1-\varepsilon. \tag{66}$$

Let K be a positive integer and w_k, $k=0,\ldots,K$, complex numbers such that

$$K=o\left(\left(\frac{\log\varrho}{\log\lambda}-1\right)^{1/3}\exp\left(\frac{1}{3}(\log\lambda)^{1/2}\right)\right),\qquad \lambda\to\infty, \tag{67}$$

and

$$|w_k|\leqslant\frac{\lambda^{1-\sigma}\log\varrho}{10K^3\log\lambda}\left(\frac{1-\dfrac{\log\lambda}{\log\varrho}}{2K}\right)^{K+1}k!(K-k)!\log^k\varrho. \tag{68}$$

Then the system of equations

$$g_k(z)=w_k,\qquad k=0,\ldots,K, \tag{69}$$

has a solution z in $\Delta_{\lambda\varrho}$ for all sufficiently large λ.

Proof. The vectors $(g_0(z),\ldots,g_K(z))$, z in $\Delta_{\lambda\varrho}$, form a convex set in C^{K+1}. Therefore the system (69) has a solution in $\Delta_{\lambda\varrho}$, if and only if for arbitrary complex numbers l_k, $k=0,\ldots,K$, there is a z in $\Delta_{\lambda\varrho}$ such that

$$\sum_{k=0}^{K}l_k g_k(z)=\sum_{k=0}^{K}l_k w_k. \tag{70}$$

It follows from (66) that, as z runs through $\Delta_{\lambda\varrho}$, the left-hand side in (70) represents every complex number of modulus

$$\leqslant\sum_{\lambda<p\leqslant\varrho}p^{-\sigma}\left|\sum_{k=0}^{K}l_k(-\log p)^k\right|.$$

On the other hand we have

$$\Big|\sum_{k=0}^{K} l_k w_k\Big| \leqslant \sum_{k=0}^{K} |l_k|\,|w_k|.$$

Thus (70) holds, if

(71) $$\frac{\lambda^{1-\sigma}\log\varrho}{10K^3\log\lambda}\left(\frac{1-\frac{\log\lambda}{\log\varrho}}{2K}\right)^{K+1}\sum_{k=0}^{K}|l_k|k!(K-k)!\log^k\varrho \leqslant \sum_{\lambda<p\leqslant\varrho} p^{-\sigma}\Big|\sum_{k=0}^{K} l_k(-\log p)^k\Big|.$$

Therefore, to prove the lemma, it suffices to show that (71) holds for any given complex numbers l_k, $k = 0, \ldots, K$, if K satisfies (67).

Now let Q_l denote the polynomial

(72) $$Q_l(\xi) = \sum_{k=0}^{K} l_k \xi^k.$$

Let $\xi_k = \log\lambda + \frac{k}{K}\log(\varrho/\lambda)$, $k = 0, \ldots, K$, and set

$$G_k(\xi) = \prod_{\substack{m=0\\ m\neq k}}^{K}(\xi-\xi_m), \qquad 0\leqslant k\leqslant K.$$

Hence we have

(73) $$|G_k^{(j)}(0)| \leqslant \frac{K!}{(K-j)!}(\log\varrho)^{K-j}, \qquad 0\leqslant j,\ k\leqslant K,$$

and

(74) $$|G_k(\xi_k)| = \prod_{\substack{m=0\\ m\neq k}}^{K}\Big|\frac{k-m}{K}\log(\varrho/\lambda)\Big| = \left(\frac{\log(\varrho/\lambda)}{K}\right)^{K} k!(K-k)!, \qquad 0\leqslant k\leqslant K.$$

Lagrange's interpolation formula tells us now that

(75) $$Q_l(\xi) = \sum_{k=0}^{K}\frac{Q_l(\xi_k)}{G_k(\xi_k)}G_k(\xi).$$

Thus it follows from (72)–(75) that

$$j!|l_j| = \Big|\sum_{k=0}^{K}\frac{Q_l(\xi_k)}{G_k(\xi_k)}G_k^{(j)}(0)\Big| \leqslant \sum_{k=0}^{K}\frac{|Q_l(\xi_k)|K!}{k!(K-k)!(K-j)!}(\log\varrho)^{K-j}\left(\frac{K}{\log(\varrho/\lambda)}\right)^{K}$$
$$\leqslant \sum_{k=0}^{K}|Q_l(\xi_k)|\frac{(\log\varrho)^{K-j}}{(K-j)!}\left(\frac{2K}{\log(\varrho/\lambda)}\right)^{K},$$

and therefore we have

(76) $$\frac{1}{K}\left(\frac{1-\frac{\log\lambda}{\log\varrho}}{2K}\right)^{K}\sum_{j=0}^{K}|l_j|j!(K-j)!\log^j\varrho \leqslant \sum_{k=0}^{K}|Q_l(\xi_k)|.$$

Now let η_k, $1\leqslant k\leqslant K$, be such that

(77) $$\max_{\xi_{k-1}\leqslant\xi\leqslant\xi_k}|Q_l(\xi)| = |Q_l(\eta_k)| \quad\text{and}\quad \xi_{k-1}\leqslant\eta_k\leqslant\xi_k.$$

Problem 83 on p. 91 in [9] says that

(78) $$\max_{-1\leqslant\xi\leqslant1}|Q^{(1)}(\xi)| \leqslant K^2\max_{-1\leqslant\xi\leqslant1}|Q(\xi)|,$$

where Q denotes a polynomial of degree K. Hence we have by (77)

(79)
$$\max_{\xi_{k-1}\leqslant\xi\leqslant\xi_k}|Q_l^{(1)}(\xi)| = \max_{-1\leqslant\xi\leqslant1}\Big|Q_l^{(1)}\Big(\xi\,\frac{\xi_k-\xi_{k-1}}{2}+\frac{\xi_k+\xi_{k+1}}{2}\Big)\Big|$$
$$= \max_{-1\leqslant\xi\leqslant1}\frac{2}{\xi_k-\xi_{k-1}}\Big|\frac{d}{d\xi}Q_l\Big(\xi\,\frac{\xi_k-\xi_{k-1}}{2}+\frac{\xi_k+\xi_{k-1}}{2}\Big)\Big|$$
$$\leqslant \frac{2K^2}{\xi_k-\xi_{k-1}}\max_{\xi_{k-1}\leqslant\xi\leqslant\xi_k}|Q_l(\xi)|$$
$$= \frac{2K^3}{\log(\varrho/\lambda)}|Q_l(\eta_k)|, \qquad 1\leqslant k\leqslant K.$$

If now

$$I_k = \Big\{\xi\,\Big|\ \xi_{k-1}\leqslant\xi\leqslant\xi_k \text{ and } |\xi-\eta_k|\leqslant\frac{\log(\varrho/\lambda)}{4K^3}\Big\},$$

it follows from (77) and (79) that for ξ in I_k

(80) $$|Q_l(\xi)| \geqslant |Q_l(\eta_k)| - |Q_l(\eta_k)-Q_l(\xi)| \geqslant \tfrac{1}{2}|Q_l(\eta_k)|, \qquad 1\leqslant k\leqslant K.$$

The length of the interval I_k is at least $\frac{\log(\varrho/\lambda)}{4K^3}$. Hence, if K satisfies (67) and $1\leqslant k\leqslant K$, we infer from the quantitative form of the prime number theorem (Theorem 23 in [4]) that there are more than

(81) $$\frac{e^{\xi_{k-1}}}{\xi_{k-1}}\,\frac{\log(\varrho/\lambda)}{5K^3}, \qquad \lambda\to\infty,$$

primes p such that $\log p$ belongs to I_k. Combining (72), (77), (80) and

(81) we see that

(82) $$\sum_{\lambda<p\leqslant\varrho} p^{-\sigma}\Big|\sum_{k=0}^{K} l_k \log^k p\Big| = \sum_{\lambda<p\leqslant\varrho} p^{-\sigma}|Q_l(\log p)|$$
$$\geqslant \frac{\lambda^{1-\sigma}\log\varrho}{10K^3\log\lambda}\Big(1-\frac{\log\lambda}{\log\varrho}\Big)\sum_{k=1}^{K}|Q_l(\eta_k)|$$
$$\geqslant \frac{\lambda^{1-\sigma}\log\varrho}{20K^3\log\lambda}\Big(1-\frac{\log\lambda}{\log\varrho}\Big)\sum_{k=0}^{K}|Q_l(\xi_k)|.$$

Thus the lemma follows from (71), (76) and (82).

Lemma 10 is a variation on the fundamental lemma in [12].

LEMMA 10. *Let $0\leqslant\sigma_2\leqslant 1-\varepsilon$ and U be a relatively compact open set in the halfplane $\sigma\leqslant\sigma_2$ of the complex s-plane. Let f be in $L_2(U)$. Then*

$$\sum_{p\leqslant\varrho}\Big|\int_U \bar f(s)p^{-s}d\sigma dt\Big| = 0$$

for all $\varrho>0$, if f is orthogonal to $P_2(U)$. If f is not orthogonal to $P_2(U)$ and $\delta>0$, there exists a sequence $(\varrho_j)_{j=1}^{\infty}$ such that $\varrho_j\to\infty$ as $j\to\infty$ and

$$\sum_{p\leqslant\varrho}\int_U\Big|\bar f(s)p^{-s}d\sigma dt\Big|\geqslant\varrho^{1-\sigma_2-\delta}$$

for $\varrho=\varrho_j$, $j=1,2,\ldots$

Proof. If

(83) $$F(z)=\int_U \bar f(s)e^{(\sigma_2+\delta-s)z}d\sigma dt,$$

then F is an entire function of exponential type. For we have by Cauchy's inequality

$$|F(z)|\leqslant e^{(\sigma_2+\delta+A)|z|}\Big(\int_U|f(s)|^2d\sigma dt\int_U d\sigma dt\Big)^{1/2},$$

where $A=\sup_{s\in U}|s|<\infty$. Since U is relatively compact, we can also write F as

(84) $$F(z)=e^{(\sigma_2+\delta)z}\sum_{k=0}^{\infty}F_k\frac{(-z)^k}{k!},$$

where

(85) $$F_k=\int_U\bar f(s)s^kd\sigma dt,\quad k=0,1,\ldots$$

Thus the first part of the lemma follows from (83)–(85). For f is orthogonal to $P_2(U)$, if and only if $F_k=0$ for $k=0,1,\ldots$, i.e. if and only if F vanishes identically. It also follows from (83) that

(86) $$|F(x)|\leqslant e^{\delta x}\Big(\int_U|f(s)|^2d\sigma dt\int_U e^{2(\sigma_2-\sigma)x}d\sigma dt\Big)^{1/2}$$
$$\leqslant e^{\delta x}\Big(\int_U|f(s)|^2d\sigma dt\int_U d\sigma dt\Big)^{1/2},\quad x \text{ real}.$$

Suppose now that there is a $\delta>0$ and a $x_0>0$ such that

(87) $$|F(x)|<e^{-\delta x}$$

for $x>x_0$. Then (86) and (87) imply that $\hat F$ given by

$$\hat F(w)=\int_{-\infty}^{\infty}F(x)e^{ixw}dx$$

is analytic in the strip $|v|<\delta$. On the other hand, since F is of exponential type and

$$\int_{-\infty}^{\infty}|F(x)|^2dx<\infty,$$

it follows from Paley–Wiener's theorem (Theorem 10, p. 13, in [8]) that $\hat F$ has a bounded support. Thus $\hat F$, and therefore also F, have to vanish identically. Hence the assumption (87) implies in view of (84) and (85) that f is orthogonal to $P_2(U)$. So, if f is not orthogonal to $P_2(U)$ and $\delta>0$, there is a sequence $(x_j)_{j=1}^{\infty}$ such that $x_j\to\infty$ as $j\to\infty$ and

(88) $$|F(x_j)|\geqslant e^{-\delta x_j},\quad j=1,2,\ldots$$

Furthermore we may assume w.l.o.g. that

(89) $$\max_{|x-x_j|\leqslant 1}e^{-(\sigma_2+\delta)x}|F(x)|=e^{-(\sigma_2+\delta)x_j}|F(x_j)|,\quad j=1,2,\ldots,$$

and

(90) $$\sigma_2+2\delta<1.$$

Now let B and E_K be defined by $B\log\dfrac{B}{eA}=1+\delta$, $B>0$, and

$$E_K(z)=\sum_{k=0}^{K}F_K\frac{(-z)^k}{k!},\quad K=0,1,\ldots$$

Then (84) and (85) imply that for $K\geqslant B|x|$

(91) $$|e^{-(\sigma_2+\delta)x}F(x)-E_K(x)|\leqslant\sum_{k>K}|F_k|\frac{|x|^k}{k!}$$
$$\leqslant\Big(\int_U|f(s)|^2d\sigma dt\int_U d\sigma dt\Big)^{1/2}\sum_{k>K}\Big(\frac{eA|x|}{K}\Big)^k$$
$$\leqslant e^{-|x|},$$

if $|x|$ is sufficiently large. Combining (88)–(91) we have

$$(92)\qquad \max_{|x-x_j|\leqslant 1}|E_K(x)| \leqslant \max_{|x-x_j|\leqslant 1} e^{-(\sigma_2+\delta)x}|F(x)| + e^{-x_j+1} \leqslant 2e^{-(\sigma_2+\delta)x_j}|F(x_j)|,$$

if $K \geqslant B(x_j+1)$ and j is sufficiently large. Thus we infer from (78), (88,) (91) and (92) by choosing a K satisfying $B(x_j+1) \leqslant K \leqslant 2Bx_j$ that

$$(93)\qquad |e^{-(\sigma_2+\delta)x}F(x)| \geqslant e^{-(\sigma_2+\delta)x_j}|F(x_j)| - |E_K(x_j) - e^{-(\sigma_2+\delta)x_j}F(x_j)| - \\ - |E_K(x) - E_K(x_j)| - |e^{-(\sigma_2+\delta)x}F(x) - E_K(x)| \\ \geqslant e^{-(\sigma_2+\delta)x_j}|F(x_j)|(1-2K^2|x-x_j|) - e^{-|x|} - e^{-|x_j|} \\ \geqslant \tfrac{1}{2}e^{-(\sigma_2+2\delta)x_j} - 3e^{-x_j} \geqslant \tfrac{1}{3}e^{-(\sigma_2+2\delta)x_j},$$

if $|x-x_j| \leqslant \frac{1}{4}(2Bx_j)^{-2}$ and j is sufficiently large. The quantitative form of the prime number theorem (Theorem 23 on p. 65 in [4]) shows that there are more than

$$(94)\qquad e^x/2x^4, \qquad x \to \infty,$$

primes p such that $x - x^{-3} \leqslant \log p \leqslant x$. Hence, by (93) and (94), we have

$$\sum_{\log p \leqslant x_j} p^{-\sigma_2-\delta}|F(\log p)| \geqslant \tfrac{1}{6}x_j^{-4}\exp\big((1-\sigma_2-2\delta)x_j\big),$$

if j is sufficiently large. In view of (83), this proves the lemma.

LEMMA 11. *Let $\frac{1}{2}+2\varepsilon < \sigma_1 < \sigma_2 < 1-2\varepsilon$. Let U be an open set in the complex s-plane such that $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ and $|t| \leqslant 1/\varepsilon$ whenever s belongs to U. Let f be an element of $L_2(U)$ which is not orthogonal to $P_2(U)$. Let Φ_f be given by* (5) *and $0 < \delta < \varepsilon$. Then there are c_1, σ_0 and a sequence $(T_j)_{j=1}^{\infty}$ such that $\sigma_1 \leqslant \sigma_0 \leqslant \sigma_2$, $T_j \to \infty$ as $j \to \infty$ and the number of positive integers $n \leqslant T/\tau$, $\tau = \log^2 T$, satisfying*

$$\max_{|y|\leqslant\tau+1}|\Phi_f(z+3n\tau i)| \leqslant (\log T)^{1-\sigma_0-x+\delta},$$

$$\max_{|y|\leqslant\tau}|\Phi_f(z+3n\tau i)| \geqslant (\log T)^{1-\sigma_0-x-\delta}$$

for $\frac{1}{2}+2\varepsilon-\sigma_1 \leqslant x \leqslant 1-\sigma_0-2\varepsilon$ *and* $T = T_j$, $j = 1, 2, \ldots$, *exceeds*

$$T\exp\left(-c_1\frac{\log T}{\log\log T}\right).$$

Proof. Apart from the use of Lemma 10, the proof of this lemma is actually a repetition of the proofs of Lemma 1, 2 (i) and 3 with the function Φ_f instead of $\log\zeta$. Therefore it will not be carried out in all detail.

We assume that

$$\int_U \bar f(s)\,d\sigma dt < 1/\varepsilon.$$

It can be shown as in Lemma 1 that

$$(95)\qquad \int_{-\tau}^{\tau}\Phi_f(z+3n\tau i)\left(\frac{\sin(dy)}{y}\right)^2\big(1+\cos(y\log\nu)\big)\,dy \\ = \frac{\pi}{4}\sum_{|\log(m/\nu)|\leqslant d}\Lambda_1(m)m^{-x-3n\tau i}\int_U \bar f(s)m^{-s}d\sigma dt\left(d-\left|\log\frac{m}{\nu}\right|\right) + O(\nu\tau^{-1})$$

for $x \geqslant \frac{1}{2}-\sigma_1$, if

$$(96)\qquad 0 < d \leqslant \tfrac{1}{2}, \quad \nu \geqslant 1, \quad T \geqslant c_2, \quad 1 \leqslant n \leqslant T/\tau, \\ \{w|\ u \geqslant x+\sigma_1,\ |v-3n\tau| \leqslant 2\tau\}$$

is contained in G, and

$$(97)\qquad \Phi_f(w) = \sum_{m=1}^{\infty}\Lambda_1(m)m^{-w}\varphi\left(\frac{m}{\mu}\right)\int_U \bar f(s)m^{-s}d\sigma dt + O(\tau^{-1/2}\log\tau)$$

for $\{w|\ u \geqslant \frac{1}{2}+2\varepsilon-\sigma_1,\ |v-3n\tau| \leqslant \tau^{1/2+1/\varepsilon}\}$, if

$$(98)\qquad \mu = \tau^{1/\varepsilon}, \quad c_2 \leqslant T, \quad 1 \leqslant n \leqslant T/\tau, \\ \{w|\ u \geqslant \tfrac{1}{2}+\varepsilon-\sigma_1,\ |v-3n\tau| \leqslant 2\tau^{1/2+1/\varepsilon}\}$$

is contained in G.

It follows from (19) and Cauchy's inequality that the right-hand sides of (95) and (97) equal

$$(99)\qquad \frac{\pi}{4}\sum_{|\log(p/\nu)|\leqslant d} p^{-x-3n\tau i}\big(d-|\log(p/\nu)|\big)\int_U \bar f(s)p^{-s}d\sigma dt + O(1), \\ x \geqslant \tfrac{1}{2}+2\varepsilon-\sigma_1,$$

and

$$(100)\qquad \sum_p p^{-w}\varphi(p/\mu)\int_U \bar f(s)p^{-s}d\sigma dt + O(1), \qquad u \geqslant \tfrac{1}{2}+2\varepsilon-\sigma_1,$$

respectively.

Now let σ_0 be such that for every $\delta > 0$ there is a sequence $(\nu_j)_{j=0}^{\infty}$ with $\nu_j \to \infty$ as $j \to \infty$,

$$(101)\qquad \int_U \bar f(s)\nu^{-s}d\sigma dt \geqslant \nu^{-\sigma_0-\delta} \quad \text{for} \quad \nu = \nu_j,\ j = 1, 2, \ldots,$$

and

$$(102)\qquad \int_U \bar f(s)\nu^{-s}d\sigma dt \leqslant \nu^{-\sigma_0+\delta} \quad \text{for} \quad \nu \geqslant \nu_0.$$

We have $\sigma_1 \leqslant \sigma_0$ by Cauchy's inequality and $\sigma_0 \leqslant \sigma_2$ by the proof of Lemma 10. Furthermore (89) and (91)–(93) show that

$$(103)\qquad \Big|\int_U \bar{f}(s)p^{-s}d\sigma dt - \int_U \bar{f}(s)\,\varrho_\nu^{-s}d\sigma dt\Big| \leqslant \tfrac{1}{3}\Big|\int_U \bar{f}(s)\,\varrho_\nu^{-s}d\sigma dt\Big|$$

for $\nu \leqslant p \leqslant \nu'$, if

$$(104)\qquad \nu' = \nu\exp(\log^{-3}\nu),\qquad \Big|\int_U \bar{f}(s)\,\varrho_\nu^{-s}d\sigma dt\Big| = \max_{\nu\leqslant\varrho\leqslant\nu'}\Big|\int_U \bar{f}(s)\,\varrho^{-s}d\sigma dt\Big|,\quad \nu\leqslant\varrho_\nu\leqslant\nu',$$

and ν is sufficiently large.

If $\nu = \nu_j$, $j = 1, 2, \ldots$ and

$$(105)\qquad \min_{l\ \text{integral}}\Big|3n\tau\frac{\log p}{2\pi} - l\Big| \leqslant 1/6$$

for all p with $|\log(p/\nu)| \leqslant d$, then it follows from (101), (103), (104) and the quantitative form of the prime number theorem (Theorem 23 in [4]) that (99) exceeds

$$(106)\qquad c_3\nu^{1-x-\sigma_0-\delta}(\log\nu)^{-7} \geqslant \nu^{1-x-\sigma_0-2\delta},\qquad \tfrac{1}{2}+2\varepsilon-\sigma_1 \leqslant x \leqslant 1-\sigma_0-2\varepsilon$$

if $\delta < \varepsilon$, $d = (\log\nu)^{-3}$ and ν is sufficiently large. Lemma 2 and 3 show that (96), (98) and (105) can be simultaneously fulfilled for more than

$$(107)\qquad \frac{T}{\tau}\exp\Big(-c_4\frac{\nu}{\log\nu}\Big) - O(T^{1-\varepsilon})$$

positive integers $n \leqslant T/\tau$. It follows from (102) and the beginning of the proof of Lemma 2 that the number of positive integers $n \leqslant T/\tau$ satisfying

$$(108)\qquad \max_{|y|\leqslant\tau+1}\Big|\sum_{p>\nu} p^{-z-3n\tau i}\varphi\Big(\frac{p}{\mu}\Big)\int_U \bar{f}(s)p^{-s}d\sigma dt\Big| \leqslant \nu^{1-\sigma_0+2\delta-x}$$

for $\frac{1}{2}+2\varepsilon-\sigma_1 \leqslant x \leqslant 1-\sigma_0-2\varepsilon$ exceeds

$$(109)\qquad \frac{T}{\tau} - O\Big(Tl!\Big(\frac{c_5}{\nu^{1+\delta}}\Big)^l + l!(c_5\nu^{-1}\tau^{1/\varepsilon})^l\Big),$$

where l is a positive integer. Since the left-hand side of (95) is dominated by

$$(110)\qquad \pi\max_{|y|\leqslant\tau}|\Phi_f(z+3n\tau i)|$$

and (102) also implies that

$$(111)\qquad \Big|\sum_{p\leqslant\nu} p^{-z-3n\tau i}\varphi\Big(\frac{p}{\mu}\Big)\int_U \bar{f}(s)p^{-s}d\sigma dt\Big| = O(\nu^{1-\sigma_0+\delta-x}),$$

Lemma 11 follows from (95)–(100) and (105)–(111), if we choose l to be the integral part of $\nu/\log\nu$ and $\nu = c_6\log T$, where c_6 is sufficiently small.

LEMMA 12. *Let δ, δ', η, c and R be positive such that $\eta > 2\delta$ and $R \geqslant 2\varepsilon$. Let h be an analytic function on*

$$W' = \{z|\ x \geqslant -R,\ |y| \leqslant 2R\} \cup \{z|\ x \geqslant 0\}$$

such that

(i) $|h(z)| \leqslant c$, *if* $x \geqslant 0$,
(ii) $|w - h(z)| \geqslant \delta'$, *if* $x \geqslant \varepsilon$ *and* $w \neq h(s)$ *for all* s *with* $\sigma \geqslant 0$,
(iii) $|h(z)| \leqslant \nu^{\eta}$, *if* z *belongs to* W',
(iv) $|h(-R+\varepsilon)| \geqslant \nu^{\eta-\delta}$.

Then $h(W')$ contains $\{w|\ |w| \leqslant \nu^{\eta-2\delta}\}$ for all $\nu \geqslant \nu_0$, where ν_0 depends only on δ, δ', ε, η, c and R.

Proof. If $w = h(z)$ for some z with $x \geqslant 0$, there is nothing to prove. Otherwise w satisfies (ii) and we assume further that

$$(112)\qquad h(z) \neq w \text{ for all } z \text{ in } W' \text{ and } |w| \leqslant \nu^{\eta-2\delta}.$$

Then

$$h_2(z) = \exp\Big(\frac{1}{\log\nu}\int_{2\varepsilon}^{z}\frac{h^{(1)}(s)}{h(s)-w}\,ds\Big) - 1$$

defines an analytic function h_2 on W'. Since (i), (ii) and Cauchy's inequalities for the derivatives of analytic functions imply that

$$\Big|\int_{2\varepsilon}^{z}\frac{h^{(1)}(s)}{h(s)-w}\,ds\Big| \leqslant \frac{\varepsilon}{\delta'}\cdot\frac{c}{\varepsilon} = \frac{c}{\delta'}$$

for $|z-2\varepsilon| \leqslant \varepsilon$, we have

$$(113)\qquad |h_2(z)| \leqslant \frac{c}{\delta'\log\nu}\exp\Big(\frac{c}{\delta'\log\nu}\Big) \leqslant \frac{2c}{\delta'\log\nu},\qquad |z-2\varepsilon| \leqslant \varepsilon,$$

if $\nu \geqslant \nu_0(\delta', c)$. From (iii) and (112) we deduce that

$$(114)\qquad |h_2(z)| \leqslant 1+\exp\Big(\frac{1}{\log\nu}\log|h(z)-w|\Big) \leqslant 1+\exp\Big(\eta+\frac{\log(1+\nu^{-2\delta})}{\log\nu}\Big) \leqslant 1+e^{2\eta}$$

for z in W', if $\nu \geqslant \nu_0(\delta, \eta)$. But (iv) and (112) imply that

$$(115)\qquad |h_2(-R+\varepsilon)| \geqslant \exp\Big(\frac{1}{\log\nu}\log|h(-R+\varepsilon)-w|\Big) - 1 \geqslant \exp\Big(\eta-\delta+\frac{\log(1-\nu^{-\delta})}{\log\nu}\Big) - 1 \geqslant \delta,$$

if $\nu \geqslant \nu_0(\delta, \eta)$. As in Lemma 8, (113)–(115) and Hadamard's 3-circle-theorem applied to h_2 with respect to the circles of center 2ε and radii ε, $R+\varepsilon$, $R+2\varepsilon$ show now that (112) can only hold for $\nu \leqslant \nu_0(\delta, \delta', \varepsilon, \eta, c, R)$. This proves the lemma.

4. Proof of the main results

Proof of Theorem 1. Let $T \geqslant c_1$, $\tau = (\log T)^2$, $c_2 \leqslant \nu \leqslant \tau$, $0 < d \leqslant 1$ and l a positive integer. Then we deduce from Lemma 2 (i) and Lemma 3 that the number of positive integers $n \leqslant T/\tau$ satisfying

$$\max_{|t|\leqslant \tau+1} |\log\zeta(s+3n\tau i)| \leqslant \frac{\nu^{1-\sigma}}{\log\nu} \tag{116}$$

and

$$\max_{|t|\leqslant\tau} |\log\zeta(s+3n\tau i)| \geqslant \frac{(d\nu)^{1-\sigma}}{\log(d\nu)} \geqslant \frac{d\nu^{1-\sigma}}{\log\nu} \tag{117}$$

for $\frac{1}{2}+3\varepsilon \leqslant \sigma \leqslant 1-2\varepsilon$ exceeds

$$\frac{T}{\tau}\exp\left(-c_3\frac{d\nu}{\log(d\nu)}\right) - O\left(T^{1-\varepsilon}+Tl!\left(\frac{\log\nu}{c_4\nu}\right)^l + l!\left(\frac{\tau^{1/l}\log\nu}{c_4\nu}\right)^l\right). \tag{118}$$

If we choose l to be the integral part of $c_4\nu/\log\nu$ and $d = \min(1, c_4/2c_3)$, then (118) is greater than

$$\frac{T}{\tau}\exp\left(-c_4\frac{\nu}{\log\nu}\right), \tag{119}$$

provided that

$$\frac{4}{c_4}\log\tau\log\log\tau \leqslant \nu \leqslant \frac{\varepsilon}{4c_3}\tau^{1/2}. \tag{120}$$

If (116) and (117) hold for n, then the function h given by

$$h(z) = \frac{\log\nu}{\nu^{1-\sigma}}\log\zeta(z+\sigma+3n\tau i), \quad \tfrac{1}{2}+3\varepsilon-\sigma \leqslant x \leqslant 1-2\varepsilon-\sigma, \quad |y| \leqslant \tau+1,$$

fulfills all conditions of Lemma 8. Thus, by combining (116)–(119) and using Lemma 8 with a sufficiently large c, we see that Theorem 1 holds for those ν satisfying (120).

On the other hand, there are c_5, c_6, c_7 such that

$$\left|\log\zeta(s+ina) - \sum_{\nu<p\leqslant 2\nu} p^{-s}\right| \leqslant \varepsilon\frac{\nu^{1-\sigma}}{\log\nu}, \quad \tfrac{1}{2}+2\varepsilon \leqslant \sigma \leqslant 1-2\varepsilon, \tag{121}$$

for more then $T\exp\left(-c_5\dfrac{\nu}{\log\nu}\right)$ positive integers $n \leqslant T$, if

$$a \text{ satisfies (1) and } c_6 \leqslant \nu \leqslant c_7\tau^{1/4}. \tag{122}$$

This results from Lemma 7 by taking $\lambda = \nu$, $\varrho = 2\nu$, $z_p = 1$ for $\lambda < p \leqslant \varrho$ and V to be a fixed large positive number. Hence the prime number theorem and (121) imply that

$$\max_{|t|\leqslant 1/2} |\log\zeta(s+ina)| \leqslant \frac{\nu^{1-\sigma}}{\varepsilon\log\nu}, \quad \tfrac{1}{2}+2\varepsilon \leqslant \sigma \leqslant 1-2\varepsilon, \tag{123}$$

and

$$\max_{|t|\leqslant 1/4} |\log\zeta(s+ina)| \geqslant \frac{\varepsilon\nu^{1-\sigma}}{\log\nu}, \quad \tfrac{1}{2}+2\varepsilon \leqslant \sigma \leqslant 1-2\varepsilon, \tag{124}$$

for more than $T\exp\left(-c_5\dfrac{\nu}{\log\nu}\right)$ positive integers $n \leqslant T$, if (122) holds. In view of Lemma 8, (123) and (124) imply Theorem 1 for the range $c_6 \leqslant \nu \leqslant c_7\tau^{1/4}$ in the same way as (116) and (117) for the range in (120).

Proof of Corollary 1. It follows from Theorem 1 by exponentiation.

Proof of Theorem 2. Let $\frac{1}{2}+2\varepsilon \leqslant \sigma_1 = \sigma-R < \sigma < 1-2\varepsilon-R$. If a, λ, ϱ, T and V satisfy the conditions of Lemma 7 and $(z_p)_{\lambda<p\leqslant\varrho}$ belongs to Δ_{λ_0}, then there are more than $\frac{1}{5}TV^{-\pi(\varrho)}$ points $s_n = \sigma+ina$, $1 \leqslant n \leqslant T$, such that

$$\log\zeta(s_n+z) = f_0(\sigma+z) + \sum_{\lambda<p\leqslant\varrho} z_p p^{-\sigma-z} + O\left(\frac{\varrho^{1-\sigma_1}}{V\log\varrho} + \frac{\lambda^{1/2-\sigma_1}}{\log\lambda}\right) \tag{125}$$

for $|z| \leqslant R$. If (125) holds, Cauchy's inequalities for the derivatives of analytic functions show that

$$(\log\zeta)^{(k)}(s_n) = f_0^{(k)}(\sigma) + \sum_{\lambda<p\leqslant\varrho} z_p p^{-\sigma}(-\log p)^k + O\left(k!R^{-k}\left\{\frac{\varrho^{1-\sigma_1}}{V\log\varrho} + \frac{\lambda^{1/2-\sigma_1}}{\log\lambda}\right\}\right), \tag{126}$$

$$|f_0^{(k)}(\sigma)| = O(k!\varepsilon^{-k}) \tag{127}$$

for $k = 0, 1, \ldots$ Now let K be a positive integer and w_k, $0 \leqslant k \leqslant K$, complex numbers satisfying

$$K = o\left(\left(\frac{\log\varrho}{\log\lambda}-1\right)^{1/3}\exp\left(\tfrac{1}{3}(\log\lambda)^{1/2}\right)\right), \quad \lambda\to\infty, \tag{128}$$

and

$$|w_k| \leqslant \frac{\lambda^{1-\sigma}\log\varrho}{10K^3\log\lambda}\left(\frac{1-\dfrac{\log\lambda}{\log\varrho}}{2K}\right)^{K+1} k!(K-k)!\log^k\varrho, \quad 0 \leqslant k \leqslant K. \tag{129}$$

Then Lemma 7, Lemma 9, (125) and (126) imply the existence of c_1, c_3, c_4 such that

$$(\log\zeta)^{(k)}(s_n) = f_0^{(k)}(\sigma) + w_k + O\left(k!R^{-k}\frac{\varrho^{1-\sigma_1}}{V\log\varrho}\right), \quad 0 \leqslant k \leqslant K, \tag{130}$$

for more than $\frac{1}{5}TV^{-\varkappa(\varrho)}$ points s_n, $1 \leqslant n \leqslant T$, if

$$(131)\qquad c_1 \leqslant T, \quad c_3 \leqslant V \leqslant \varrho^{1/2}\min\left((\log\varrho)^{-1/2}, \frac{\log\lambda}{\log\varrho}\left(\frac{\lambda}{\varrho}\right)^{\sigma_1-1/2}\right) \quad \text{and}$$
$$\varrho = c_4(\log T\log^{1/2}V)^{1/2}(V\log^{1/2}V)^{-\frac{5}{2(2\sigma_1-1)}}.$$

As a function of λ, the right-hand side of (129) takes on its maximum near the maximum of the function

$$\lambda \mapsto \lambda^{1-\sigma}\left(1-\frac{\log\lambda}{\log\varrho}\right)^K$$

when λ varies from ϱ^{ε} to ϱ. Therefore we choose

$$(132)\qquad \lambda = \varrho\exp\left(-\frac{K}{1-\sigma}\right)$$

and assume now that

$$(133)\qquad K \leqslant \delta(1-\sigma)\log\varrho, \quad \delta \leqslant 1-\varepsilon.$$

Hence K satisfies (128),

$$(134)\qquad \frac{\log\lambda}{\log\varrho}\varrho^{1/2}\left(\frac{\lambda}{\varrho}\right)^{\sigma_1-1/2} \geqslant \varepsilon\varrho^{1/2}\exp\left(-K\frac{\sigma_1-\frac{1}{2}}{1-\sigma}\right)$$

and the right-hand side of (129) is larger than

$$(135)\qquad \tfrac{1}{10}\varrho^{1-\sigma}K^{-3}\big(2e(1-\sigma)\big)^{-K}(K-k)!\,k!\,(\log\varrho)^{k-K-1}.$$

Since

$$\frac{l!}{(\log\varrho)^l} \geqslant \left(\frac{l}{e\log\varrho}\right)^l \geqslant \left(\frac{L}{e\log\varrho}\right)^L, \quad 1 \leqslant l \leqslant L \leqslant \log\varrho,$$

we have

$$(136)\qquad \varepsilon^{-k} \leqslant \varrho^{\frac{1}{2}(1-\sigma)}\big(2e(1-\sigma)\big)^{-K}\frac{(K-k)!}{(\log\varrho)^{K-k}}$$

for $0 \leqslant k \leqslant K$, if δ in (133) is such that

$$\tfrac{1}{2} = \delta\log\left(\frac{2e^2}{\delta\varepsilon}\right).$$

Thus the general part of the theorem follows from (127)–(136), if we use (130) with $w_k - f_0^{(k)}(\sigma)$ instead of w_k. We deduce the particular case $K \leqslant 1/\varepsilon$ from it by taking $V = c_5(\varrho/\log\varrho)^{1/2}$ and $R = 1/\log\varrho$.

Proof of Corollary 2. Let $K \geqslant 2$ and w_k', $0 \leqslant k \leqslant K$, complex numbers such that

$$(137)\qquad |w_k'| \leqslant \eta^k/3K, \quad k = 1, \ldots, K,$$

for some $\eta > 1$. Define w_k, $0 \leqslant k \leqslant K$, recursively by

$$(138)\qquad w_0 = w_0', \quad w_1 = w_1',$$
$$w_l' = w_l + \sum_{k=1}^{l-1}\left(1-\frac{k}{l}\right)w_{l-k}w_k', \quad l = 2, \ldots, K.$$

It follows from (137) and (138) that

$$(139)\qquad |w_k| \leqslant \frac{1}{2K}\eta^k, \quad k = 1, \ldots, K.$$

For, by induction, we have

$$|w_l| \leqslant |w_l'| + \sum_{k=1}^{l-1}\left(1-\frac{k}{l}\right)|w_{l-k}w_k'| \leqslant \frac{\eta^l}{3K}\left(1+\sum_{k=1}^{l-1}\left(1-\frac{k}{l}\right)\frac{1}{2K}\right)$$
$$= \frac{\eta^l}{3K}\left(1+\frac{l-1}{4K}\right) \leqslant \frac{\eta^l}{2K}, \quad l = 2, \ldots, K.$$

Since

$$\zeta^{(l)}(s) = \sum_{k=0}^{l-1}\binom{l-1}{k}(\log\zeta)^{(l-k)}(s)\,\zeta^{(k)}(s), \quad l = 1, 2, \ldots,$$

we obtain from (138)

$$(140)\qquad \frac{\zeta^{(l)}(s)}{l!\,\zeta(s)} - w_l'$$
$$= \frac{(\log\zeta)^{(l)}(s)}{l!} - w_l + \sum_{k=1}^{l-1}\left(1-\frac{k}{l}\right)\left\{\frac{(\log\zeta)^{(l-k)}(s)\,\zeta^{(k)}(s)}{(l-k)!\,k!\,\zeta(s)} - w_{l-k}w_k'\right\}$$
$$= \frac{(\log\zeta)^{(l)}(s)}{l!} - w_l +$$
$$+ \sum_{k=1}^{l-1}\left(1-\frac{k}{l}\right)\left\{\left(\frac{(\log\zeta)^{(l-k)}(s)}{(l-k)!} - w_{l-k}\right)w_k' + \left(\frac{\zeta^{(k)}(s)}{k!\,\zeta(s)} - w_k'\right)w_{l-k} +\right.$$
$$\left.+\left(\frac{(\log\zeta)^{(l-k)}(s)}{(l-k)!} - w_{l-k}\right)\left(\frac{\zeta^{(k)}(s)}{k!\,\zeta(s)} - w_k'\right)\right\}, \quad 2 \leqslant l \leqslant K.$$

Next we assume the existence of positive numbers δ, R such that

$$(141)\qquad \left|\frac{(\log\zeta)^{(k)}(s)}{k!} - w_k\right| \leqslant \delta R^{-k}, \quad 0 \leqslant k \leqslant K,$$

for some s. If $\delta \leqslant 1$ and $R\eta \geqslant 10$, we conclude that

$$(142)\qquad |\log\zeta(s) - w_0'| \leqslant \delta \quad \text{and} \quad \left|\frac{\zeta^{(l)}(s)}{l!\,\zeta(s)} - w_l'\right| \leqslant \frac{\delta}{R}\eta^{l-1}, \quad 1 \leqslant l \leqslant K.$$

For the first inequality and the case $l = 1$ in (142) hold trivially by (138) and (141), whereas for $2 \leqslant l \leqslant K$ we deduce from (137), (139) and (140) that

$$\left|\frac{\zeta^{(l)}(s)}{l!\,\zeta(s)} - w_l'\right| \leqslant \delta R^{-l} + \sum_{k=1}^{l-1}\left(1-\frac{k}{l}\right)\left\{\delta R^{k-l}\frac{\eta^k}{3K} + \frac{\delta}{R}\eta^{k-1}\frac{\eta^{l-k}}{2K} + \delta R^{k-l}\frac{\delta}{R}\eta^{k-1}\right\}$$
$$\leqslant \frac{\delta}{R}\eta^{l-1}\left\{(R\eta)^{-l+1} + \frac{1}{3K}\sum_{k=1}^{l-1}(R\eta)^{k-l+1} + \frac{l-1}{2K} + \frac{\delta}{R\eta}\sum_{k=1}^{l-1}(R\eta)^{k-l+1}\right\}$$
$$\leqslant \frac{\delta}{R}\eta^{l-1}\left\{\frac{1}{R\eta} + \frac{1}{3K(1-1/R\eta)} + \tfrac{1}{2} + \frac{\delta}{R\eta-1}\right\}$$
$$\leqslant \frac{\delta}{R}\eta^{l-1}\left\{\frac{1}{10} + \frac{1}{6(9/10)} + \tfrac{1}{2} + \frac{1}{9}\right\} \leqslant \frac{\delta}{R}\eta^{l-1}.$$

Theorem 2 shows that for complex numbers w_k satisfying (139) there are more than $\frac{1}{5}TV^{-n(\varrho)}$ points $s_n = \sigma + ina$, $1 \leqslant n \leqslant T$, for which (141) holds with

$$\delta = c_9\frac{\varrho^{1/2-\sigma_1}}{V\log\varrho} \quad \text{and} \quad R > 0, \tag{143}$$

if $T \geqslant c_1$, $K \leqslant c_2\log\log T$, ϱ and V satisfy (2) and

$$|w_0| \leqslant \frac{1}{10}\varrho^{1-\sigma}(\log\varrho)^{-K-1}K^{-3}K!\big(2e(1-\sigma)\big)^{-K},$$
$$\eta = \left(\frac{\varrho^{1-\sigma}}{5K^2\log\varrho}\right)^{1/K}\big(4e(1-\sigma)\big)^{-1}. \tag{144}$$

For

$$\eta^K \leqslant \frac{\varrho^{1-\sigma}}{5K^2\log\varrho}\big(2e(1-\sigma)\big)^{-K}\varrho^{-1/\eta}$$
$$\leqslant \frac{\varrho^{1-\sigma}}{5K^2\log\varrho}\big(2e(1-\sigma)\big)^{-K}(K-k)!\left(\frac{\eta}{\log\varrho}\right)^{K-k}, \quad 1 \leqslant k \leqslant K,$$

if c_2 is sufficiently small. Since $R\eta \geqslant 10$ for $R \geqslant 40e(1-\sigma)\left(\dfrac{\varrho^{1-\sigma}}{5K^2\log\varrho}\right)^{-1/K}$, the first part of the corollary follows from (137) and (141)–(144). The particular case $K \leqslant 1/\varepsilon$ follows from it as in Theorem 2.

Proof of Theorem 3. First let f be orthogonal to $P_2(U)$, i.e.

$$\int_U \bar{f}(s)s^n\,d\sigma\,dt = 0$$

for $n = 0, 1, \ldots$ We have

$$\log\zeta(s+z) = \sum_{n=0}^{\infty} b_n(z)s^n,$$

if $x \geqslant 2/\varepsilon$ and s belongs to U. Hence

$$\Phi_f(z) = \sum_{n=0}^{\infty} b_n(z)\int_U \bar{f}(s)s^n\,d\sigma\,dt = 0, \qquad x \geqslant 2/\varepsilon,$$

what shows that Φ_f vanishes identically.

So let us assume now that f is not orthogonal to $P_2(U)$. By (102), the Dirichlet series for Φ_f

$$\sum_{m=1}^{\infty} \Lambda_1(m)m^{-z}\int_U \bar{f}(s)m^{-s}\,d\sigma\,dt$$

converges absolutely for $x > 1-\sigma_0$. Since in Lemma 12 condition (i) and the analyticity of h imply condition (ii) for some $\delta' > 0$, the functions

$$z \mapsto \Phi_f(z+\tfrac{1}{2}+iv), \qquad v \text{ real},$$

satisfy (i) and (ii) in Lemma 12 uniformly in v. Combining these remarks with Lemma 11 and 12 we obtain Theorem 3.

Proof of Theorem 4. Part (i) is a direct consequence of Lemma 7.

It is clear that the $\Gamma_{\lambda\varrho}$ are convex circled sets. Hence their projections are closed discs of radius

$$r_f(\lambda, \varrho) = \max_{z\in\Lambda_{\lambda\varrho}}\left|\sum_{\lambda<p\leqslant\varrho} z_p\int_U \bar{f}(s)p^{-s}\,d\sigma\,dt\right| = \sum_{\lambda<p\leqslant\varrho}\left|\int_U \bar{f}(s)p^{-s}\,d\sigma\,dt\right|.$$

Thus Lemma 10 implies the properties of $r_f(\lambda, \varrho)$ stated in (ii). Since

$$\Gamma_{\lambda\varrho} \subset \Gamma_{\lambda\nu} \tag{145}$$

for $\varrho \leqslant \nu$, the union $\bigcup_{\varrho>\lambda}\Gamma_{\lambda\varrho}$ and its closure $\Gamma_\lambda(U)$ in $L_2(U)$ are also convex circled sets. Since the elements of $\Gamma_{\lambda\varrho}$ are entire functions, $\Gamma_\lambda(U)$ must be contained in $P_2(U)$. As a closed subspace of the Hilbert space $L_2(U)$, $P_2(U)$ is isomorphic to its dual. Thus, by a consequence of the theorem of Hahn–Banach (Prop. 3, p. 120, [3]) an element of $P_2(U)$ belongs to $\Gamma_\lambda(U)$ if and only if its projection with respect to any f in $P_2(U)$ belongs to the projection of $\Gamma_\lambda(U)$ with respect to f. But the projection of $\Gamma_\lambda(U)$ with respect to any f in $P_2(U)$ is the whole complex plane in view of (145) and the lower bounds for $r_f(\lambda, \varrho)$.

Theorem 3 shows that the functions

$$s \mapsto \log\zeta(s+ina) \tag{146}$$

are in $P_2(U)$, if ina belongs to G_U. Combining Theorem 4 (i) and (ii) with $V = (\varrho/\log\varrho)^{1/2}$ and λ large enough we conclude that the functions in (146) are everywhere dense in $P_2(U)$. Since every element of $P_\infty(C)$ is the uniform limit of elements in $P_2(U)$ restricted to C, the functions in (146) are also everywhere dense in $P_\infty(C)$. For, by Cauchy's theorem, two elements of $P_2(U)$ are close with respect to the supremum norm on C, if they are close with respect to the L_2-norm on U.

For part (iv) we observe first that

$$\sum_{\lambda<p\leqslant\varrho} z_p p^{-s} - \sum_{k=0}^{K}(s-\sigma_0)^k \sum_{\lambda<p\leqslant\varrho} z_p p^{-\sigma_0}\frac{(-\log p)^k}{k!} \tag{147}$$

$$= O\left(\sum_{\lambda<p\leqslant\varrho} p^{-\sigma_0}\sum_{k>K}\frac{(r\log p)^k}{k!}\right) = O\left(\frac{\varrho^{1-\sigma_0}}{\log\varrho}\left(\frac{er\log\varrho}{K}\right)^K\right) = O\left(\frac{\varrho^{-\alpha(1-\sigma_0)}}{\log\varrho}\right)$$

for all $(z_p)_{\lambda<p\leqslant\varrho}$ in $\Delta_{\lambda\varrho}$ and $|s-\sigma_0|\leqslant r$, if

$$K = \delta(1-\sigma_0)\log\varrho,\quad r = r'(1-\sigma_0)\quad\text{and}\quad \alpha = \delta\log\left(\frac{\delta}{er'}\right)-1. \tag{148}$$

If

$$0<\delta\leqslant 1-\varepsilon,$$

$$\lambda = \varrho\exp\left(-\frac{K}{1-\sigma_0}\right) = \varrho^{1-\delta},\quad \varrho \text{ sufficiently large} \tag{149}$$

and w_k, $0\leqslant k\leqslant K$, are complex numbers satisfying

$$|w_k|\leqslant\frac{1}{10}\varrho^{1-\sigma_0}(\log\varrho)^{k-K-1}K^{-3}(K-k)!\,(2e(1-\sigma_0))^{-K},\quad 0\leqslant k\leqslant K, \tag{150}$$

Lemma 9 and (128)–(135) show the existence of a $(z_p)_{\lambda<p\leqslant\varrho}$ in $\Delta_{\lambda\varrho}$ such that

$$\sum_{k=0}^{K}(s-\sigma_0)^k\sum_{\lambda<p\leqslant\varrho} z_p p^{-\sigma_0}\frac{(-\log p)^k}{k!} = \sum_{k=0}^{K}(s-\sigma_0)^k w_k. \tag{151}$$

If

$$R = R'(1-\sigma_0)>0\quad\text{and}\quad \beta = 1-R'-\delta\log\left(\frac{2e}{R'}\right), \tag{152}$$

we have

$$\varrho^{1-\sigma_0}(2e(1-\sigma_0))^{-K}R^{K-k}\frac{(K-k)!}{(R\log\varrho)^{K-k}} \geqslant \varrho^{1-\sigma_0-R}\left(\frac{R}{2e(1-\sigma_0)}\right)^K R^{-k} = \varrho^{\beta(1-\sigma_0)}R^{-k}. \tag{153}$$

Now let f be an analytic function of the form

$$f(z) = \sum_{k=0}^{\infty} w_k z^k, \tag{154}$$

where

$$|w_k|\leqslant\frac{\varrho^{\beta(1-\sigma_0)}}{\log^4\varrho}R^{-k},\quad k = 0,1,\ldots \tag{155}$$

Then, by (147)–(155), there is a g in $\Gamma_{\lambda\varrho}$ such that

$$g(s) = f(s-\sigma_0)+O\left(\frac{\varrho^{-\alpha(1-\sigma_0)}}{\log\varrho}\right)+O\left(\frac{\varrho^{\beta(1-\sigma_0)}}{\log^4\varrho}\sum_{k>K}\left(\frac{r'}{R'}\right)^k\right) \tag{156}$$

$$= f(s-\sigma_0)+O\left(\frac{\varrho^{-\alpha(1-\sigma_0)}}{\log\varrho}\right)+O\left(\frac{\varrho^{\beta(1-\sigma_0)}}{\log^4\varrho}\left(\frac{r'}{R'}\right)^K\right) = f(s-\sigma_0)+O\left(\frac{\varrho^{-\alpha(1-\sigma_0)}}{\log\varrho}\right)$$

for $|s-\sigma_0|\leqslant r$, if (148), (149) and (152) hold with a positive β. For then we have

$$\delta(1-\sigma_0)\leqslant\frac{\delta}{2}\leqslant\frac{1}{2\log\left(\frac{2e}{R'}\right)}\leqslant\frac{1}{2\log(2e)}\leqslant\frac{1}{3}$$

and

$$\beta+\delta\log\left(\frac{r'}{R'}\right) = \beta+\delta\log\left(\frac{\delta}{eR'}\right)-1-\alpha = -R'-\delta\log\left(\frac{2e}{R'}\right)+\delta\log\left(\frac{\delta}{eR'}\right)-\alpha$$

$$= -R'-\delta\log\left(\frac{2e^2}{\delta}\right)-\alpha < -\alpha.$$

Moreover α is positive, if $r'<\frac{\delta}{e}e^{-1/\delta}$. Thus part (iv) follows from part (i) and (154)–(156).

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On the generalized Ramanujan–Nagell equation I

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Introduction. In this paper we shall study the diophantine equation $x^2-D = 2^n$ $(D \in \mathbf{Z})$ in the positive integers x, n. The equation $x^2+7 = 2^n$ is known as the Ramanujan–Nagell equation. It was solved by several authors (see Hasse [6]) and has five solutions, namely $(x, n) = (1, 3)$, $(3, 4)$, $(5, 5)$, $(11, 7)$, $(181, 15)$.

In 1960 Apéry [1] proved that the equation $x^2-D = 2^n$ $(D < 0$, $D \neq -7)$ has at most two solutions. Browkin and Schinzel [4] conjectured that this equation has two solutions if and only if $D = -23$ or $1-2^k$ for some $k > 3$. Schinzel ([7], p. 212) partly resolved this conjecture by proving that, unless $D = 1-2^k$, the equation has at most one solution with $n > 80$. In Theorem 2 of the present paper we prove the Browkin–Schinzel conjecture.

Theorems 3 and 4 deal with the equation $x^2-D = 2^n$ $(D > 0)$. In Theorem 4 we prove that this equation has at most four solutions. Surprisingly it turns out to be possible to construct infinitely many equations each one admitting precisely four solutions. In Theorem 3 a complete classification is given for those equations with $0 < D < 10^{12}$ having exactly three or four solutions. I have not found any reference to the case $D > 0$ except for a remark by Hasse ([6], p. 100) and a few congruence considerations by Browkin, Schinzel ([4], p. 311).

Theorems 2, 3 and 4 depend on Corollary 1 of Theorem 1 which states that $n < 435+10(\log|D|/\log 2)$ for any solution (x, n). This result makes it possible to solve a given equation $x^2-D = 2^n$ in finitely many steps. Theorem 1 gives a good lower bound for the approximation to $\sqrt{2}$ by rational numbers whose denominators are a power of two. The proof of this theorem uses so-called hypergeometric functions. In 1937 Siegel [8] introduced these functions in the theory of diophantine approximations. By refining Siegel's method Baker [2] succeeded in giving a good lower bound for the rational approximations to $\sqrt[3]{2}$. The proof of Theorem 1 is in fact an adaptation of Siegel's method.