Thus

\[ T = (1 - \lambda) \frac{\psi(x^{-1})}{x} \frac{1}{n^{s-1}} + O\left( \frac{\log n}{n^{s-1}} \right). \]

Combining (a) and (b) we see that

\[ X' = \psi(x^{-1}) \frac{1}{n^{s-1}} + \frac{(1 - \lambda)\psi'(x^{-1})}{x} \frac{1}{n^{s-1}} + O\left( \frac{\log n}{n^{s-1}} \right). \]

Relation (29) combined with relations (30), (31) and (32) gives the desired relation (28) and hence completes the proof of Theorem 1.

**On an extension of a theorem of S. Chowla**

**by**

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1. **Introduction.** In [4] S. Chowla proved that if \( p \) is an odd prime, then the \( (p-1)/2 \) real numbers \( \cot(2\pi a/p) \), \( a = 1, 2, \ldots, (p-1)/2 \) are linearly independent over the field \( \mathbb{Q} \) of rational numbers. Other proofs were given by H. Haase [5], R. Ayoub [1], [2] and T. Okada [8].

The purpose of this note is to show the following theorem, which is an extension of S. Chowla's theorem mentioned above.

**Theorem.** Let \( k = 2 \) and \( q \) be integers with \( k > 0 \) and \( q > 2 \). Let \( T \) be a set of \( \psi(q)/\phi \) representatives modulo \( q \) such that the union \( \{ T, -T \} \) is a complete set of residues prime to \( q \). Then the real numbers \( D^{k-1}(\psi(q)/\phi)(x) \), \( x \in T \) are linearly independent over \( \mathbb{Q} \), where \( \psi \) is the Euler totient function and \( D = d/dz \).

In the case \( k = 2 \), this corresponds to the result of H. Jager and H. W. Lenstra, Jr. [6].

2. **Preliminary results.** We put

\[ F_k(x) = \begin{cases} \frac{k}{(-2\pi i)^k} D^{k-1}(\psi(x)) & \text{if } z \text{ is not an integer}, \\ 0 & \text{if } z \text{ is an integer and } k \text{ is odd}, \\ B_k & \text{if } z \text{ is an integer and } k \text{ is even}, \end{cases} \]

where \( B_k \) is the \( k \)th Bernoulli number. Then we have the following partial fraction decomposition of \( F_k(x) \):

\[ F_k(x) = \frac{k!}{(2\pi i)^k} \sum_{n=-\infty}^{\infty} \frac{1}{(x + n)^k}, \]

where the dash ' means that the term with \( n = -x \) is omitted if \( x \) is an integer. (If \( k = 1 \), we interpret the sum as grouping the corresponding positive and negative terms together.)

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**References**


Let \( \psi \) be an arithmetical function which is periodic mod \( q \). Then we have from (1)

\[
\sum_{n=-\infty}^{\infty} \frac{\psi(n)}{n^k} = \sum_{m=0}^{\infty} \frac{\psi(m)}{m^k} \sum_{n=-\infty}^{\infty} \frac{1}{(mq + n)^k} = \frac{1}{q^k} \sum_{m=0}^{q-1} \psi(m) \sum_{n=-\infty}^{\infty} \frac{1}{(m/q + n)^k}
\]

If we put

\[
\psi_*(m) = \frac{1}{q} \sum_{n=-\infty}^{q-1} \psi(n) e \left( \frac{-mn}{q} \right),
\]

where we write \( e^{i\theta} \) for \( e^{\text{int} \theta} \), then \( \psi_* \) is also periodic mod \( q \) and the following inversion formula holds:

\[
\psi(n) = -\frac{1}{q} \sum_{m=0}^{q-1} \psi_* (m) e \left( \frac{mn}{q} \right).
\]

From this we get

\[
\sum_{n=-\infty}^{\infty} \frac{\psi(n)}{n^k} = \sum_{m=0}^{\infty} \psi_* (m) \sum_{n=-\infty}^{\infty} \frac{1}{mn^k} = \frac{2\pi i k}{k!} \sum_{m=0}^{q-1} \psi_* (m) P_k \left( \frac{m}{q} \right),
\]

where

\[
P_k(x) = \begin{cases} 0 & \text{if } h = 1 \text{ and } x \text{ is an integer}, \\ B_k(x - [x]) & \text{otherwise}, \end{cases}
\]

is the \( k \)th Bernoulli function, \( B_k(x) \) denoting the \( k \)th Bernoulli polynomial (cf. [9], p. 16).

Letting \( \psi \) be the characteristic function of the set \( \{nq+b\} \), \( n = 0, \pm 1, \pm 2, \ldots \) and noting that \( \psi_*(m) = \frac{1}{q} \sum_{n=-\infty}^{\infty} e \left( \frac{-nm}{q} \right) \), we have from (1) and (3)

\[
P_k \left( \frac{b}{q} \right) = -\frac{k!}{(2\pi i)^k} \sum_{n=-\infty}^{\infty} \frac{1}{(b/q + n)^k} = -\frac{k!q^k}{(2\pi i)^k} \sum_{n=-\infty}^{\infty} \frac{\psi(n)}{n^k}
\]

\[
= q^{k-1} \sum_{m=0}^{q-1} e \left( \frac{-mb}{q} \right) P_k \left( \frac{m}{q} \right).
\]

Since \( P_k (-x) = (-1)^k P_k (x) \), we have from (5)

\[
F_k \left( \frac{-b}{q} \right) = (-1)^k F_k \left( \frac{b}{q} \right).
\]

We say that \( \psi \) is even (resp. odd) if \( \psi (-n) = \psi (n) \) (resp. \( \psi (-n) = -\psi (n) \)) for all integers \( n \). We note that if both \( k \) and \( \psi \) are even (or odd), we have

\[
\sum_{n=-\infty}^{\infty} \frac{\psi(n)}{n^k} = 2 \sum_{n=0}^{\infty} \frac{\psi(n)}{n^k}.
\]

We shall need the following lemma in the next section, which follows easily from the well-known Frobenius determinant relation (cf. [7], p. 284, Th. 5).

**Lemma.** Let \( G \) be a finite abelian group and let \( H \) be a subgroup of \( G \). Let \( \lambda \) be a character of \( H \) and let \( A \) be the set of all characters of \( G \) whose restriction to \( H \) is equal to \( \lambda \). Then for each (complex valued) function \( f \) on \( G \) with

\[
f(ab) = \lambda(h) f(a) \quad (a \in G, h \in H),
\]

we have

\[
\det f(a^{-1}) = \prod_{\alpha \in A} \left( \sum_{\beta \in \mathcal{Z}} \alpha \beta f(a) \right),
\]

where \( T \) is a complete representative system of \( G \) by \( H \) and \( \mathcal{Z} \) is the complex conjugate of \( \alpha \).

3. Proof of Theorem. Let \( \zeta \) denote a primitive \( q \)th root of unity. Then the Galois group of \( Q(\zeta) \) over \( Q \) is given by the mappings \( \sigma_q : \zeta \mapsto \zeta^a \), where \( a \) runs through a complete set of residues prime to \( q \). Since \( B_k(x) \) is a polynomial in \( x \) with coefficients in \( Q \), it follows from (4) that \( P_k (x) \in Q \)

for all \( x \in Q \). Hence the equation (5) shows that \( P_k \left( \frac{b}{q} \right) \in Q(\zeta) \) for all integers \( b \) and that

\[
\left( P_k \left( \frac{b}{q} \right) \right)^a = P_k \left( \frac{ab}{q} \right).
\]

To prove our theorem it suffices to show that \( P_k \left( \frac{b}{q} \right) \), \( b \in T \) are linearly independent over \( Q \).

Suppose that there exist \( G_b \in Q \) such that

\[
\sum_{b \in T} G_b P_k \left( \frac{b}{q} \right) = 0.
\]

Then applying the mappings \( \sigma_q : (a \in T) \), we have

\[
\sum_{b \in T} G_b P_k \left( \frac{ab}{q} \right) = 0,
\]
where $a$ is defined by $\overline{a} = 1 \mod q$. Now (8) together with (6) calls for the application of our lemma with

$G$: the group of reduced residue classes mod $q$,

$H = \{1, -1\}$,

$\lambda(1) = (-1)^k$,

$\Delta$: the set of all even or odd Dirichlet characters mod $q$ according as $k$ is even or odd,

$f(b) = F_k\left(\frac{b}{q}\right)$,

$T$: the set occurring in our theorem.

We obtain

$$\det F_k\left(\frac{\alpha b}{q}\right) = \prod_{\alpha \in \Delta} \left(\sum_{a \in \mathbb{Z}} \frac{\chi(a)}{\alpha} F_k\left(\frac{a}{\alpha q}\right)\right).$$

Here from (2) and (7) we have for any $\chi \in \Delta$

$$\sum_{a \in \mathbb{Z}} \frac{\chi(a)}{\alpha} F_k\left(\frac{a}{\alpha q}\right) = \frac{1}{2} \sum_{m = a}^{q-1} F_k\left(\frac{m}{q}\right) = -\frac{k!q^k}{(2\pi i)^k} \sum_{\alpha \in \Delta} L(k, \overline{\chi}).$$

From this and (9) we have

$$\det F_k\left(\frac{\alpha b}{q}\right) = \left(-\frac{k!q^k}{(2\pi i)^k}\right)^{\omega(\alpha)} \prod_{\alpha \in \Delta} L(k, \overline{\chi}) \neq 0.$$ 

This together with (8) shows that $C_b = 0$ for all $b \in T$, which completes the proof of our theorem.

4. Corollaries. Let $\Phi_q$ denote the $q$th cyclotomic polynomial.

**Corollary 1 (cf. Baker–Birch–Wirsing [3], Th. 1).** If $\psi$ is a non-vanishing arithmetical function with period $q$ such that (i) $\psi$ is even or odd according as $k$ is even or odd, (ii) $\psi(n) = 0$ if $(n, q) > 1$, (iii) $\Phi_q$ is irreducible over $\mathbb{Q}(\psi(1), \ldots, \psi(q))$, then

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^k} \neq 0.$$ 

**Proof.** We have from (7), (2), and the conditions (i), (ii)

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^k} = \frac{1}{2} \sum_{m = 1}^{q-1} \psi(m) \sum_{n = 1}^{\infty} \frac{1}{n^k} = \frac{(2\pi i)^k}{k!q^k} \sum_{m = 1}^{q-1} \psi(m) F_k\left(\frac{m}{q}\right) = -\frac{(2\pi i)^k}{k!q^k} \sum_{b \in \mathbb{Z}} \psi(b) F_k\left(\frac{b}{q}\right).$$

We see from the condition (iii) that the $q$th cyclotomic field $\mathbb{Q}(\zeta)$ and the field $\mathbb{Q}(\psi(1), \ldots, \psi(q))$ are linearly disjoint over $\mathbb{Q}$, so our theorem implies that $F_k\left(\frac{b}{q}\right), b \in T$, are linearly independent over $\mathbb{Q}(\psi(1), \ldots, \psi(q))$. 

Therefore

$$\sum_{b \in \mathbb{Z}} \psi(b) F_k\left(\frac{b}{q}\right) \neq 0,$$

as required.

**Corollary 2 (cf. [3]; Cor. 1 to Th. 1).** Let $\{q, q(g)\} = 1$ and let $A$ be the set of all even or odd Dirichlet characters mod $q$ according as $k$ is even or odd. Then the numbers $L(k, \overline{\chi})$, $\chi \in A$ are linearly independent over $\mathbb{Q}$.

**Proof.** This follows immediately from Corollary 1 on noting that any

$$\psi = \sum_{\alpha \in \Delta} a_{\alpha} \chi$$

with rational $a_\alpha$ fulfills the conditions of Corollary 1, since $\Phi_q$ is irreducible over the $q(g)$-th cyclotomic number field and the matrix $[\chi(a)](a \in T, \chi \in A)$ is nonsingular.

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References


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