[9] W. Forman and H. N. Shapiro, *Abstract prime number theorems*, Comm. Pure Appl. Math. 7 (1954), p. 587–619.
[10] G. Halász, *Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen*, Acta Math. Acad. Sci. Hungar. 19 (1968), p. 365–403.
[11] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience, 1974.
[12] H. Müller, *Ein Beitrag zur abstrakten Primzahltheorie*, J. Reine Angew. Math. 259 (1973), p. 171–182.
[13] D. V. Widder, *The Laplace transform*, Princeton University Press, 1946.

DÉPARTEMENT DE MATHÉMATIQUES
U.E.R. DES SCIENCES
123, rue Albert Thomas
87100 Limoges, France

Reçu le 25.1.1978
et dans la forme modifiée le 2.5.1978 (1035)

ACTA ARITHMETICA
XXXVIII (1980)

A note on Dirichlet's L-functions

by

R. BALASUBRAMANIAN (Bombay)

1. Introduction. The aim of the paper is to prove an asymptotic formula for $\sum_{\chi}|L(\frac{1}{2}+it,\chi)|^2$ (see Theorem 1 below). This is an improvement of result of Gallagher [3], who proves an upper bound for the sum. He remarks that it is possible to get by his method an asymptotic formula in the range $|t|<q^{O(1)}$. (The condition $|t|<q^{o(1)}$ as given in the paper is a misprint.) He also remarks that Selberg and Paley have asymptotic formulae for $|t|<q^{1/4-\varepsilon}$. We prove below the asymptotic formula in the range $|t|<q^{3/4-\varepsilon}$. In the range $|t|>q^{3/4-\varepsilon}$, Gallagher's result is better than ours. It is no doubt possible to deduce Gallagher's result also by our method, but since the proof is essentially the same as that of Gallagher, we are not giving the proof. For some other results, see § 5.

This paper can be considered as a continuation of my paper *A note on Hurwitz's Zeta function* [1].

2. Statement of the theorem.

THEOREM. *The asymptotic formula*

$$\sum_{\chi}|L(\tfrac{1}{2}+it,\chi)|^2 = \frac{\varphi^2(q)}{q}\log qt + O\big(q(\log\log q)^2\big) + O\big(te^{10\sqrt{\log q}}\big) + O\big(q^{1/2}t^{2/3}e^{10\sqrt{\log q}}\big)$$

holds uniformly for all values of q, and $t\geqslant 3$.

COROLLARY. *If $|t|\leqslant q^{3/4-\varepsilon}$, then*

$$\sum_{\chi}|L(\tfrac{1}{2}+it,\chi)|^2 \sim \frac{\varphi^2(q)}{q}\log qt \quad \textit{for all } t\geqslant 3.$$

3. Notation. To avoid minor complications, we make the following convention. A sum of the form $\sum_{a<m\leqslant b} f(m)$ is defined to be zero if either $b\leqslant a$ or the semiclosed interval $(a, b]$ does not contain even one integer. In

the sequel, $f = O(g)$ will mean $|f| \leqslant K|g|$ for some absolute constant K, independent of every parameter, $f = O_{\sigma,t,\varepsilon}(g)$ will mean $|f| \leqslant K|g|$ where K may depend on σ, t and ε. Both $f \ll g$ and $g \gg f$ will mean the same as $f = O(g)$. $f(q) \sim g(q)$ will mean $\dfrac{f(q)}{g(q)} \to 1$ as $q \to \infty$.

4. Proof.

LEMMA 1. *We have*

$$L(s,\chi) = \frac{1}{q^s}\sum_{a=1}^{q}\chi(a)\zeta\left(s;\frac{a}{q}\right),$$

where $\zeta(s; a/q)$ is the Hurwitz's zeta function.

Proof. This is easily proved in $\operatorname{Re} s \geqslant 2$ and extended by analytic continuation.

LEMMA 2. *We have*

$$\sum_{\chi}|L(s,\chi)|^2 = \frac{\varphi(q)}{q^{2\sigma}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\zeta\left(s;\frac{a}{q}\right)\right|^2,$$

where $\sigma = \operatorname{Re} s$.

Proof. This follows from Lemma 1 and the fact that

$$\sum_{\chi}\chi(m)\overline{\chi(n)} = \begin{cases}\varphi(q) & \text{if } \ m \equiv n \,(\operatorname{mod} q) \text{ and } (m,q)=1,\ (n,q)=1,\\ 0 & \text{otherwise.}\end{cases}$$

LEMMA 3. *We have*

$$\sum_{a=1}^{q}\left|\zeta\left(\tfrac{1}{2}+it,\frac{a}{q}\right)\right|^2 = q\log qt + O(q) + I_1 + \bar{I}_1$$

where

$$I_1 = q\sum_{k=1}^{T}\sum_{m=1}^{q(T-k+1)}\frac{1}{m^{\frac12+it}}\,\frac{1}{(m+kq)^{\frac12-it}}.$$

Proof. First of all, observe that

$$\zeta\left(\tfrac12+it,\frac{a}{q}\right) = \sum_{0\leqslant n\leqslant T}\frac{1}{\left(n+\frac{a}{q}\right)^{\frac12+it}} + O(T^{-1/2})$$

where T is the integer nearest to t. Using the bound

$$\left|\sum_{0\leqslant n\leqslant T}\frac{1}{\left(n+\frac{a}{q}\right)^{\frac12+it}}\right| = O(T^{1/6}\log^2 T) + O\left(\left(\frac{q}{a}\right)^{1/2}\right)$$

we have,

$$\sum_{a=1}^{q}\left|\zeta\left(\tfrac12+it,\frac{a}{q}\right)\right|^2 = \sum_{a=1}^{q}\left|\sum_{0\leqslant n\leqslant T}\frac{1}{\left(n+\frac{a}{q}\right)^{\frac12+it}}\right|^2 + O(qT^{-1/3}\log^2 T).$$

Now

$$\sum_{a=1}^{q}\left|\sum_{0\leqslant n\leqslant T}\frac{1}{\left(n+\frac{a}{q}\right)^{\frac12+it}}\right|^2 = q\sum_{a=1}^{q}\sum_{0\leqslant n\leqslant T}\sum_{0\leqslant m\leqslant T}\frac{1}{(qn+a)^{\frac12+it}}\,\frac{1}{(qm+a)^{\frac12-it}}.$$

The terms corresponding to $m = n$ give the main term $q\log qt$. The terms corresponding to $m \neq n$ give

$$q\sum_{a=1}^{q}\sum_{0\leqslant m\leqslant T}\sum_{n<m}\frac{1}{(qn+a)^{\frac12+it}}\,\frac{1}{(qm+a)^{\frac12-it}} +$$

$$+\,q\sum_{a=1}^{q}\sum_{0\leqslant m\leqslant T}\sum_{T\geqslant n>m}\frac{1}{(qn+a)^{\frac12+it}}\,\frac{1}{(qm+a)^{\frac12-it}}.$$

Obviously the second term is the conjugate of the first. We write the first term as

$$q\sum_{a=1}^{q}\sum_{k=1}^{T}\sum_{n=0}^{T-k}\frac{1}{(qn+a)^{\frac12+it}}\,\frac{1}{(qn+qk+a)^{\frac12-it}}$$

$$= q\sum_{k=1}^{T}\sum_{n=0}^{T-k}\sum_{a=1}^{q}\frac{1}{(qn+a)^{\frac12+it}}\,\frac{1}{(qn+qk+a)^{\frac12-it}}$$

$$= q\sum_{k=1}^{T}\sum_{n=0}^{T-k}\sum_{m=qn+1}^{qn+q}\frac{1}{m^{\frac12+it}}\,\frac{1}{(m+kq)^{\frac12-it}}.$$

Hence the lemma.

LEMMA 4. *If I_1 is defined as in Lemma 3, then*

$$I_1 = I_2 + I_3 + O(T^{2/3}\log^2 T) + O(T/q^{1/2})$$

where

$$I_2 = q\sum_{k=1}^{T}\sum_{m\leqslant 2T}\frac{1}{m^{\frac12+it}}\,\frac{1}{(m+kq)^{\frac12-it}}$$

and

$$I_3 = q\sum_{k=1}^{T}\sum_{2T<m\leqslant q(T-k+1)}\frac{1}{m^{\frac12+it}}\,\frac{1}{(m+kq)^{\frac12-it}}.$$

Proof. We have only to prove that (see § 3), if k is such that $q(T-k+1) \leqslant 2T$, then replacing the upper limit by $2T$ gives an error $O(T^{2/3}\log^2 T) + O(T/q^{1/2})$. If $q = 1, 2, 3$ or 4 it is trivial to prove. If $q > 5$, the error is

$$O\Big(q \sum_{k > T-2T/q} \frac{1}{(kq)^{1/2}} \sum_{m \leqslant 2T} \frac{1}{m^{1/2}}\Big)$$

and observe that $T - 2T/q \geqslant T$. This gives the lemma.

LEMMA 5. *We have* $I_3 = O(q\log\log t)$.

Proof. Using Lemma 4.10 (page 66) of Titchmarsh [5] we prove that

$$\sum_{2T < m \leqslant q(T-k+1)} \frac{1}{m^{1/2}(m+kq)^{1/2}} e^{-it(\log m - \log(m+kq))}$$
$$= \int_{2T}^{q(T-k+1)} \frac{1}{u^{1/2}(u+kq)^{1/2}} e^{-it(\log u - \log(u+kq))}\,du + O\big(q^{-1/2}(kt)^{-1/2}\big)$$
$$= \int_{2T/q}^{T-k+1} \frac{1}{u^{1/2}(u+k)^{1/2}} e^{-it(\log u - \log(u+k))}\,du + O\big(q^{-1/2}(kt)^{-1/2}\big).$$

Hence

$$I_3 = q\sum_{k=1}^{T} \int_{2T/q}^{T-k+1} \frac{1}{u^{1/2}(u+k)^{1/2}} e^{-it(\log u - \log(u+k))}\,du + O(q^{1/2}).$$

Now an application of Lemma 4.3 (page 61) of Titchmarsh [5] gives that

$$q\sum_{k=1}^{T} \int_{1}^{2T/q} \frac{1}{u^{1/2}(u+k)^{1/2}} e^{-it(\log u - \log(u+k))}\,du = O(q^{1/2}).$$

Hence it follows that

$$I_3 = q\sum_{k=1}^{T} \int_{1}^{T-k+1} \frac{1}{u^{1/2}(u+k)^{1/2}} e^{-it(\log u - \log(u+k))}\,du + O(q^{1/2}) = qI_4 + O(q^{1/2})$$

where

$$I_4 = \sum_{k=1}^{T} \int_{1}^{T-k+1} \frac{1}{u^{1/2}(u+k)^{1/2}} e^{-it(\log u - \log(u+k))}\,du.$$

$I_4 = O(\log\log t)$ has been proved in Balasubramanian [1].

As we will show below, this almost completes the proof of the theorem except for the fact that we have to get a 'good' bound for I_2. Here I am not able to get anything better than the trivial bound. Any improvement in the estimate for I_2 proves directly the validity of the asymptotic formula in a wider range.

LEMMA 6. *We have* $I_2 = O(q^{1/2}T^{2/3}\log^2 T)$.

Proof.

$$I_2 = q^{\frac12+it}\sum_{m \leqslant 2T} \frac{1}{m^{\frac12+it}} \sum_{k=1}^{T} \frac{1}{(k+m/q)^{\frac12-it}} = q^{\frac12+it}\sum_{m \leqslant 2T} \frac{1}{m^{\frac12+it}}\zeta_1(\bar{s}; m/q) + O(q^{1/2}).$$

Now use the bound $\zeta_1(\bar{s}; m/q) = O(T^{1/6}\log^2 T)$ and take the trivial estimate. This proves the lemma.

LEMMA 7. *We have the asymptotic formula*

$$\sum_{a=1}^{q} |\zeta(\tfrac12+it, a/q)|^2 = q\log qt + O(q\log\log q) +$$
$$+ O(q\log\log t) + O(q^{1/2}T^{2/3}\log^2 T) + O(T/q^{1/2}).$$

Proof. The result follows from Lemmas 3, 4, 5 and 6.

LEMMA 8. *We have*

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} |\zeta(\tfrac12+it, a/q)|^2 = \varphi(q)\log qt + O\big(q(\log\log q)^2\big) +$$
$$+ O\big(q(\log\log q)(\log\log t)\big) + O(q^{1/2}e^{10\sqrt{\log q}}T^{2/3}) + O(Te^{10\sqrt{\log q}}).$$

Proof.

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} |\zeta(\tfrac12+it, a/q)|^2 = \sum_{a=1}^{q}{}' |\zeta(\tfrac12+it, a/q)|^2 \sum_{\substack{d|q\\d|a}} \mu(d)$$
$$= \sum_{d|q} \mu(d) \sum_{a=1}^{q/d} |\zeta(\tfrac12+it, a/(q/d))|^2$$

and use Lemma 7. We have also to use the facts like

$$\sum_{d|q} \frac{\mu(d)\log d}{d} = O\big((\log\log q)^2\big),$$

$$\sum_{d|q} \frac{1}{d} = O(\log\log q),$$

$$\sum_{d|q} \frac{1}{d^{1/2}} = O(e^{10\sqrt{\log q}}).$$

Lemma 2 and Lemma 8 complete the proof of the theorem.

5. Few more results. As can be easily seen, our result improves the result of Elliott [2], particularly the second part of Theorem 1 (ii). It is not difficult to improve his other results also and prove the following theorems. The proof which we have given for Theorem 1 can be adopted to prove Theorem 2 and 3 given below. We are giving an outline of an alternative proof. Accordingly we prove the following theorems.

THEOREM 2. *If q is an integer > 2, s any complex number in the strip $3/4 \geqslant \sigma > 1/2$, we have, for $\varepsilon > 0$*

$$\sum_{\chi\neq\chi_0} |L(s,\chi)|^2 = \varphi(q)\prod_{p|q}\Big(1-\frac{1}{p^{2\sigma}}\Big)\zeta(2\sigma)+O_{\sigma,t,\varepsilon}\big(q^{(2+\varepsilon)(1-\sigma)}\big).$$

THEOREM 3. (i) *We have*

$$\sum_{\chi\neq\chi_0} |L(\tfrac{1}{2}+it,\chi)|^2 = \frac{\varphi^2(q)}{q}\log q + O_\varepsilon\big(q(\log q)^\varepsilon\big)$$

uniformly in $|t| \leqslant (\log q)^\varepsilon$.

(ii) *If $1/4 \leqslant \sigma \leqslant 1/2$ then*

$$\sum_{\chi\neq\chi_0} |L(\sigma+it,\chi)|^2 = O_t(q^{2-2\sigma}\log q).$$

(iii) *Let δ_q be a function of q which is*

$$O\bigg(\Big(\frac{\varphi(q)}{q}\Big)^4 (\log q)^{-1}\bigg).$$

For each character χ, let s_χ be a point in the disc $|s-\frac{1}{2}| \leqslant \delta_q$. Then

$$\sum_{\chi\neq\chi_0} |L(s_\chi,\chi)|^2 \sim \frac{\varphi^2(q)}{q}\log q.$$

THEOREM 4. (i) *If q is sufficiently large, there are $\gg q(\log q)^{-1000}$ L-functions to the modulus q, which are such that*

$$|L(s,\chi)| \geqslant C\Big(\frac{\varphi(q)}{q}\Big)^{1/2}(\log q)^{1/2}$$

in the circle $|s-\frac{1}{2}| \leqslant \delta_q$ where C is a small positive constant. In particular they are zero free in the circle $|s-\frac{1}{2}| \leqslant \delta_q$.

(ii) *If q is a sufficiently large prime, there are $\gg q(\log q)^{-2}$ L-functions to the modulus q, which are such that*

$$|L(s,\chi)| \geqslant C\Big(\frac{\varphi(q)}{q}\Big)^{1/2}(\log q)^{1/2}$$

in the circle $|s-\frac{1}{2}| \leqslant \delta_q$ where C is a small positive constant. In particular they are zero free in the circle $|s-\frac{1}{2}| \leqslant \delta_q$.

6. Proof. Let $T = |t|+2$. Let χ be any non-principal character. Let $\varepsilon > 0$ be given. If $\sigma > 1/2$ we start with

$$h\sum_{n=1}^{\infty} n^{-s}\chi(n)e^{-\left(\frac{n}{X}\right)^h} = \frac{1}{2\pi i}\int_{\operatorname{Re} w=2} L(s+w,\chi)X^w\Gamma\Big(\frac{w}{h}\Big)dw,$$

$$X = (qT)^{1+\frac{\varepsilon(2\sigma-1)}{100}}, \qquad h = \Big[\frac{50\,000}{\varepsilon(2\sigma-1)}\Big]+1$$

and move the line of integration to $\operatorname{Re} w = -h/2$. This gives

LEMMA 9. *For any non-principal character χ, we have*

$$\sum_{n=1}^{\infty} n^{-s}\chi(n)e^{-\left(\frac{n}{X}\right)^h} = L(s,\chi)+O_h\big((qT)^{(h+1)/2}X^{-h/2}\big) \quad \textit{if} \quad 1/4 \leqslant \sigma \leqslant 3/4.$$

Using the fact that

$$\sum_{\chi}\chi(m)\overline{\chi(n)} = \begin{cases}\varphi(q) & \text{if} \quad m\equiv n \,(\operatorname{mod} q),\ (m,q)=1,\ (n,q)=1;\\ 0 & \text{otherwise},\end{cases}$$

and Lemma 9, we prove

$$\begin{aligned}\sum_{\chi\neq\chi_0}|L(s,\chi)|^2 &= \varphi(q)\sideset{}{'}\sum_{\substack{n=1\\ n\equiv m\,(\operatorname{mod} q)}}^{\infty}\sideset{}{'}\sum_{m=1}^{\infty}\frac{1}{(mn)^\sigma}e^{-\left(\frac{n}{X}\right)^h-\left(\frac{m}{X}\right)^h}\Big(\frac{m}{n}\Big)^{it} -\\ &\quad -\sideset{}{'}\sum_{n=1}^{\infty}\sideset{}{'}\sum_{m=1}^{\infty}\frac{1}{(mn)^\sigma}e^{-\left(\frac{n}{X}\right)^h-\left(\frac{m}{X}\right)^h}\Big(\frac{m}{n}\Big)^{it}+O(q^{-100}T^{-100})\\ &= \varphi(q)\sideset{}{'}\sum_{\substack{n\leqslant q\\ m\equiv n\,(\operatorname{mod} q)}}\sideset{}{'}\sum_{m\leqslant q}+\varphi(q)\sideset{}{'}\sum_{\substack{n\leqslant q\\ m\equiv n\,(\operatorname{mod} q)}}\sideset{}{'}\sum_{m>q}+\\ &\quad +\varphi(q)\sideset{}{'}\sum_{\substack{n>q\\ m\equiv n\,(\operatorname{mod} q)}}\sideset{}{'}\sum_{m\leqslant q}+\varphi(q)\sideset{}{'}\sum_{\substack{n>q\\ m\equiv n\,(\operatorname{mod} q)}}\sideset{}{'}\sum_{m>q}-\sideset{}{'}\sum_{n=1}^{\infty}\sideset{}{'}\sum_{m=1}^{\infty}+O(q^{-100}T^{-100})\\ &= \Sigma_1'+\Sigma_2'+\Sigma_3'+\Sigma_4'-\Sigma_5'+O(q^{-100}T^{-100}), \text{ say}\end{aligned}$$

where $\sum'$ denotes the summation over those integers which are relatively prime to q.

To prove Theorem 2, take $\sigma > 1/2$, s any fixed point and observe that

$$\sum_{n=1}^{\infty}\frac{1}{n^\sigma}e^{-\left(\frac{n}{Y}\right)^h} = O_h(Y^{1-\sigma}+1) \quad \text{in} \quad 1/2 \leqslant \sigma \leqslant 3/4.$$

This gives

$$\Sigma_1' = \varphi(q)\prod_{p|q}\left(1-\frac{1}{p^{2\sigma}}\right)\zeta(2\sigma)+O_\sigma(q^{2-2\sigma}),$$

$$\Sigma_j' = O_{\sigma,t,\varepsilon}(q^{(2+\varepsilon)(1-\sigma)}), \quad j = 2, 3, 4 \text{ or } 5.$$

This completes the proof of Theorem 2.

To prove Theorem 3(i), we proceed exactly in the same way, taking $\sigma = 1/2$. By taking $1/4 \leqslant \sigma \leqslant 1/2$, and proceeding in the same way, we prove Theorem 3(ii). In these cases we take $X = (qt)^{1+\frac{\varepsilon}{100}}$, $h = \left[\frac{5000}{\varepsilon}\right]$.

7. Proof of Theorem 3(iii). We argue as on page 226 of [2]. Let $0 < R < 1/3$ and $R = R_q$ be such that $\delta_q R_q^{-1} \to 0$ as $q \to \infty$. Let Γ be a circle with centre $1/2$ and radius R traversed in the anti clockwise direction. Then from the formula

$$L^2(s_\chi, \chi) - L^2(\tfrac{1}{2}, \chi) = \frac{1}{2\pi i}\int_\Gamma L^2(w, \chi)\left(\frac{1}{w-s_\chi} - \frac{1}{w-\frac{1}{2}}\right)dw$$

the theorem follows in view of Theorem 3(ii) provided we impose on R the condition

$$\frac{\delta}{R}\, q^{2-2(\frac{1}{2}-R)}\log q = O\left(\frac{\varphi^2(q)}{q}\log q\right);$$

we can choose for instance

$$R = \left(\log q\left(\frac{q}{\varphi(q)}\right)^4\right)^{-1}\left(\delta_q \log q\left(\frac{q}{\varphi(q)}\right)^4\right)^{1/2}$$

and this completes the proof.

8. Proof of Theorem 4.

LEMMA 10. *Let for character $\chi \pmod q$, points $s_{\chi,r} = \sigma_{\chi,r} + it_{\chi,r}$ be given which satisfy the following conditions:*

$$\tfrac{1}{2} \leqslant \sigma_{\chi,r} \leqslant 1, \qquad -T \leqslant t_{\chi,1} < t_{\chi,2} < \ldots < t_{\chi,r} \leqslant T,$$
$$|t_{\chi,r}| \geqslant 1 \quad \textit{if } \chi \textit{ is principal},$$
$$|t_{\chi,j+1} - t_{\chi,j}| \geqslant 1 \quad \textit{if} \quad R_\chi \geqslant 2.$$

Then

$$\sum_{\chi \pmod q}^{*}\sum_{r=1}^{R_\chi}\left|\frac{L(\sigma_{\chi,r}+it_{\chi,r}, \chi)}{C_1(\log qT)^{100}}\right|^{\frac{2}{1-\sigma_{\chi,r}}} \ll qT \quad (T \geqslant 3),$$

and

$$\sum_{\chi \pmod q}\sum_{r=1}^{R_\chi}\left|\frac{L(\sigma_{\chi,r}+it_{\chi,r}, \chi)}{C_1(\log qT)^{100}}\right|^{\frac{2-1/100}{1-\sigma_{\chi,r}}} \ll qT \quad (T \geqslant 3).$$

Proof. This is Theorem 5 of Ramachandra [4]. (Incidentally, there is a typographical mistake in the statement of the theorem, viz. Lemma 3 should read as Theorem 3. Further $*$ is missing in the first result.)

LEMMA 11. *For every $\chi \pmod q$, let s_χ be a point satisfying*

$$\tfrac{1}{2} \leqslant \sigma_\chi \leqslant \tfrac{1}{2} + \frac{1}{\log q} \quad \textit{and} \quad |t_\chi| \leqslant (\log q)^4.$$

Then for sufficiently large q

$$\sum_{\substack{\chi \pmod q\\ \chi \neq \chi_0}} |L(\sigma_\chi + it_\chi, \chi)|^3 \ll q(\log q)^{400}.$$

Proof. Take $R_\chi = 1$ in the second result of Lemma 10 and apply Hölder's inequality.

From Lemma 11, we deduce

LEMMA 12. *For every χ, let s_χ be a point in the disc*

$$|s - \tfrac{1}{2}| \leqslant \frac{1}{\log q} \quad \textit{and} \quad \sigma_\chi \geqslant \tfrac{1}{2}.$$

Then

$$\sum_{\chi \neq \chi_0} |L(s_\chi, \chi)|^3 \ll q(\log q)^{400}.$$

Now we prove

LEMMA 13. *For every χ, let s_χ be a point in the disc*

$$|s - \tfrac{1}{2}| \leqslant \frac{1}{\log q} \quad \textit{and} \quad \sigma_\chi \leqslant \tfrac{1}{2}.$$

Then

$$\sum_{\chi \neq \chi_0} |L(s_\chi, \chi)|^3 \ll q(\log q)^{400}.$$

Proof. Applying maximum modulus principal to the function

$$F(w) = L(w, \chi)\, e^{(w-s_\chi)^2} q^{2(w-s_\chi)}$$

in the rectangle

$$-2 \leqslant \sigma \leqslant \tfrac{1}{2} + \frac{1}{\log q}, \qquad |t - t_\chi| \leqslant \log^2 q,$$

we can prove that

$$|F(s_\chi)| \ll 1+|F(s'_\chi)|,$$

where s'_χ is a point in $\sigma = \frac{1}{2}+1/\log q$; $|t-t_\chi| \leqslant \log^2 q$ where the maximum of the function is attained. Hence

$$\sum_{\chi\neq\chi_0} |F(s_\chi)|^3 \ll q + \sum_{\chi\neq\chi_0} |F(s'_\chi)|^3.$$

Applying Lemma 11 to the right-hand side, the result follows.

From Lemma 12 and 13, we deduce

LEMMA 14. *For every* χ, *let* s_χ *be a point in the disc* $|s-\frac{1}{2}| \leqslant 1/\log q$. *Then*

$$\sum_{\chi\neq\chi_0} |L(s_\chi, \chi)|^3 \ll q(\log q)^{400}.$$

Now Theorem 4(i) follows from the facts

$$\sum_{\chi\neq\chi_0} |L(s_\chi, \chi)|^2 \ll \frac{\varphi^2(q)}{q}\log q \quad \text{(Theorem 3(i))}$$

and

$$\sum_{\chi\neq\chi_0} |L(s_\chi, \chi)|^3 \ll q(\log q)^{400} \quad \text{(Lemma 14)}.$$

In case q is a prime, we have a stronger version of Lemma 14, viz.

LEMMA 15. *For every* $\chi(\bmod q)$, q *prime, let* s_χ *be a point in the disc* $|s-\frac{1}{2}| \leqslant 1/\log q$. *Then*

$$\sum_{\chi\neq\chi_0} |L(s_\chi, \chi)|^4 \ll q(\log q)^4.$$

This lemma is easily deduced from the special case of the theorem given in the appendix of [4], which we state as

LEMMA 16. *If* q *is a prime* $\geqslant 2$, $|\sigma-\frac{1}{2}| \leqslant 1/\log q$,

$$\sum_{\chi\neq\chi_0} \int_{|t|\leqslant 30} |L(\sigma+it, \chi)|^4 dt \ll q(\log q)^4.$$

This completes the proof of Theorem 4(ii).

References

[1] R. Balasubramanian, *A note on Hurwitz's zeta function*, Ann. Acad. Sci. Fenn. Ser. A I, 4(1978-79), pp. 41–44.

[2] P. D. T. A. Elliott, *On the distribution of the values of Dirichlet L-series in the half plane* $\sigma > \frac{1}{2}$, Indag. Math. 33 (1971), pp. 222–234.

[3] P. X. Gallagher, *Local mean value and density estimates for Dirichlet L-functions*, ibid. 37 (1975), pp. 259–264.

[4] K. Ramachandra, *A simple proof of the mean fourth power estimate for* $\zeta(\frac{1}{2}+it)$ *and* $L(\frac{1}{2}+it, \chi)$, Ann. Scuola Norm. Sup. Pisa, Serie IV, 1 (1974), pp. 81–97.

[5] E. C. Titchmarsh, *The theory of the Riemann Zeta-function*, Oxford 1951.

SCHOOL OF MATHEMATICS
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Bombay, India

Received on 4.4.1978 (1059)